Uniform Resurgence and Large N

Gerald Dunne

University of Connecticut

eNLarge Horizons Research Program, IFT Madrid
June 1, 2015

GD & G. Başar, 1501.05671

GD, G. Başar & M. Ünsal, to appear

- asymptotics is a versatile analytic tool for QFT
- resurgent asymptotics is more powerful than classical Poincaré asymptotics
- quantitatively useful for physics applications

Resurgent Trans-Series

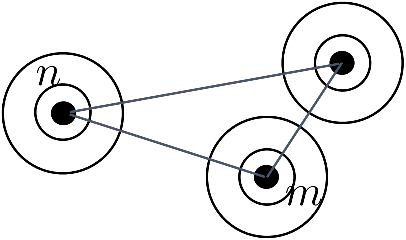
- trans-series expansion in QM and QFT applications:

$$f(g^2) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zero-modes}}$$

- new: analytic continuation encoded in trans-series
- new: trans-series coefficients $c_{k,l,p}$ highly correlated
- new: exponential asymptotics

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



local analysis may encode more global information than one might naïvely think

Graded Resurgence Triangle

- resurgence is a basic property of all-orders steepest descents contour integrals
- physics: sectors with the same quantum numbers can mix, and "cure" ambiguities

$$\mathbf{1} P_{[0,0],\pm}$$

$$\mathcal{I} P_{[1,1],\pm} \qquad \bar{\mathcal{I}} P_{[1,-1],\pm}$$

$$[\mathcal{I}^2] P_{[2,2],\pm} \qquad [\mathcal{I}\bar{\mathcal{I}}]_\pm P_{[2,0],\pm} \qquad [\bar{\mathcal{I}}^2] P_{[2,-2],\pm}$$

$$[\mathcal{I}^3] P_{[3,3],\pm} \qquad [\mathcal{I}^2\bar{\mathcal{I}}]_\pm P_{[3,1],\pm} \qquad [\mathcal{I}\bar{\mathcal{I}}^2]_\pm P_{[3,-1],\pm} \qquad [\bar{\mathcal{I}}^3] P_{[3,-$$

Resurgence in Path Integrals (GD, Ünsal, 1401.5202)

- periodic potential: $V(x) = \frac{1}{g^2}\,\sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n! \left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\text{Im}\,E \sim \pi\,e^{-2\frac{1}{2g^2}}\left(1 - \frac{5}{2}\cdot g^2 - \frac{13}{8}\cdot g^4 - \dots\right)$$

- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n! \left(1 - \frac{53}{6}\cdot\frac{1}{3}\cdot\frac{1}{n} - \frac{1277}{72}\cdot\frac{1}{3^2}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\text{Im}\,E \sim \pi\,e^{-2\frac{1}{6g^2}}\left(1 - \frac{53}{6}\cdot g^2 - \frac{1277}{72}\cdot g^4 - \dots\right)$$

Uniform WKB and Resurgent Trans-Series

- in fact, there is even more resurgent structure

- in certain non-trivial QM models, the perturbative series determines 1-instanton fluctuations, and beyond: this propagates through entire trans-series

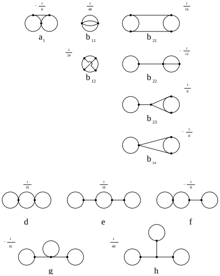
$$E(N,g^2) = E_{\text{pert}}(N,g^2) \pm \frac{\left(\frac{2}{g^2}\right)^N}{\sqrt{2\pi g^2}\, N!} e^{-S/g^2}\, \mathcal{P}(N,g^2) + \dots$$

$$\mathcal{P}(N,g^2) = \frac{\partial E(N,g^2)}{\partial N} \exp\left[S \int_0^{g^2} \frac{dg^2}{g^4} \left(\frac{\partial E(N,g^2)}{\partial N} - 1 + \frac{\left(N+\frac{1}{2}\right)\, g^2}{S}\right)\right]$$

$\Rightarrow$ perturbation theory $E(N,g^2)$ encodes everything !

(GD, Ünsal, 1306.4405, 1401.5202)

- fluctuations about $\mathcal{I}$ (or $\bar{\mathcal{I}}$) saddle are determined by those about the vacuum saddle, to all fluctuation orders



- "QFT computation": fluctuation about $\mathcal{I}$ for double-well (and periodic Mathieu potential):

3-loop (Escobar-Ruiz/Shuryak/Turbiner, arXiv:1501.03993)

$$e^{-\frac{S_0}{g}}\left[1-\frac{71}{72}\,g-0.607535\,g^2-\dots\right]$$

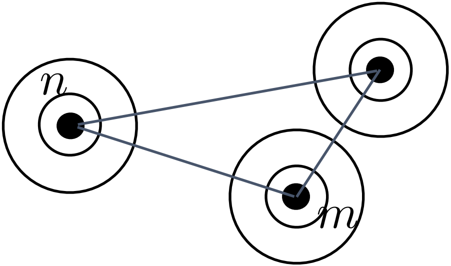
resurgence $\Rightarrow e^{-\frac{S_0}{g}}\left[1+\frac{1}{72}\,g\left(-102N^2-174N-71\right)\right.$

$$\left.+\frac{1}{10368}g^2\left(10404N^4+17496N^3-2112N^2-14172N-6299\right)+\dots\right]$$

- known for all N and to essentially any loop order, directly from perturbation theory !

