Squashed fuzzy extra dimensions in $\mathcal{N} = 4$ SYM with large rank

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H.S., J. Zahn arXiv:1409.1440, H.S. arXiv:1504.05703

Motivation

- mechanism in large-N Yang-Mills with adjoint scalars:
  dynamical generation of extra dimensions $\mathbb{R}^4 \times \mathcal{K}_N$
  (Higgs mechanism!)

  Madore, Myers, Arkani-Hamed etal, HS-Zoupanos-Chatzistavrakidis, Aschieri, Manousselis, O'Connor etal, Polchinski-Strassler, Andrews-Dorey,...
- aspects of string theory (intersecting branes, KK modes ...) realized in 4-D gauge theory

  not holographic, "direct"
- here: $\mathcal{N} = 4$ SYM, deformed by cubic potential

outline

- $SU(N)$ $\mathcal{N} = 4$ SYM with cubic (flux) terms
- new vacuum solutions with $SU(3)$ structure
- geometry: squashed (fuzzy) coadjoint orbits of $SU(3)$

  zero modes & chiral fermions
- stacks of branes,
  approaching physically interesting configurations

$\mathcal{N} = 4$ SYM and squashed fuzzy branes

starting point: $SU(N)$ $\mathcal{N} = 4$ SUSY

$$\boxed{\begin{aligned} S &= \int d^4x \frac{1}{4g^2} tr\Big(- F^{\mu\nu}F_{\mu\nu} - 2D^\mu\Phi^a D_\mu\Phi_a + [\Phi^a,\Phi^b][\Phi_a,\Phi_b]\Big) \\ &\quad + tr\Big(\bar\Psi\gamma^\mu(i\partial_\mu + [A_\mu,.])\Psi + \bar\Psi\Gamma^a[\Phi_a,\Psi]\Big) \end{aligned}}$$

- $N = 1$ SYM in 10 D dimensionally reduced to 4D
- 6 scalar fields Φ^a, global $SO(6)_R$
- $D_\mu\Phi^a = (\partial_\mu + i[A_\mu,.])\Phi^a$
- $(\gamma^\mu, \Gamma^a) = \Gamma^A$ ... 10D Clifford generators, $\Psi \to$ 4 Weyl fermions
- most symmetric 4D gauge theory, UV finite

$\mathcal{N} = 4$ SYM beautiful but too "round" for physics

more structure:

- spontaneous symmetry breaking (SSB): still no "interesting" (chiral) low-energy physics
- add soft susy breaking terms to potential

H.S., J. Zahn arXiv:1409.1440, H.S. arXiv:1504.05703

scalar potential: $\mathcal{V}[\Phi] = \big(V_4[\Phi] + V_{\text{soft}}[\Phi]\big)$,

$$\boxed{\begin{aligned} V_4[\Phi] &= -\tfrac{1}{4}\text{tr}[\Phi_a, \Phi_b][\Phi^a, \Phi^b], \\ V_{\text{soft}}[\Phi] &= \text{tr}\big(im\, f_{abc}\Phi^a\Phi^b\Phi^c + M^2_{ab}\Phi^a\Phi^b\big) \end{aligned}}$$

f_{abc} ... tot. antisymm., soft SUSY breaking

- known: $f_{abc} \sim \epsilon_{abc}$: $\rightarrow$ fuzzy sphere solutions $\Phi^a \sim J^a$
- new: $f_{abc} \sim$ truncated $su(3)$ structure constants (roots)
  $\rightarrow$ squashed coadjoint $SU(3)$ orbits (4D, 6D)

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complex dimensionless fields $\Phi_a = mY_a$,

$$\begin{aligned} Y_1^\pm &= Y_4 \pm iY_5 \equiv Y_{\pm\alpha_1}, \\ Y_2^\pm &= Y_6 \mp iY_7 \equiv Y_{\pm\alpha_2}, \\ Y_3^\pm &= Y_1 \mp iY_2 \equiv Y_{\pm\alpha_3} \end{aligned}$$

let $f_{abc} = c_{abc}|_{a,b,c\neq 3,8}$ ... structure constants of $\mathfrak{su}(3)$ without Cartan

observe

$$\mathrm{Tr}\big(if_{abc}Y^aY^bY^c\big) \sim \mathrm{Tr}\big(\varepsilon_{ijk}Y_i^+Y_j^+Y_k^+ + h.c.\big)$$

