The SO(10, 10) matrix model

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Motivation

Matrix models

- ✔ Strong relations to string theory *and* noncommutative geometry.

  Banks, Fischler, Shenker, Susskind '96; Ishibashi, Kawai, Kitazawa, Tsuchiya '96

  Connes, Douglas, Schwarz '97; Berenstein, Maldacena, Nastase '02; etc.

- ✔ Address both conceptual and phenomenological questions.

  particle physics models ACh, Steinacker, Zoupanos '10-'12; Aoki '10-'13, . . .

  cosmology (early and late) Kim, Nishimura, Tsuchiya '11-'14

- ✔ Notably, address emergence of spacetime.
- ✔ Combine both analytical and numerical techniques.

Motivation

Phase space and (Quantum) Gravity

- Arguments that quantum gravity requires dynamical theory of phase space.
- Born reciprocity and string theory. Freidel, Leigh, Minic '14
- The gravitational field is encoded in noncommutative phase space:

$$[\hat{x}^a, \hat{p}_i] = i\hbar e_i^a(\hat{x}) \ .$$

  Noncommutative frame formalism. Madore '00; Buric, Madore '11
- In the IIB matrix model: Einstein equations, when matrices are interpreted as differential operators on commutative space. Hanada, Kawai, Kimura '05

Is there a matrix model that could capture the dynamics of phase space?

The plan

1. The type IIB matrix model
2. The SO(10,10) model and its origin
3. Noncommutative phase spaces with nontrivial frame
4. Towards a relation to string theory
5. Take-home messages

The type IIB model

Ishibashi, Kawai, Kitazawa, Tsuchiya '96

Definition

Matrix path integral

$$Z = \int dA d\Psi e^{-S} ,$$

with action

$$S_{\text{IIB}} = -\frac{1}{4g^2} g^{MM'} g^{NN'} \text{Tr}\, [A_M, A_N][A_{M'}, A_{N'}] - \frac{1}{2g^2} \text{Tr}\, \bar{\Psi} \Gamma^M [A_M, \Psi] \; .$$

A_M: 10 $N \times N$ Hermitian matrices (large N) .
Ψ: fermionic superpartners.

Such integrals converge for certain dimensions (incl. 10) and gauge groups.
Krauth, Staudacher '98; Austing, Wheater '01

Origin

- Reduction of 10D sYM to a point. Eguchi, Kawai; Parisi, Zhang; Gross, Kitazawa '82
  Bosonic sector:
  $$\int \mathrm{d}^{10}x \,\mathrm{Tr}\, F \wedge \star F \ ,$$
  where
  $$F = \tfrac{1}{2}(\partial_M A_N - \partial_N A_M + i[A_M, A_N])\mathrm{d}x^M \wedge \mathrm{d}x^N \ ,$$
  and reduction implemented via $\partial_M A_N = 0$.
- Regularization of the (Green-Schwarz) IIB superstring action (more later).

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Symmetries

- Translational $A_M \to A_M + c_M \mathbb{1} \ , \ c_M \in \mathbb{R}$.
- Gauge $A_M \to U A_M U^{-1}, \ U \in U(N)$.
- Global rotational $A_M \to L_M^N A_N, \ L \in$ SO(10) or SO(9, 1) .
- $\mathcal{N} = 2$ supersymmetry

Classical solutions

Equations of motion ($\Psi = 0$):

$$g^{MM'}[A_M, [A_{M'}, A_N]] = 0 \ .$$

- Moyal type solutions

$$[A_M, A_N] = i\theta_{MN}$$

- Classification of Lie type solutions ACh '11

$$[A_M, A_N] = iC_{MNP}A_P$$

  $\rightsquigarrow$ no semisimple, but nilpotent (7) and solvable (2) Lie algebras.
  Motivated models for the expansion of the universe Kim, Nishimura, Tsuchiya '11
- Solutions with "split noncommutativity", $\mathbb{R}^{3,1} \times_\theta K$. Steinacker '11
- General prescription to find Lorentzian solutions Kim, Nishimura, Tsuchiya '12
- Also, quantized minimal surfaces, e.g. catenoid Arnlind, Hoppe '12
- ...

Standard interpretations

- A_M as quantized coordinates, fluctuations as gauge fields
  $\rightsquigarrow$ emergent spacetime and NC Yang-Mills   Aoki et al. '98-'99
- A_M as noncommutative momenta, differential operator viewpoint.
  Connes, Douglas, Schwarz '97, . . .

Extension of the model that implements both pictures and describes full phase space.