Connecting Perturbative and Non-Perturbative Sector

in certain non-trivial QM models, all orders of multi-instanton trans-series are encoded in perturbation theory of fluctuations about perturbative vacuum



(Drukker et al, 1007.3837; Mariño, 1104.0783; Aniceto, Russo, Schiappa, 1410.5834)

- certain protected quantities in especially symmetric QFTs can be reduced to matrix models $\Rightarrow$ resurgent asymptotics

- 3d Chern-Simons on $\mathbb{S}^3$ $\rightarrow$ matrix model

$$Z_{CS}(N,g) = \frac{1}{\text{vol}(U(N))} \int dM \exp\left[-\frac{1}{g}\,\text{tr}\left(\frac{1}{2}\left(\ln M\right)^2\right)\right]$$

- ABJM: $\mathcal{N}=6$ SUSY CS, $G = U(N)_k \times U(N)_{-k}$

$$Z_{ABJM}(N,k) = \sum_{\sigma\in S_N} \frac{(-1)^{\epsilon(\sigma)}}{N!} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi k} \frac{1}{\prod_{i=1}^{N} 2\text{ch}\left(\frac{x_i}{2}\right)\,\text{ch}\left(\frac{x_i - x_{\sigma(i)}}{2k}\right)}$$

- $\mathcal{N}=4$ SUSY Yang-Mills on $\mathbb{S}^4$

$$Z_{SYM}(N,g^2) = \frac{1}{\text{vol}(U(N))} \int dM \exp\left[-\frac{1}{g^2}\,\text{tr}M^2\right]$$

- uniform resurgence with two parameters: N and g^2

Physics Motivation

- large N expansion:

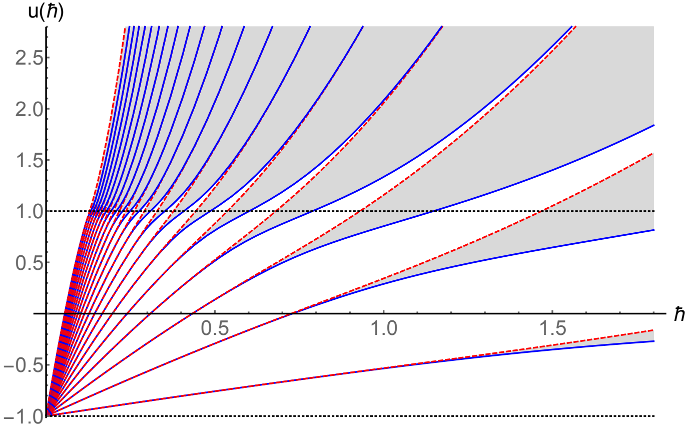
$$
\begin{aligned}
F(N, g^2) &= \sum_{h=0}^{\infty} \frac{1}{N^{2h-2}} f_h(N\, g^2) \\
&= \sum_n g^{2n} p_n^{(0)}(N) + e^{-\frac{1}{g^2}} \sum_n g^{2n} p_n^{(1)}(N) + \dots \\
&= \sum_k \frac{1}{g^{2k}} c_k(N)
\end{aligned}
$$

- how does a divergent trans-series at weak coupling turn into a convergent series at strong-coupling?
- what happens to the resurgent structure?
- separated by a phase transition: “instantons condense”

Mathieu Equation Spectrum: ($\hbar$ plays role of g^2)

- 3 different spectral regions: $-\frac{\hbar^2}{2}\,\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$

$$u = u(N, \hbar)$$



- exact quantization condition, valid everywhere !

Gross-Witten-Wadia matrix model

$$Z(N,g^2) = \int dU \; \exp\left[\frac{1}{g^2}\,\mathrm{tr}\left(U + U^\dagger\right)\right] = \det\left(I_{k-l}\left(\frac{2N}{\lambda}\right)\right)_{k,l=1,\ldots N}$$

$$F(N,g^2) = \frac{1}{N^2}\ln Z(N,g^2) = \sum_{h=0}^{\infty}\frac{1}{N^{2h-2}}\,f_h(N\,g^2)$$

- 't Hooft parameter $\lambda = N\,g^2$
- soluble for finite N
- large N phase transition at $\lambda_c = 2$
- related to Mathieu spectrum (Wadia, Neuberger, ...)
- (convergent) strong coupling expansion coefficients have poles (de Wit, 't Hooft; Goldschmidt, Samuel, ...)

Other Physics Motivation

$$F(N, g^2) = \sum_{h=0}^{\infty} \frac{1}{N^{2h-2}} f_h(N\, g^2)$$

- hermitean matrix models
- 2d Yang-Mills on sphere (Gross-Matytsin; Witten, ...)

$$Z(N, A) = \sum_R (\dim R)^2 \exp\left[-\frac{A}{2N} C_2(R)\right] \qquad A_c = \pi^2$$

- Chern-Simons on S^3: $Z(N, k) \longrightarrow$ matrix integral
- ABJM models on S^3: $Z(N, k) \longrightarrow$ matrix integral
- localizable 4d QFT: $\longrightarrow$ matrix integral
- ODE/Integrable Model correspondence
- ...

Ionization in Time-Dependent Electric Fields

- Keldysh (1964): atomic ionization in $E(t) = \mathcal{E} \cos(\omega t)$
- adiabaticity parameter: $\gamma \equiv \frac{\omega \sqrt{2mE_b}}{e\mathcal{E}}$
- WKB $\Rightarrow$ $P_{\text{ionization}} \sim \exp\left[-\frac{4}{3}\frac{\sqrt{2m}E_b^{3/2}}{e\hbar\mathcal{E}}\, g(\gamma)\right]$

$$P_{\text{ionization}} \sim \begin{cases} \exp\left[-\frac{4}{3}\frac{\sqrt{2m}E_b^{3/2}}{e\hbar\mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(non-perturbative)} \\ \left(\frac{e\mathcal{E}}{2\omega\sqrt{2mE_b}}\right)^{2E_b/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(perturbative)} \end{cases}$$

- semi-classical instanton interpolates between non-perturbative “tunneling ionization” and perturbative “multi-photon ionization”