breaks $SO(6)_R$ to $SU(3)_R$

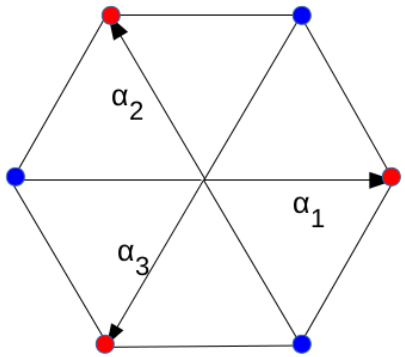
<u>eom:</u>

$$\begin{aligned} 0 &= \big(\Box_4 + m^2(\Box_Y + 4M_i^2)\big)Y_i^+ + 4m^2\varepsilon_{ijk}Y_j^-Y_k^- \\ \Box_Y &\equiv [Y^a,[Y_a,.]] \end{aligned}$$

squashed $SU(3)$ brane solutions

$$Y_{\pm\alpha_i} = r_i\,\pi(T_{\pm\alpha_i})$$

... root generators of $\mathfrak{su}(3)$, any rep.



(soft) SUSY breaking:

action is *almost* $\mathcal{N} = 1^*$ deformation of $\mathcal{N} = 4$ SYM:
consider superpotential

$$W = \frac{\sqrt{2}}{g}\mathrm{tr}([\Phi_1^+, \Phi_2^+]\Phi_3^- - m\Phi_3^-\Phi_3^-) \qquad ...\mathcal{N} = 1^*$$

(declare $\Phi_1^+\Phi_2^+, \Phi_3^-$ as holomorphic coords)

$\rightarrow$ effective potential

$$\begin{aligned} V(\Phi) &= -\tfrac{1}{4g^2}\mathrm{tr}[\Phi^\alpha, \Phi^\beta][\Phi_\alpha, \Phi_\beta] \\ &\quad +4\tfrac{1}{g^2}\mathrm{tr}\big(- m[\Phi_1^+, \Phi_2^+]\Phi_3^+ - m[\Phi_2^-, \Phi_1^-]\Phi_3^- + 2m^2\Phi_3^+\Phi_3^-\big). \end{aligned}$$

$\equiv$ present potential,
however, mass $M_3^2 = 2m^2$ too large for squashed $SU(3)$ brane solutions

Symmetries:

solutions $Y_{\pm\alpha_i} = T_{\pm\alpha_i}$ break $SU(3)_R \nrightarrow U(1) \times U(1) \cong$ gauge trafo

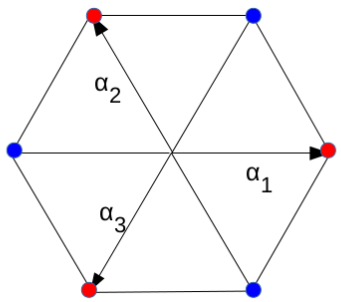
residual symmetry: $\big(U(1) \times U(1)\big)_{K_i}$ generated by

$$\begin{aligned} K_i &:= 2\tau_i - [H_{\alpha_i}, .], \\ \tau_i \phi_\alpha &= \tfrac{1}{2}(\alpha_i, \alpha)\phi_\alpha \ , \quad \text{generates } U(1)_i \subset U(3)_R \end{aligned}$$

$\Rightarrow$ 8-2 = 6 Goldstone bosons from $SU(3)/_{U(1)\times U(1)}$

note:
the $Y_i^\pm \sim T_{\pm\alpha_i}$ generate $SU(3)_Y \subset SU(N)$, is not part of the $SO(6)_R$ symmetry

transform as $\begin{cases} (3) & \text{under } SU(3)_R \\ (8) & \text{under } SU(3)_Y \end{cases}$



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$\rightarrow$ massive KK modes!

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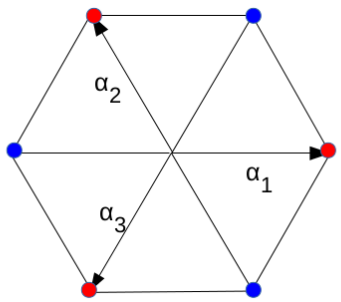

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Fluctuation modes & KK modes in $\mathcal{N} = 4$ SYM

all fluctuations (scalar, gauge fields) take values in

$$\mathfrak{u}(N) = End(\mathcal{H}_\mu) \cong \mathcal{H}_\mu \otimes \mathcal{H}_\mu^* = \oplus_\Lambda \mathcal{H}_\Lambda$$

$\cong$ functions on squashed $\mathcal{C}_N[\mu]$ ... KK modes !