The SO(10, 10) matrix model

ACh '14

Bosonic action of the model:

$$
\begin{aligned}
S \quad = \quad & -\tfrac{1}{4}\mathrm{Tr}\Big(g^{MM'} g^{NN'} [A_M, A_N][A_{M'}, A_{N'}] \\
& + \tilde{g}_{MM'} \tilde{g}_{NN'} [V^M, V^N][V^{M'}, V^{N'}] \\
& + 2\, g^{MM'} \tilde{g}_{NN'} [A_M, V^N][A_{M'}, V^{N'}] \\
& - 2 g^{MP} g^{M'Q} b_{QN} b_{PN'} [A_M, V^N][A_{M'}, V^{N'}] \\
& + 2 g^{MP} g^{NQ} b_{PM'} b_{QN'} [A_M, A_N][V^{M'}, V^{N'}] \\
& + 4 g^{MM'} g^{NP} b_{N'P} [A_M, A_N][A_{M'}, V^{N'}] \\
& + 4 g^{MP} \tilde{g}_{NN'} b_{M'P} [A_M, V^N][V^{M'}, V^{N'}] \Big) \, .
\end{aligned}
$$

10 + 10 hermitian matrices A_M and V^M.

b: antisymmetric (const.) 2-tensor and $\tilde{g} = g - b g^{-1} b$.

Origin: Reduction of “Generalized Yang-Mills” theory to a point.

Elements of Courant algebroids

Liu, Weinstein, Xu '95

Algebroids: Merge the notions of algebra and vector bundle.

"Algebras whose structure constants are not constant".

Courant algebroids merge Drinfel'd doubles and generalized tangent bundles.

Elements of Courant algebroids

Liu, Weinstein, Xu '95

The simplest such structure: standard Courant algebroid:

$$(\mathcal{T}M = \mathrm{TM} \oplus \mathrm{T}^\star \mathrm{M}, [\cdot,\cdot]_C, \langle\cdot,\cdot\rangle, \rho : \mathcal{T}\mathrm{M} \to \mathrm{TM})$$

- Sections: $\mathfrak{X} \in \Gamma(\mathcal{T}M)$: $\mathfrak{X} = X + \eta$,
- Courant bracket: $[\mathfrak{X}_1, \mathfrak{X}_2]_C = [X_1, X_2]_{\mathrm{Lie}} + \mathcal{L}_{X_1}\eta_2 - \mathcal{L}_{X_2}\eta_1 - \frac{1}{2}\mathrm{d}\big(X_1(\eta_2) - X_2(\eta_1)\big)$.
- Pairing: $\langle\mathfrak{X}_1, \mathfrak{X}_2\rangle = \frac{1}{2}\big(X_1(\eta_2) + X_2(\eta_1)\big)$.
- Anchor: $\rho(\mathfrak{X}) = X$.

Connection (gauge field) and curvature (field strength)

Generalized connection:

$$\mathcal{D} = \mathrm{d} + A + V = \partial_M \mathrm{d}x^M + A_M \mathrm{d}x^M + V^M \partial_M \, .$$

Curvature defined as usual:

$$\mathcal{F}(\mathfrak{X}_1, \mathfrak{X}_2) = [\mathcal{D}_{\mathfrak{X}_1}, \mathcal{D}_{\mathfrak{X}_2}] - \mathcal{D}_{[\mathfrak{X}_1, \mathfrak{X}_2]} \, .$$

For the generalized connection in question:

$$\mathcal{F} = F + DV + [V, V] \, ,$$

where we use the notation

$$\begin{aligned} F &= \tfrac{1}{2} F_{MN} \mathrm{d}x^M \wedge \mathrm{d}x^N \, , \\ DV &= (\partial_M V^N + i[A_M, V^N]) \mathrm{d}x^M \wedge \partial_N \, , \\ [V, V] &= \tfrac{i}{2} [V^M, V^N] \partial_M \wedge \partial_N \, , \end{aligned}$$

the bracket being the gauge commutator.

An issue and two solutions

We face a problem in defining YM theory

- The Courant bracket does not satisfy the Jacobi identity.
- Elements of $\wedge^\bullet \mathcal{T}\mathrm{M}$, such as $\mathcal{F}$, do not transform as tensors.

Two ways out

- ✔ Use Dirac structures Courant '90, i.e. subbundles $L \subset \mathcal{T}\mathrm{M}$ such that

$$\dim L = \tfrac{1}{2}\dim \mathcal{T}\mathrm{M}, \quad \langle \mathfrak{X}_L, \mathfrak{Y}_L \rangle = 0, \quad [\mathfrak{X}_L, \mathfrak{Y}_L] \in \Gamma(L), \forall\, \mathfrak{X}_L, \mathfrak{Y}_L \in \Gamma(L)\ .$$

  ACh, Gautason '14; cf. Fournel, Lazzarini, Masson '12
- ✔ New idea: Reduce to a point.