Keldysh Approach in QED

Brézin/Itzykson, 1970; Popov, 1971

- Schwinger effect in $E(t) = \mathcal{E}\cos(\omega t)$
- adiabaticity parameter: $\gamma \equiv \frac{m\,c\,\omega}{e\,\mathcal{E}}$
- WKB $\quad\Rightarrow\quad P_{\rm QED} \sim \exp\left[-\pi\,\frac{m^2\,c^3}{e\,\hbar\,\mathcal{E}}\,g(\gamma)\right]$

$$P_{\rm QED} \sim \begin{cases} \exp\left[-\pi\,\frac{m^2\,c^3}{e\,\hbar\,\mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(non-perturbative)} \\ \left(\frac{e\,\mathcal{E}}{\omega\,m\,c}\right)^{4mc^2/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(perturbative)} \end{cases}$$

- semi-classical instanton interpolates between non-perturbative "tunneling pair-production" and perturbative "multi-photon pair production"

Resurgence of $\mathcal{N}=2$ SUSY SU(2) (Başar, GD, 1501.05671)

- moduli parameter: $u = \langle \mathrm{tr}\, \Phi^2 \rangle$
- electric: $u \gg 1$; magnetic: $u \sim 1$; dyonic: $u \sim -1$
- $a = \langle \text{scalar} \rangle$, $a_D = \langle \text{dual scalar} \rangle$, $a_D = \frac{\partial \mathcal{W}}{\partial a}$
- Nekrasov twisted superpotential:

$$\mathcal{W}^{inst} \sim \frac{\hbar^2}{2\pi i}\left(\frac{\Lambda^4}{16a^4} + \frac{21\Lambda^8}{256a^8} + \dots\right) + \frac{\hbar^4}{2\pi i}\left(\frac{\Lambda^4}{64a^6} + \frac{219\Lambda^8}{2048a^{10}} + \dots\right) + \dots$$

$$\mathcal{W}^{class} + \mathcal{W}^{pert} \sim -\frac{a^2}{2\pi i}\log\frac{a^2}{\Lambda^2} - \frac{\hbar^2}{48\pi i}\log\frac{a^2}{2\Lambda^2} + \hbar^2 \sum_{n=1}^{\infty} d_{2n}\left(\frac{\hbar}{a}\right)^{2n}$$

- Matone relation:

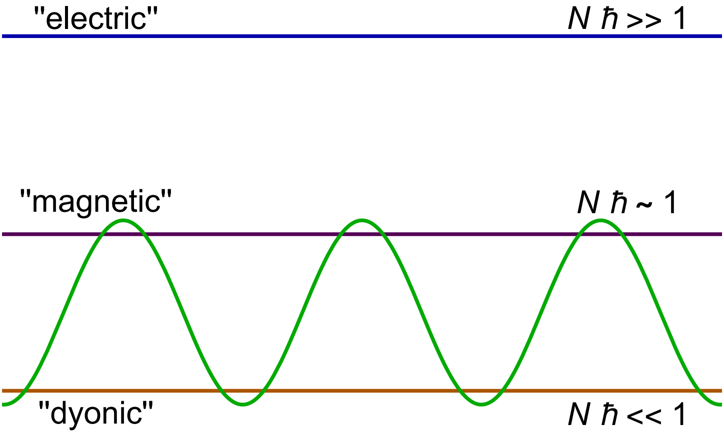
$$u(a,\hbar) = \frac{i\pi}{2}\Lambda\frac{\partial \mathcal{W}(a,\hbar,\Lambda)}{\partial \Lambda} - \frac{\hbar^2}{48}$$

- Mathieu equation:

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \Lambda^2\cos(x)\,\psi = u\,\psi \quad , \quad a \equiv \frac{N\hbar}{2}$$

Mathieu Equation Spectrum: ($\hbar$ plays role of g)

- 3 different spectral regions: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$



Perturbative/Non-perturbative connection (Başar, GD, 1501.05671)

- Zinn-Justin/Jentschura: $B(u,\hbar)$, $A(u,\hbar) \to$ trans-series
- GD, Ünsal: $u(B,\hbar)$ encodes $A(B,\hbar)$:

$$\frac{\partial u}{\partial B} = -\frac{\hbar}{16}\left(2B + \hbar\frac{\partial A}{\partial \hbar}\right)$$

- simple proof from Nekrasov $\mathcal{W}$ and Matone relation

$$u \sim \Lambda\frac{\partial \mathcal{W}}{\partial \Lambda} \quad \Rightarrow \quad \frac{\partial u}{\partial a} \sim \Lambda\frac{\partial}{\partial \Lambda}\frac{\partial \mathcal{W}}{\partial a} = \Lambda\frac{\partial a_D}{\partial \Lambda}$$

- identifications:

$$a \leftrightarrow \frac{\hbar}{2}B \quad , \quad a_D \leftrightarrow \frac{\hbar}{4\pi}A + \text{shift} \quad , \quad \Lambda \sim \frac{1}{\hbar}$$

- quantum geometry: $a(u,\hbar)$ and $a_D(u,\hbar)$ related

Connecting weak and strong coupling

important physics question:

does weak coupling analysis contain enough information to extrapolate to strong coupling ?

... even if the degrees of freedom re-organize themselves in a very non-trivial way?

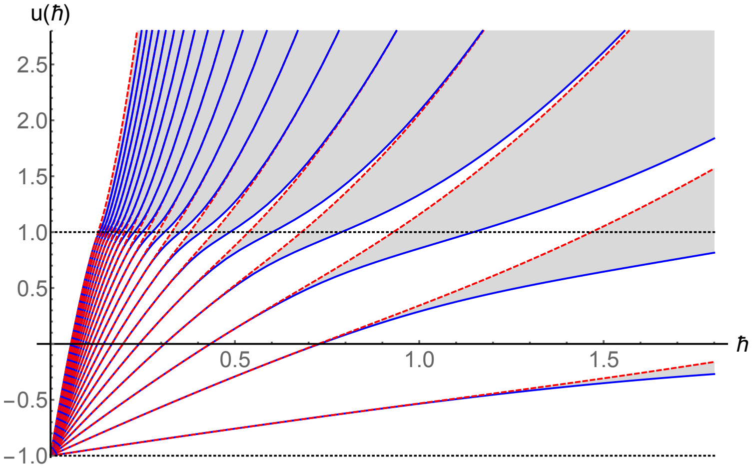
what about a QFT in which the vacuum re-arranges itself in a non-trivial manner?

classical (Poincaré) asymptotics is clearly not enough:

is resurgent asymptotics enough?