$\rightarrow$ 8- or 10-dim gauge theory, below Λ_{UV}

expand fields into $SU(3)_Y$ harmonics

gauge fields:

$$A_\mu(x) = \sum A_\mu^{(M,\Lambda)}(x)\,\hat{Y}_M^\Lambda$$

scalar fields:

$$\Phi_a = mY^a + \varphi^a(x)$$

$$\varphi_a(x) = \sum \varphi_a^{(M,\Lambda)}(x)\,\hat{Y}_M^\Lambda$$

fermions ... (similar)

massive gauge bosons

example: gauge field modes $A_\mu(x)$

kinetic term $tr\big(D_\mu Y_a D^\mu Y^a\big)$ gives mass term,

$$\int \mathrm{tr}(D_\mu Y_a)^\dagger D_\mu Y_a = \int \mathrm{tr}(\partial_\mu Y_a^\dagger \partial_\mu Y_a + \sum_{\Lambda,M} m^2_{\Lambda,M} A^\dagger_{\mu,(\Lambda M)} A^\mu_{(\Lambda,M)}) + S_{int}$$

using $D_\mu \phi_a = (\partial_\mu + i[A_\mu, .])\phi_a$ and decomposition

$$A_\mu(x) = \sum A_\mu^{(M,\Lambda)}(x)\,\hat{Y}^\Lambda_M \equiv A_\mu(x,y) \qquad \text{on } \mathbb{R}^4 \times \mathcal{C}_N[\mu]$$

into eigenmodes of

$$\Box_Y \hat{Y}^\Lambda_M \equiv [Y^a, [Y_a, \hat{Y}^\Lambda_M]] = m^2_{(\Lambda M)} \hat{Y}^\Lambda_M$$

$\Rightarrow$ tower of massive KK modes $A_{\mu,(\Lambda M)}(x)$, mass

$$m^2_{(\Lambda M)} \sim (\langle \Lambda+\rho, \Lambda+\rho\rangle - \langle \rho,\rho\rangle - \langle M,M\rangle) > 0$$

(no massless gauge modes)

internal geometry: squashed fuzzy coadjoint orbits

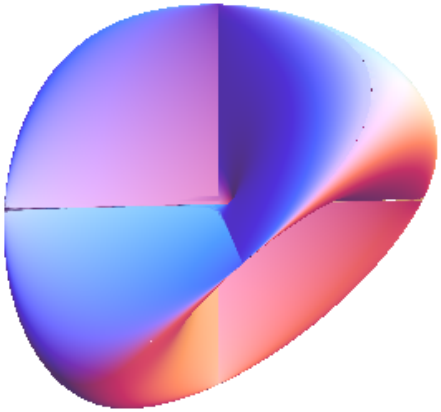
claim: $Y_a = \pi_\mu(T_a)$ ... quantized coadjoint $SU(3)$ orbit,
projected along Cartan directions

$$\begin{array}{ccccc} \mathcal{C}[\mu] & \hookrightarrow & \mathbb{R}^8 & \xrightarrow{\Pi} & \mathbb{R}^6 \\ (y^a)_{a=1,\dots,8} & & & \mapsto & (y^a)_{a=1,2,4,5,6,7} \end{array}$$

4- or 6-dimensional variety, self-intersecting embedding in $\mathbb{R}^6$
multiple covering at origin, spanning all 6 directions

H.S., J. Zahn arxiv:1409.1440

e.g: 3-dim. section of squashed $\mathbb{C}P^2$ through $y_2 = y_5 = y_7 = 0$:



triple self-intersection at origin

classical coadjoint orbits of $SU(3)$

$$\mathcal{C}[\mu] = \{p = g^{-1}\mu g;\ g \in SU(3)\} \ \cong\ SU(3)/_{\mathcal{K}} \ \subset \mathfrak{su}(3)^* \cong \mathbb{R}^8$$

$\mathcal{K}$ ... stabilizer group of weight $\mu \in \mathfrak{g}_0^*$

- μ 3 EV $\rightarrow$ $\mathcal{C}[\mu] \cong SU(3)/_{U(1)\times U(1)}$ ... 6-dim.
- μ 2 EV $\rightarrow$ $\mathcal{C}[\mu] \cong SU(3)/_{SU(2)\times U(1)} \ \cong\ \mathbb{C}P^2$ ... 4-dim.