Consider the Lagrangian: $\frac{1}{4}\mathcal{H}^{MM'}\mathcal{H}^{NN'}\mathcal{F}_{MN}\mathcal{F}_{M'N'}$,

formed with the generalized metric on $\mathcal{T}\mathrm{M}$: $\mathcal{H} = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix}$.

Its point reduction yields the SO(d,d) matrix model.

Back to the matrix model

Symmetries

- Translational and gauge symmetries for both A_M and V^M.
- Extended rotational $\mathrm{SO}(10,10)$.
- Note the symmetry: $A^M \to V_M$ and $V_M \to -A^M$.

Equations of motion

$$\Box A_M = 0 \ , \quad \Box V^M = 0 \ ,$$

where we defined the generalized Laplacian operator

$$\begin{aligned} \Box \cdot \quad = \quad & g^{MM'}[A_M,[A_{M'},\cdot]] + (g - bg^{-1}b)_{MM'}[V^M,[V^{M'},\cdot]] \\ & + g^{MP} b_{M'P}\big([A_M,[V^{M'},\cdot]] + [V^{M'},[A_M,\cdot]]\big) \ . \end{aligned}$$

Solutions?

Emergent phase space

For $b = 0$ a large class of solutions is obtained from the vacuum Ansatz

$$A_a = e_a^i(\hat{x})\hat{p}_i \ , \quad V^a = \hat{x}^a \ , \quad a = 1, \dots, 2m, \ 2m \leq d \ ,$$
$$A_{2m+1} = \cdots = A_d = V^{2m+1} = \cdots = V^d = 0 \ ,$$

with phase space algebras of the general form:

$$\begin{aligned} [\hat{x}^a, \hat{x}^b] &= i\theta^{ab} \ , \\ [\hat{x}^a, \hat{p}_i] &= i\hbar e^a_i \ , \\ [\hat{p}_i, \hat{p}_j] &= M_{ij} + N_{ij}{}^k \hat{p}_k + P_{ij}^{kl} \hat{p}_k \hat{p}_l \ , \end{aligned}$$

with known coefficients M, N, P.

Agreement with general result of noncommutative frame formalism. Madore '00

$\rightsquigarrow$ Non-commutative spaces with a nontrivial frame $e^i{}_a(\hat{x})$. cf. the work of Burić, Madore

A class of nontrivial examples - Nilmanifolds

Associated to nilpotent Lie algebras.

Compact case: Iterated toroidal fibrations.

Nipotency step $\sim$ fiber iteration.
Step 1 $\rightarrow$ torus (Kähler nilmanifold).

Benson, Gordon '88

Step 2 to d-1$\rightarrow$ non-Kähler, often symplectic.

$\exists$ global basis $e^i \rightsquigarrow$ parallelizable.

Classical frame components:

$$e^i{}_a = \delta^i{}_a + \tfrac{1}{2} f^i{}_{ab} x^b + \mathcal{O}(x^2) \ .$$

$$\begin{tikzcd}
T^{d_n} \arrow[r, hook] & \mathcal{M}^{d=\sum_i d_i} \arrow[d] \\
 & \vdots \arrow[d] \\
T^{d_3} \arrow[r, hook] & \mathcal{M}^{d_1+d_2+d_3} \arrow[d] \\
T^{d_2} \arrow[r, hook] & \mathcal{M}^{d_1+d_2} \arrow[d] \\
 & T^{d_1}
\end{tikzcd}$$

Explicit nontrivial examples with quadratic phase space algebra do exist.

Revisiting the origin of the model

Recall that the IIB model is obtained as "regularization" of the Green-Schwarz action:

$$S_{\text{GS}} = -T \int \mathrm{d}^2\sigma \sqrt{-G} \xrightarrow{\text{Schild's trick}} \int \mathrm{d}^2\sigma \sqrt{g} \tfrac{1}{4} \{X^\mu, X^\nu\}^2 + \ldots \xrightarrow{\text{Q/C correspondence}} S_{\text{IIB}}$$

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<u>work in progress</u>

The SO(10,10) model is related to the T-duality symmetric worldsheet action: Hull '04

$$S = -T \int \mathcal{H}_{MN} \mathrm{d}\mathcal{X}^M \wedge \star \mathrm{d}\mathcal{X}^N \,,$$

written in the context of the doubled formalism and being closely related to Tseytlin's formulation of closed superstrings. Tseytlin '90

Take-home messages

The SO(10,10) matrix model is a well motivated theory.

- Originates from a generalized YM theory on a Courant algebroid by reduction.
- It is related to T-duality symmetric formulations of closed string theory.
- Noncommutative phase spaces with nontrivial frame as classical solutions.
- It should address the dynamics of full phase space, not just spacetime.

Several open issues.

- Frame is encompassed, but how does gravity emerge? Einstein equations?
- General solutions and effective action?
- Quantization?

The best is yet to come