Mathieu Equation Spectrum: ($\hbar$ plays role of g^2)

- 3 different spectral regions: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$



Mathieu Equation Spectrum: deep inside the wells

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$

- divergent, non-Borel-summable perturbation theory:

$$\begin{aligned} u(N,\hbar) &\sim -1+\hbar\left[N+\frac{1}{2}\right]-\frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2+\frac{1}{4}\right] \\ &\quad -\frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3+\frac{3}{4}\left(N+\frac{1}{2}\right)\right]-\dots \\ &\equiv \sum_{n=0}^{\infty} u_n(N)\,\hbar^n \end{aligned}$$

- non-Borel-summable: $\longrightarrow$ resurgent trans-series
- but, energy is really a function of two variables:

$$u=u(N,\hbar)$$

- formal pert. series $\to$ resurgent trans-series for $N\,\hbar \ll \frac{8}{\pi}$

Mathieu Equation Spectrum: deep inside the wells

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$$

- exponentially narrow energy bands

$$\Delta u^{\rm band}(N,\hbar) \quad \sim \quad \frac{32}{\sqrt{\pi}\,N!}\left(\frac{32}{\hbar}\right)^{N-1/2} \exp\left[-\frac{8}{\hbar}\right] \times \left(\frac{\partial u}{\partial N}\right) \exp\left[-8\int_0^{\hbar}\frac{d\hbar}{\hbar^2}\left(\frac{1}{\hbar}\frac{\partial u}{\partial N} - 1 + \frac{1}{8}\left(N+\frac{1}{2}\right)\hbar\right)\right]$$

- blue fluctuation factor is new
- all orders in $\hbar$, $\forall N$, fluctuations determined by $u(N,\hbar)$
- generates entire trans-series, from perturbation theory
- extreme example of resurgence:
  perturbative saddle encodes everything!

All-orders WKB for the Mathieu equation (Mironov/Morozov)

- all-orders WKB action: (Dunham, 1932)

$$a(u) = \frac{\sqrt{2}}{2\pi}\left(\oint_C \sqrt{u-V}\,dx - \frac{\hbar^2}{2^6}\oint_C \frac{(V')^2}{(u-V)^{5/2}}dx - \frac{\hbar^4}{2^{13}}\oint_C \left(\frac{49(V')^4}{(u-V)^{11/2}} - \frac{16V'V'''}{(u-V)^{7/2}}\right)dx - \dots\right)$$

$$\Rightarrow \quad a(u) = \sum_{n=0}^{\infty} \hbar^{2n}\, a_n(u)$$

- Bohr-Sommerfeld near well minima: $u \sim -1$

$$a_0(u) \sim \frac{u+1}{2} + \frac{(u+1)^2}{32} + \frac{3(u+1)^3}{512} + \frac{25(u+1)^4}{16384} + \dots$$

$$a_1(u) \sim \frac{1}{128} + \frac{5(u+1)}{2048} + \frac{35(u+1)^2}{32768} + \frac{525(u+1)^3}{1048576} + \dots$$

- invert $\quad a(u) = \frac{\hbar}{2}\left(N + \frac{1}{2}\right) \quad \Longrightarrow \quad u = u(N,\hbar) = u(a,\hbar)$

- non-Borel-summable expression for band centers

All-orders WKB for the Mathieu equation

- band widths: tunneling action $a^D(u)$ over dual cycle $\tilde{C}$
- all-orders WKB actions:

$$a(u) = \sum_{n=0}^{\infty} \hbar^{2n}\, a_n(u) \qquad , \qquad a^D(u) = \sum_{n=0}^{\infty} \hbar^{2n}\, a_n^D(u)$$

- $a(u)$ determines location of center of band
- $a^D(u)$ determines width of band
- Bohr-Sommerfeld near well minima: $u \sim -1$

$$a_0^D(u) \sim \frac{4i}{\pi} - \frac{i(u+1)}{2\pi}\left(\log\left(\frac{u+1}{32}\right) - 1\right) - \dots$$

Keller-Weinstein (1987):

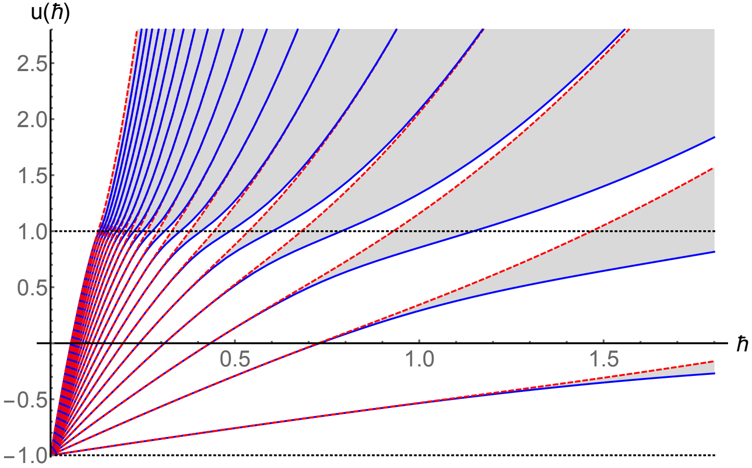
$$\Delta u \sim \frac{2}{\pi}\frac{\partial u}{\partial N}\, e^{-\frac{2\pi}{\hbar}\mathrm{Im}\, a_0^D} \sim \frac{\hbar}{\pi}\frac{\partial u}{\partial a_0}\, e^{-\frac{2\pi}{\hbar}\mathrm{Im}\, a_0^D}$$