embedding of $\mathcal{C}[\mu] \subset \mathbb{R}^8$:

$$p = y^a \lambda_a \quad \in \mathcal{C}[\mu], \qquad a = 1, ..., 8$$

$$\boxed{y^a : \quad \mathcal{C}[\mu] \hookrightarrow \mathbb{R}^8 \ \cong\ \mathfrak{su}(3)}$$

quantized (fuzzy) coadjoint orbits:

$\mathcal{C}[\mu]$ is symplectic (Kirillov-Kostant) $\rightarrow$ quantize it:

$\mu = (n_1, n_2)$ ... dominant integral weight

$\mathcal{H}_\mu$ ... corresp. highest weight irrep

fuzzy $\mathcal{C}_N[\mu]$:

$$Y^a = \pi_\mu(T^a)$$

generates matrix algebra

$$\mathcal{A}_N = End(\mathcal{H}_\mu)$$

can show (in general):

decomposition into harmonics of $SU(3)$ agrees with classical harmonic analysis,

$$Pol(\mathcal{C}[\mu]) \cong \oplus_\Lambda m_\Lambda \mathcal{H}_\Lambda \cong \mathcal{H}_\mu \otimes \mathcal{H}_\mu^* = End(\mathcal{H}_\mu) = \mathcal{A}_N$$

up to cutoff Λ_{max}

quantization map

$$\begin{aligned} \mathcal{Q}: \quad Pol(\mathcal{C}[\mu]) &\to \mathcal{A}_N \\ Y^\Lambda_M &\mapsto \hat{Y}^\Lambda_M \end{aligned}$$

respects $SU(3)$, truncated at cutoff Λ_{max}.

$Y^a = \mathcal{Q}(y^a) = \pi_\mu(T^a)$ interpreted as quantized embedding functions

$$Y^a \sim y^a : \quad \mathcal{C}[\mu] \hookrightarrow \mathbb{R}^8.$$

commutation relations:

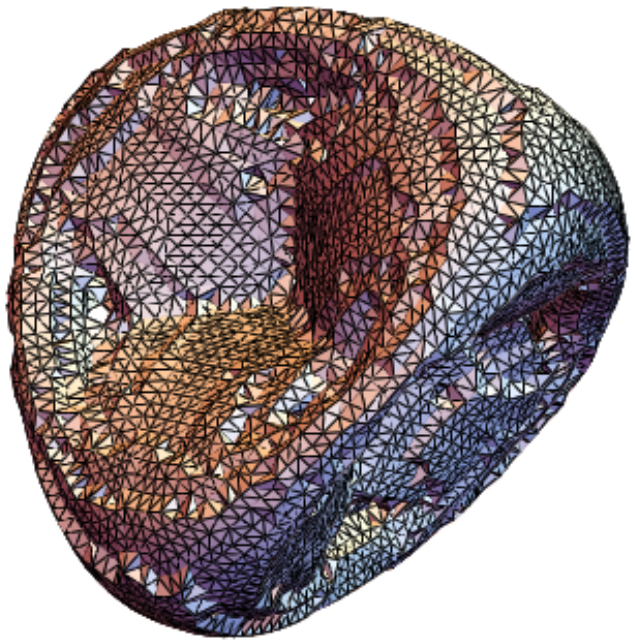
$$[Y^a, Y^b] = i\, c^{ab}_c\, Y^c$$

... quantizes Poisson structure (Kirillov-Kostant)

$$\{y^a, y^b\} = c^{ab}_c\, y^c$$

$Y^a = \pi_\mu(T^a), \quad a = 1, ..., 8$ ... quantized coadjoint orbit $\mathcal{C}_N[\mu]$
$Y^a = \pi_\mu(T^a), \quad a \neq 3, 8$ ... squashed $\mathcal{C}_N[\mu]$
= solution of (deformed) $\mathcal{N} = 4$ SU(N) SYM

computer measurement of $Y^a = \pi_\mu(T^a)$:



squashed $\mathcal{C}_N[\mu]$ for $\mu = (20, 0)$.

L. Schneiderbauer, H.S. (unpublished)

scalar fluctuation modes

scalar fluctuations: $\phi_a = mY_a + \varphi_a$ governed by

$$\begin{aligned} V_2[\varphi] &= \mathrm{tr}\varphi^\alpha \left(\Box_Y + 2\not{D}_{\mathrm{diag}} - 2\not{D}_{\mathrm{mix}}\right)\varphi_\alpha, \\ (\not{D}_{\mathrm{mix}}\varphi)_i^+ &= -\varepsilon_{ikj}[Y_k^-, \varphi_j^-], \qquad \tau\not{D}_{\mathrm{mix}} = -\not{D}_{\mathrm{mix}}\tau \\ (\not{D}_{\mathrm{diag}}\varphi)_\alpha &= [H_\alpha, \varphi_\alpha] \qquad \text{(no sum)} \end{aligned}$$

can show:

- no negative modes (!!)
- $\exists$ zero modes (= flat deformations), corresponding to

$$\not{D}_{\mathrm{mix}}\phi_\alpha^{(0)} = 0.$$

- regular zero modes: 6 zero modes for each $\mathcal{H}_\Lambda$

  (corresponding to extremal weights Λ' in $\mathcal{H}_\Lambda \otimes (8)$)

  with distinct $U(1)_{K_i}$ charges

  geometric interpretation:

  string between coincident sheets of $\mathcal{C}[\mu]$ at/near origin

  e.g. $\varphi_\Lambda = |\Omega\mu\rangle\langle\mu|$, $|\mu\rangle$ ... coherent states

- exceptional zero modes:

  in particular: 6 Goldstone bosons

(verified numerically)

fermions

Dirac operator on squashed $\mathcal{C}_{\mathcal{N}}[\mu]$:

$$\not{D}_{(6)}\Psi \quad = 2\sum_{i=1}^{3}\left(\gamma_i[Y_i^+,.] + \gamma_i^\dagger[Y_1^-,.]\right)$$

spinorial ladder operators $\{\gamma_i,\gamma_j^\dagger\} = \delta_{ij}$ ... $SO(6)$ Clifford algebra

zero modes:

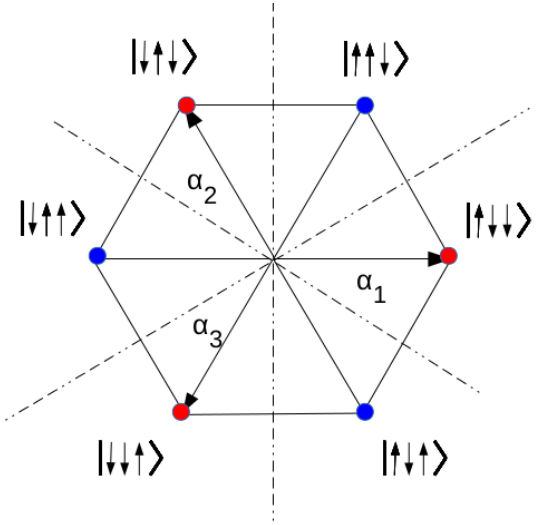
$$\not{D}_{(6)}\Psi_\Lambda = 0$$

- in one-to-one correspondence to regular zero modes

  (fermionic strings connecting branes!)
- but not corresponding to exceptional zero modes
- have distinct chirality determined by gauge charge (weight Λ')

chiral fermions

well-defined chirality



$$\Gamma^{(6)}\Psi_{w\Lambda} = (-1)^{|w|}\Psi_{w\Lambda} \ .$$

$$\Psi = \psi_+ \otimes \chi_+ + \psi_- \otimes \chi_-,$$

$$\not{D}_{(4)}\psi_\pm = 0, \qquad \gamma_5\psi_\pm = \pm\psi_\pm, \qquad \psi_\pm^C = \psi_\mp.$$

3 generations of Weyl/ Majorana spinors on $\mathbb{R}^4$ (Weyl rotations $\frac{2\pi}{3}$) distinguished by $U(1)_{K_i}$ charges

towards interesting physics:

stacks of squashed $\mathcal{D}_i = \mathcal{C}[\mu_i]$ branes (=reducible rep. of $\mathfrak{su}(3)$)

$$Y^a = \begin{pmatrix} Y^a_{\mu_1} & \\ & Y^a_{\mu_2} \end{pmatrix}$$

off-diagonal fermions

$$\Psi = \begin{pmatrix} 0 & \Psi_{12} \\ \Psi_{21} & 0 \end{pmatrix}$$

transform in bi-fundamental $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ of $U(N_1) \times U(N_2)$
$\rightarrow$ zero-modes as above