Dunne-Ünsal (2012): extends to all orders in $\hbar$

- everything determined by perturbative saddle $u(N, \hbar)$

Mathieu Equation Spectrum: far above the barrier

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$



now consider the high energy spectrum: $u(N,\hbar)\gg 1$

Mathieu Equation Spectrum: far above the barrier

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi \qquad , \quad u\gg 1 \quad , \quad N\hbar\gg 1$$

- gap edges $u_N^{(\pm)}(\hbar)$

$$\begin{aligned}
u_0 &= \frac{\hbar^2}{8}\left(0-\frac{1}{\hbar^2}+\frac{7}{4\hbar^6}-\frac{58}{9\hbar^{10}}+\frac{68687}{2304\hbar^{14}}+\dots\right)\\
u_1^{(-)} &= \frac{\hbar^2}{8}\left(1-\frac{4}{\hbar^2}-\frac{2}{\hbar^4}+\frac{1}{\hbar^6}-\frac{1}{6\hbar^8}-\frac{11}{36\hbar^{10}}+\frac{49}{144\hbar^{12}}+\dots\right)\\
u_1^{(+)} &= \frac{\hbar^2}{8}\left(1+\frac{4}{\hbar^2}-\frac{2}{\hbar^4}-\frac{1}{\hbar^6}-\frac{1}{6\hbar^8}+\frac{11}{36\hbar^{10}}+\frac{49}{144\hbar^{12}}+\dots\right)\\
u_2^{(-)} &= \frac{\hbar^2}{8}\left(4-\frac{4}{3\hbar^4}+\frac{5}{54\hbar^8}-\frac{289}{19440\hbar^{12}}+\frac{21391}{6998400\hbar^{16}}+\dots\right)\\
u_2^{(+)} &= \frac{\hbar^2}{8}\left(4+\frac{20}{3\hbar^4}-\frac{763}{54\hbar^8}+\frac{1002401}{19440\hbar^{12}}-\frac{1669068401}{6998400\hbar^{16}}+\dots\right)
\end{aligned}$$

- convergent ("strong-coupling") expansions!

Mathieu Equation Spectrum: far above the barrier

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi \qquad , \quad u\gg 1 \quad , \quad N\hbar\gg 1$$

- convergent ("strong-coupling") expansions for all N

$$u(N,\hbar)\sim\frac{\hbar^2}{8}\left(N^2+\frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4+\frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8\right.$$
$$\left.+\frac{9N^4+58N^2+29}{64(N^2-1)^5(N^2-4)(N^2-9)}\left(\frac{2}{\hbar}\right)^{12}+\dots\right)$$

- where are the gaps ?

- poles in coefficients, just like de Wit/'t Hooft in strong-coupling expansions in unitary matrix integrals

Beyond Large N: Multi-instantons at strong coupling

$$u(N,\hbar) \sim \frac{\hbar^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 + \dots\right.$$

- this expansion, with poles, comes from a convergent continued fraction representation
- there is a multi-instanton trans-series structure hidden in these convergent expressions, when viewed as functions of two variables

All-orders WKB for Mathieu spectrum: far above barrier

- all-orders WKB actions:

$$a(u) = \sum_{n=0}^{\infty} \hbar^{2n}\, a_n(u) \qquad , \qquad a^D(u) = \sum_{n=0}^{\infty} \hbar^{2n}\, a_n^D(u)$$

- Bohr-Sommerfeld high in the spectrum: $u \gg 1$

$$a_0(u) \sim \sqrt{2u}\left(1 - \frac{1}{16u^2} - \frac{15}{1024u^4} - \frac{105}{16384u^6} - \dots\right)$$

$$a_1(u) \sim -\frac{1}{16\,(2u)^{5/2}}\left(1 + \frac{35}{32u^2} + \frac{1155}{1024u^4} + \frac{75075}{65536u^6} + \dots\right)$$

- $\Rightarrow$ energy $u \propto N^2 + \dots$
- here, Bohr-Sommerfeld $\Rightarrow$ convergent expansions
- what about the gaps?

Mathieu Equation Spectrum: far above the barrier

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$

- narrow gaps high in the spectrum: complex instantons
- Dyhkne: same formula for band/gap splittings

$$\Delta u \sim \frac{2}{\pi}\frac{\partial u}{\partial N}\, e^{-\frac{2\pi}{\hbar}\,\mathrm{Im}\, a_0^D}$$

$$\Delta u_N^{\rm gap} \sim \frac{\hbar^2}{4}\frac{1}{\left(2^{N-1}(N-1)!\right)^2}\left(\frac{2}{\hbar}\right)^{2N}\left[1+O\left(\left(\frac{2}{\hbar}\right)^4\right)\right]$$

$$\sim \frac{N\,\hbar^2}{2\pi}\left(\frac{e}{N\,\hbar}\right)^{2N} \quad , \quad N\gg 1$$

- same formula, but $a_0(u)$ & $a_0^D(u)$ expanded near $u\sim\infty$

$$a_0^D(u)\sim i\frac{\sqrt{2u}}{\pi}\left(-2+\log(8u)+\frac{1-\log(8u)}{16u^2}+\frac{47-30\log(8u)}{2048u^4}+\dots\right)$$

- everything (still) encoded in perturbative saddle $u(N,\hbar)$

Mathieu Equation Spectrum: far above the barrier

high in the spectrum, the expansions are convergent!