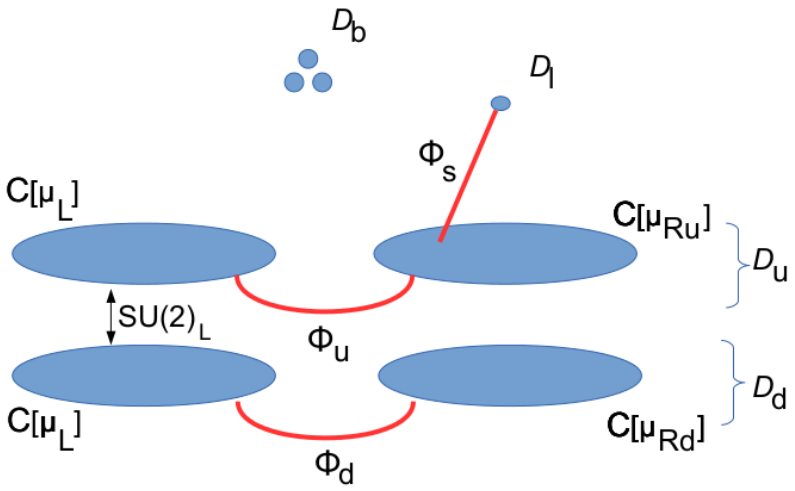
e.g. branes linked to 3 point-branes: $\mathcal{H}_2 \cong \mathbb{C}^3 \quad \rightarrow \quad$ "quarks"

$$\Psi_{12} \in \mathcal{H}_\mu \otimes \mathbb{C}^3$$

$\rightarrow$ 3+3 chiral zero-modes attached to extremal weights

get quiver gauge theory corresponding to stack of branes,
zero modes connecting branes = chiral superfields (+ exceptionals)

A standard-model-like brane configuration



- 3 generations ($\mathbb{Z}_3$ Weyl rotations!),
  chiralities distinguished by (massive) gauge modes
- can realize all standard model fermions (+ more ...)
  e.g: quarks link $\mathcal{C}[\mu]$ with 3 coincident point branes ($\rightarrow$ color)

same solutions in IKKT matrix model, $\rightarrow$ dynamical (NC) space-time, (emergent) gravity ?

off-diagonal fermionic zero modes $\rightarrow$ (extended) SSM matter:

$$\Psi = \begin{pmatrix} *_2 & \widetilde{H}_d & \widetilde{H}_u & l_L & Q_L \\ & * & e' & e_R & d_R \\ & & * & \nu_R & u_R \\ & & & * & u' \\ & & & & *_3 \end{pmatrix}.$$

- correct quantum numbers, no exotics ($u' \sim u_R,\ e' \sim e_R$)
- $\exists$ cubic potential for bosonic ("Higgs") modes ϕ_u, ϕ_d
  $\rightarrow$ further SSB expected, complicated H.S. arXiv:1504.05703
- appropriate symmetry-breaking pattern possible, leaving $SU(3) \times U(1)_Q$ in light sector (assuming suitable Higgs)
- $\exists$ mirror fermions (opposite chirality), distinguished coupling to (mirror) Higgs, $\rightarrow$ may have (much?) higher masses

spinning squashed branes

$\mathcal{C}_N[\mu]$ branes can be stabilized by rotation in pure $\mathcal{N} = 4$ SYM (no flux!) H.S, arXiv:1411.3139

ansatz:

$$\boxed{Y_i^{\pm} = r_i e^{\pm i\omega_i x}\, \pi_\mu(T_i^{\pm})}$$

$\omega_i = \omega_{i,\mu}$ ... 3 time-like vectors ("frequencies")

zero mode sector has Lorentz-invariant kinematics

conclusion

- $\exists$ rich class of background solutions of $SU(N)$ $\mathcal{N} = 4$ SYM
  ... self-intersecting $SU(3)$ branes $\mathbb{R}^4 \times \mathcal{C}[\mu]$
- interesting low-energy physics:
  space-filling $\rightarrow$ chiral fermions, scalars & gauge fields
- can be surprisingly similar to standard model in broken phase (?!)
- open issues:
  generalizations, further SSB
  relation with in string theory?
  strong coupling?
  gravity in matrix model version ?

coherent states on $\mathbb{C}[\mu]$: let $p \in \mathcal{C}[\mu]$ ... north pole

$$\begin{aligned} SU(3) &\to \mathcal{C}[\mu] \\ g &\mapsto g \triangleright p \end{aligned}$$

stabilizer $\mathcal{K} \subset SU(3) \Rightarrow \mathcal{C}[\mu] \cong SU(3)/\mathcal{K}$ ... coadjoint orbit

let $|\mu\rangle \in \mathcal{H}_\mu$ ... highest weight vector in $\mathcal{H}_\mu$

$$\langle\mu|\vec{y}|\mu\rangle = \vec{p} \qquad \text{...localized at north pole}$$

def. $|\psi_g\rangle := g \triangleright |\mu\rangle$

projector $\Pi_p = |\psi_g\rangle\langle\psi_g| \in End(\mathcal{H}_\mu)$ ... independent of $\mathcal{K} \subset SU(3)$

$$\begin{aligned} \mathcal{C}[\mu] \cong SU(3)/\mathcal{K} &\to End(\mathcal{H}_\mu) \\ p &\mapsto \Pi_p := |\psi_g\rangle\langle\psi_g| \quad =: \delta_N(y-p) \end{aligned}$$

$$\langle p|\vec{y}|p\rangle = tr(\Pi_p \vec{y}) = \vec{p} \qquad \text{... sweeps out } \mathcal{C}[\mu]$$

example: fuzzy $\mathbb{C}P^2_N$ (Grosse & Strohmaier, Balachandran etal, ...)

arises for $\mu = N\Lambda_1$ or $\mu = N\Lambda_2$, stabilizer $\mathcal{K} = SU(2) \times U(1)$

satisfy the relations

$$\begin{aligned} [Y^a, Y^b] &= \tfrac{i}{2}\Lambda_N f_{abc}\, Y^c \\ \delta_{ab} Y^a Y^b &= R^2, \\ d^c_{ab} Y^a Y^b &= R\, \tfrac{2N/3+1}{\sqrt{\frac{1}{3}N^2+N}}\, Y^c. \end{aligned}$$

harmonic decomposition

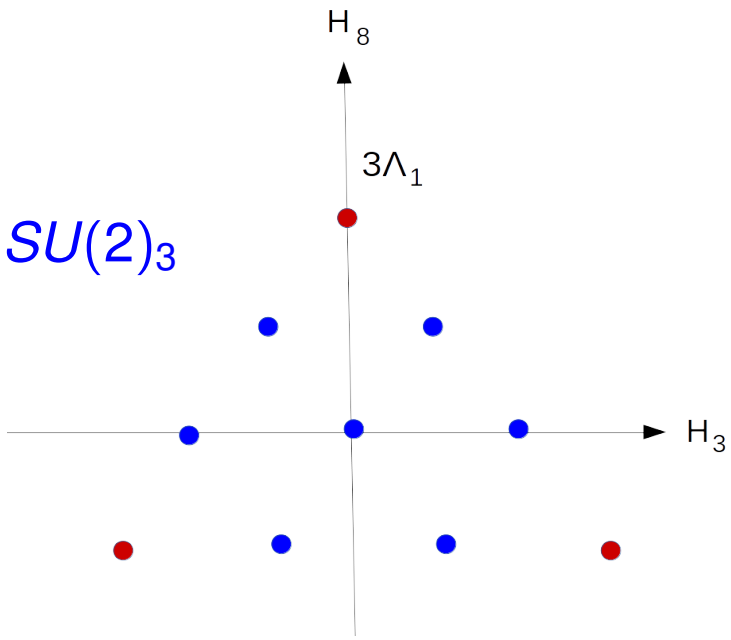
$$\mathcal{A} = End(\mathcal{H}_\mu) = \bigoplus_{n=0}^{N} \mathcal{H}_{(n,n)}.$$

squashed $\mathbb{C}P^2_N$:

projection $\Pi \equiv$ take away H_3, H_8

- $\langle\mu|\vec{y}|\mu\rangle = \vec{0}$, projected to 0 by Π
- acting with $SU(3)$ on $|\mu\rangle$: stabilizer $SU(2)_3$

  $\Rightarrow$ 4D orbit along 4567 plane
- analogous for 3 Weyl images of μ



- three 4D sheets intersecting at origin
- extremal weight states $|\mu\rangle$ = coherent states on different sheets at origin