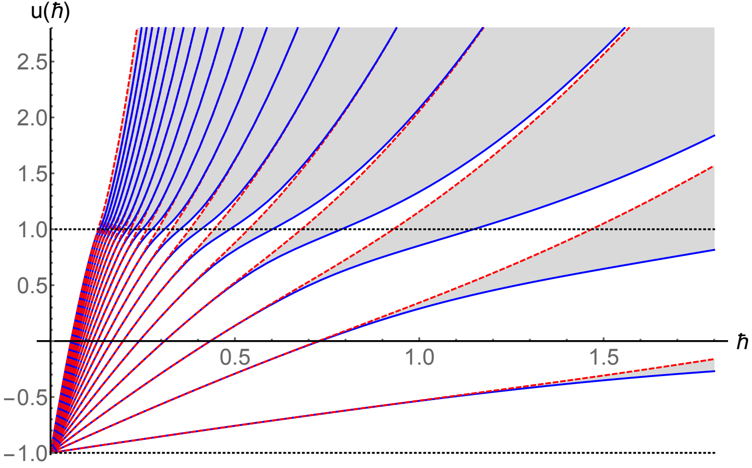
but there are narrow gaps

non-perturbative gap widths given by complex instantons

what happened to the trans-series for the energy deep inside the wells ?

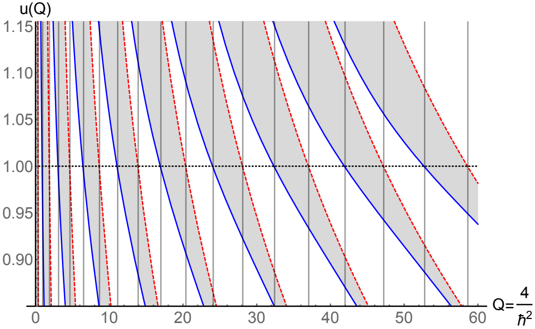
how to instantons condense?

3 different spectral regions: $-\frac{\hbar^2}{2}\,\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$



Mathieu spectrum: near the potential barrier

- in this region instantons are large
- bands and gaps are of equal width



- degrees of freedom re-organize from tight-binding 'atomic' states to 'nearly-free' scattering states
- smooth transition between divergent trans-series and convergent expansions

Mathieu spectrum: near the potential barrier

- near $u \sim 1$

$$a_0 \sim \frac{4}{\pi} + \frac{u-1}{2\pi}\left[\ln\left(\frac{32}{u-1}\right)+1\right]+\dots \quad \sim \frac{\hbar}{2}\,N$$

$$a_0^D \sim \frac{i}{2}\,(u-1)+\dots$$

- leading asymptotic estimate: $\qquad N \sim \frac{8}{\pi\,\hbar}$

- leading term in weak-coupling expansion:

$$u \sim -1 + \frac{8}{\pi}\left[1 - \frac{1}{16}\frac{8}{\pi} - \frac{1}{2^8}\left(\frac{8}{\pi}\right)^2 - \frac{5}{2^{14}}\left(\frac{8}{\pi}\right)^3 - \frac{33}{2^{18}}\left(\frac{8}{\pi}\right)^4 - \dots\right] + O(\hbar)$$

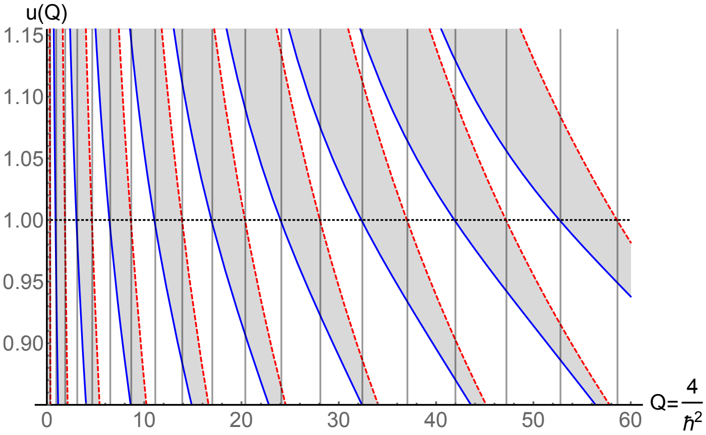
- leading term in strong-coupling expansion:

$$u \sim \frac{1}{2}\left[\left(\frac{4}{\pi}\right)^2 + \frac{1}{2}\left(\frac{\pi}{4}\right)^2 + \frac{5}{32}\left(\frac{\pi}{4}\right)^6 + \frac{9}{64}\left(\frac{\pi}{4}\right)^{10} + \dots\right] + O(\hbar)\dots$$

- both equal to 1

Mathieu spectrum: near the potential barrier

- near $u \sim 1$, leading asymptotic estimate: $N \sim \frac{8}{\pi\,\hbar}$
- more refined estimate: $a_0 \sim \frac{\hbar}{2}\left(N \pm \frac{1}{4}\right) \longrightarrow N \pm \frac{1}{4} \sim \frac{8}{\pi\hbar}$



- bands and gaps are equal width here

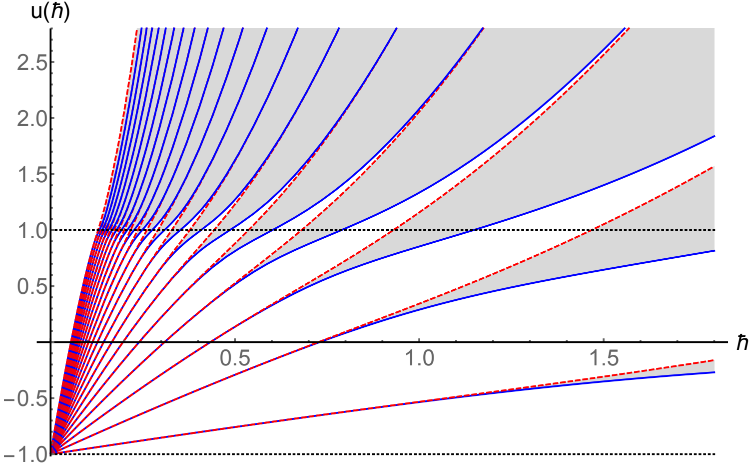
Mathieu Equation Spectrum: far above the barrier

new: resurgent structure high in the spectrum, in the convergent region

Basar, GD, Ünsal, 2015

Mathieu Equation Spectrum

$$-\frac{\hbar^2}{2}\,\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$



Beyond Large N: Multi-instantons at strong coupling

- convergent expansion, but coefficients have poles:

$$u(N,\hbar) \sim \frac{\hbar^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 + \dots\right.$$

$$u_2^{(+)} = \frac{\hbar^2}{8}\left(4 + \frac{20}{3\hbar^4} - \frac{763}{54\hbar^8} + \frac{1002401}{19440\hbar^{12}} - \frac{1669068401}{6998400\hbar^{16}} + \dots\right)$$

$$u_2^{(-)} = \frac{\hbar^2}{8}\left(4 - \frac{4}{3\hbar^4} + \frac{5}{54\hbar^8} - \frac{289}{19440\hbar^{12}} + \frac{21391}{6998400\hbar^{16}} + \dots\right)$$

- average: $\frac{20-4}{2\cdot 3} = \frac{8}{3} = \frac{2^4}{2\cdot(4-1)}$

Beyond Large N: Multi-instantons at strong coupling

- convergent expansion, but coefficients have poles:

$$u(N,\hbar) \sim \frac{\hbar^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 + \dots\right.$$

$$u_2^{(+)} = \frac{\hbar^2}{8}\left(4 + \frac{20}{3\hbar^4} - \frac{763}{54\hbar^8} + \frac{1002401}{19440\hbar^{12}} - \frac{1669068401}{6998400\hbar^{16}} + \dots\right)$$

$$u_2^{(-)} = \frac{\hbar^2}{8}\left(4 - \frac{4}{3\hbar^4} + \frac{5}{54\hbar^8} - \frac{289}{19440\hbar^{12}} + \frac{21391}{6998400\hbar^{16}} + \dots\right)$$

- average: $\frac{20-4}{2\cdot 3} = \frac{8}{3} = \frac{2^4}{2\cdot(4-1)}$

$$\frac{8}{\hbar^2}u_2^{(\pm)} = \left(4 + \frac{8}{3\hbar^4}\right) \pm \frac{4}{\hbar^4}\left(1 - \frac{16}{9\hbar^4}\right) - \frac{379}{54\hbar^8}\left(1 - \frac{62632}{17055\hbar^4}\right) \pm \frac{11141}{432\hbar^{12}}\left(1 - \right.$$

- this is an instanton expansion
- pole develops at 2-instanton order in all fluctuations

Beyond Large N: Multi-instantons at strong coupling

- convergent expansion, but coefficients have poles:

$$u(N,\hbar) \sim \frac{\hbar^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 + \frac{9N^4+58N^2+29}{64(N^2-1)^5(N^2-4)(N^2-9)}\left(\frac{2}{\hbar}\right)^{12} + \dots\right)$$

$$u_4^{(+)} = \frac{\hbar^2}{8}\left(16 + \frac{8}{15\hbar^4} + \frac{433}{3375\hbar^8} - \frac{45608}{5315625\hbar^{12}} + \dots\right)$$

$$u_4^{(-)} = \frac{\hbar^2}{8}\left(16 + \frac{8}{15\hbar^4} - \frac{317}{3375\hbar^8} + \frac{80392}{5315625\hbar^{12}} + \dots\right)$$

- average: $\frac{433-317}{2\cdot 3375} = \frac{58}{3375} = \frac{2^8(5\cdot 16+7)}{32\cdot 15^3\cdot 12}$

$$\frac{8}{\hbar^2}u_4^{(\pm)} = \left(16 + \frac{8}{15\hbar^4} + \frac{58}{3375\hbar^8} + \frac{17932}{5315625\hbar^{12}}\right) \pm \frac{1}{9\hbar^8}\left(1 + \frac{8}{75\hbar^4} + \dots\right)$$

- pole develops at 2-instanton order: instanton expansion

Beyond Large N: Multi-instantons at strong coupling

$$u(N,\hbar) \sim \frac{\hbar^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 + \dots\right.$$

- re-organize as a multi-instanton expansion

$$u_N^{(\pm)}(\hbar) = \frac{\hbar^2 N^2}{8}\sum_{n=0}^{N-1}\frac{\alpha_n(N)}{\hbar^{4n}} \pm \frac{\hbar^2}{8}\frac{1}{(2^{N-1}(N-1)!)^2}\left(\frac{2}{\hbar}\right)^{2N}\sum_{n=0}^{N-1}\frac{\beta_n(N)}{\hbar^{4n}} + \dots$$

- fluctuation series are very similar
- 1-instanton gap splitting: (Basar, GD, Unsal, 2015)

$$\Delta u_N \equiv \frac{1}{(2^{N-1}(N-1)!)^2}\frac{\partial u}{\partial N}e^{A(N,\hbar)} \quad \Rightarrow \quad \frac{\partial A}{\partial \hbar^2} = -\frac{4}{\hbar^4}\frac{\partial u}{\partial N}$$

- 1-inst. flucts. determined by pert. exp. (polynomial !)

- resurgent multi-instanton structure in convergent region

Uniform Expansions: Small $\hbar$ and Large N

- often we study theories with both $\hbar$ and N
- 't Hooft limit: $\hbar \to 0$ with $\lambda \equiv N\,\hbar$ fixed
- but $\lambda \ll 1$, $\lambda \sim 1$ and $\lambda \gg 1$ are all different
- mathematical analogy: Bessel functions

$$I_N\left(\frac{1}{\hbar}\right) = I_N\left(N\frac{1}{N\hbar}\right) \sim \begin{cases} \sqrt{\frac{\hbar}{2\pi}}\, e^{1/\hbar} & , \quad \hbar \to 0,\ N \text{ fixed} \\ \frac{1}{\sqrt{2\pi N}} \left(\frac{e}{2N\hbar}\right)^N & , \quad N \to \infty,\ \hbar \text{ fixed} \end{cases}$$

Uniform Expansions: Small $\hbar$ and Large N

- often we study theories with both $\hbar$ and N
- 't Hooft limit: $\hbar \to 0$ with $\lambda \equiv N\,\hbar$ fixed
- but $\lambda \ll 1$, $\lambda \sim 1$ and $\lambda \gg 1$ are all different
- mathematical analogy: Bessel functions

$$I_N\left(\frac{1}{\hbar}\right) = I_N\left(N\frac{1}{N\hbar}\right) \sim \begin{cases} \sqrt{\frac{\hbar}{2\pi}}\, e^{1/\hbar} & , \quad \hbar \to 0,\ N \text{ fixed} \\ \frac{1}{\sqrt{2\pi N}}\left(\frac{e}{2N\hbar}\right)^N & , \quad N \to \infty,\ \hbar \text{ fixed} \end{cases}$$

- uniform asymptotics:

$$I_N\left(N\frac{1}{N\hbar}\right) \sim \frac{\exp\left[\sqrt{N^2+\frac{1}{\hbar^2}}\right]}{\sqrt{2\pi}\left(N^2+\frac{1}{\hbar^2}\right)^{\frac{1}{4}}}\left(\frac{\frac{1}{N\,\hbar}}{1+\sqrt{1+\frac{1}{(N\,\hbar)^2}}}\right)^N$$

- can be made uniformly resurgent

Mathieu spectrum: all regions

- $N\hbar \ll 1$, deep inside wells: resurgent trans-series

$$u^{(\pm)}(N,\hbar) \sim \sum_{n=0}^{\infty} c_n(N)\hbar^n \pm \frac{32}{\sqrt{\pi}\, N!}\left(\frac{32}{\hbar}\right)^{N-1/2} e^{-\frac{8}{\hbar}} \sum_{n=0}^{\infty} d_n(N)\hbar^n + \dots$$

- Borel poles at two-instanton location

- $N\hbar \gg 1$, far above barrier: convergent series

$$u^{(\pm)}(N,\hbar) = \frac{\hbar^2 N^2}{8} \sum_{n=0}^{N-1} \frac{\alpha_n(N)}{\hbar^{4n}} \pm \frac{\hbar^2}{8} \frac{\left(\frac{2}{\hbar}\right)^{2N}}{(2^{N-1}(N-1)!)^2} \sum_{n=0}^{N-1} \frac{\beta_n(N)}{\hbar^{4n}} + \dots$$

- coefficients have poles at two-instanton location

- $N\hbar \sim \frac{8}{\pi}$, near barrier top

$$u^{(\pm)}(N,\hbar) \sim 1 - \frac{c_N}{\left(N \pm \frac{1}{4}\right)^2}\left(\frac{4}{\hbar^2} - \frac{\pi^2}{16}\left(N \pm \frac{1}{4}\right)^2\right) \quad , \quad c_N \sim O(1)$$

Resurgence of $\mathcal{N}=2$ SUSY SU(2) (Başar, GD, 1501.05671)

- moduli parameter: $u = \langle \mathrm{tr}\, \Phi^2 \rangle$
- electric: $u \gg 1$; magnetic: $u \sim 1$; dyonic: $u \sim -1$
- $a = \langle \text{scalar} \rangle$, $a_D = \langle \text{dual scalar} \rangle$, $a_D = \frac{\partial \mathcal{W}}{\partial a}$
- Nekrasov twisted superpotential:

$$\mathcal{W}^{inst} \sim \frac{\hbar^2}{2\pi i}\left(\frac{\Lambda^4}{16a^4} + \frac{21\Lambda^8}{256a^8} + \dots\right) + \frac{\hbar^4}{2\pi i}\left(\frac{\Lambda^4}{64a^6} + \frac{219\Lambda^8}{2048a^{10}} + \dots\right) + \dots$$

$$\mathcal{W}^{class} + \mathcal{W}^{pert} \sim -\frac{a^2}{2\pi i}\log\frac{a^2}{\Lambda^2} - \frac{\hbar^2}{48\pi i}\log\frac{a^2}{2\Lambda^2} + \hbar^2\sum_{n=1}^{\infty} d_{2n}\left(\frac{\hbar}{a}\right)^{2n}$$

- Mathieu equation:

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \Lambda^2\cos(x)\,\psi = u\,\psi \quad , \quad a \equiv \frac{N\hbar}{2}$$

- Matone relation:

$$u(a,\hbar) = \frac{i\pi}{2}\Lambda\frac{\partial \mathcal{W}(a,\hbar,\Lambda)}{\partial \Lambda} - \frac{\hbar^2}{48}$$

Resurgence in $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ Theories

- $\mathcal{N} = 2^*$ theories correspond to elliptic Lamé potential

- but Bohr-Sommerfeld is not the whole story ... it just gives the locations of bands and gaps, not their widths

- new non-perturbative contributions to twisted super-potential

(Başar, GD, 1501.05671)

$$u \to u_{\rm formal} + O\left(e^{-\frac{\rm periods}{\hbar}}\right) \quad \to \quad \mathcal{W} \to \mathcal{W}_{\rm formal} + O\left(e^{-\frac{\rm periods}{\hbar}}\right)$$

- exact WKB in large N region and conjectured world sheet interpretation (Kashani-Poor, Troost, 1504.08324)

Conclusions

- the Mathieu spectrum is extremely rich, both physically and mathematically
- still holds some surprises
- trans-series structure in weak-coupling and strong-coupling regions, and in the transition region
- uniform trans-series capture large N physics: real and complex instantons; phase transitions; instanton condensation; ...
- even more structure for $\mathcal{N} = 2^*$ theories: Lamé potential
- implications for gauge theory and integrable models ?