The spectrum of Large-N gauge theories

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Lattice action

Path integral

$$Z = \int (\mathcal{D}U_\mu(i))\,(\det M(U_\mu))^{N_f}\, e^{-S_g(U_{\mu\nu}(i))}$$

with

$$U_\mu(i) = Pexp\left(ig \int_i^{i+a\hat{\mu}} A_\mu(x)\mathrm{d}x \right) \ , \ U_{\mu\nu}(i) = U_\mu(i) U_\nu(i+\hat{\mu}) U^\dagger_\mu(i+\hat{\nu}) U^\dagger_\nu(i)$$

Gauge part

$$S_g = \beta \sum_{i,\mu} \left(1 - \frac{1}{N} \mathcal{R}\mathrm{e}\ \mathrm{Tr}(\mathrm{U}_{\mu\nu}(\mathrm{i})) \right) \qquad , \qquad \textit{with } \beta = 2N/g_0^2$$

Invariance under SU(N) gauge transformations
$\tilde{U}_\mu(i) = G^\dagger(i) U_\mu(i) G(i+\hat{\mu})$

Wilson fermions

Take the naive Dirac fermions and add an irrelevant term that goes like the Laplacian

$$M_{\alpha\beta}(ij) = (m + 4r)\delta_{ij}\delta_{\alpha\beta} - \frac{1}{2}\left[(r - \gamma_\mu)_{\alpha\beta} U_\mu(i)\delta_{i,j+\mu} + (r + \gamma_\mu)_{\alpha\beta} U^\dagger_\mu(j)\delta_{i,i-\mu}\right]$$

This formulation breaks explicitly chiral symmetry

Define the hopping parameter

$$\kappa = \frac{1}{2(m + 4r)}$$

Chiral symmetry recovered in the limit $\kappa \to \kappa_c$ (κ_c to be determined numerically)

Quenched approximation

For an observable $\mathcal{O}$

$$\langle \mathcal{O} \rangle = \frac{\int (\mathcal{D}U_\mu(i)) \, (\det M(U_\mu))^{N_f} f(M) e^{-S_g(U_{\mu\nu}(i))}}{\int (\mathcal{D}U_\mu(i)) \, (\det M(U_\mu))^{N_f} e^{-S_g(U_{\mu\nu}(i))}}$$

Assume $\det M(U_\mu) \simeq 1$ i.e. fermions loops are removed from the action

The approximation is exact in the $m \to \infty$ and $N \to \infty$ limit ($g^2 N$ is fixed)
$\hookrightarrow$ the large N spectrum is quenched for $m \neq 0$

As N increases, unquenching effects are expected for smaller quark masses

Extracting the spectrum

Trial operators $\Phi_1(t), \ldots, \Phi_n(t)$ with the quantum numbers of the state of interest

$$
\begin{aligned}
C_{ij}(t) &= \langle 0 | \left(\Phi_i(0)\right)^\dagger \Phi_j(t) | 0 \rangle \\
&= \langle 0 | \left(\Phi_i(0)\right)^\dagger e^{-Ht} \Phi_j(0) e^{Ht} | 0 \rangle \\
&= \sum_n \langle 0 | \left(\Phi_i(0)\right)^\dagger | n \rangle \langle n | e^{-Ht} \Phi_j(0) e^{Ht} | 0 \rangle \\
&= \sum_n e^{-\Delta E_n t} \langle 0 | \left(\Phi_i(0)\right)^\dagger | n \rangle \langle n | \Phi_j(0) | 0 \rangle = \sum_n c^*_{in} c_{jn} e^{-\Delta E_n t}
\end{aligned}
$$

Masses can be extracted from diagonal correlators

$$
C_{ii} = \sum_n |c_{in}|^2 e^{-a m_n t} \underset{t \to \infty}{\rightarrow} |c_{i1}|^2 e^{-a m_1 t}
$$

Variational calculations

Using the information of the whole correlation matrix allows us to better control the groundstate mass and to determine masses of excitations with a variational procedure

1. Find the eigenvector v that minimises

$$am_1(t_d) = -\frac{1}{t_d} \log \frac{v_i^* C_{ij}(t_d) v_j}{v_i^* C_{ij}(0) v_j}$$

   for some t_d
2. Fit $v(t)$ with the law $Ae^{-m_1 t}$ to extract m_1
3. Find the complement to the space generated by $v(t)$
4. Repeat 1-3 to extract $m_2, \ldots, m_n$

Need a good overlap with the state of interest

Fermionic operators

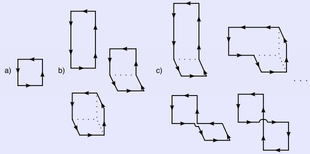
For isotriplet states (flavour index $\alpha \neq \beta$):

Particle	Bilinear	J^{PC}
a_0	$\bar{\psi}_\alpha \psi_\beta$	0^{++}
π	$\bar{\psi}_\alpha \gamma_5 \psi_\beta$, $\bar{\psi}_\alpha \gamma_0 \gamma_5 \psi_\beta$	0^{-+}
ρ	$\bar{\psi}_\alpha \gamma_i \psi_\beta$, $\bar{\psi}_\alpha \gamma_0 \gamma_i \psi_\beta$	1^{--}
a_1	$\bar{\psi}_\alpha \gamma_5 \gamma_i \psi_\beta$	1^{++}
b_1	$\bar{\psi}_\alpha \gamma_i \gamma_j \psi_\beta$	1^{+-}

Flavour singlet states more difficult to study

Glueball operators

- Using traced Wilson loops of various shapes, build operators $O_i(t)$ that are eigenstates of P and C and under the (discrete) group of rotations transform according to irreducible representations
- Examples of basic contours used



- Continuous spin reconstructed by looking the decomposition of the irreducible representations of the diedric group in irreps of SO(3)

String tension

Confining potential: $V = \sigma R$

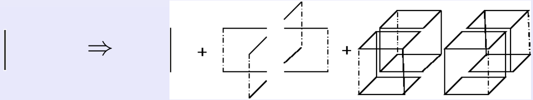
Polyakov loop $P_k(i) = \frac{1}{N} \mathrm{Tr} \prod_{j=0}^{L} U_k(i + j\hat{k})$

$$
\begin{aligned}
P(t) &= \sum_{\vec{n},k} P_k(\vec{n}, t) \\
C(t) &= \langle (P(0))^\dagger P(t) \rangle = \sum_j |c_j|^2 e^{-am_j t} \underset{t \to \infty}{\rightarrow} |c_l|^2 e^{-am_l t} \\
am_l &\simeq a^2 \sigma L - \frac{\pi (D-2)}{6L}
\end{aligned}
$$

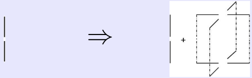
Building extended operators

Starting from links, we can built extended loop operators via

- Smearing



- Blocking



For mesons, extended operators can be built by smearing source and sink operators over a few lattice spacings with a weight function (e.g. Gaussian smearing, wall smearing etc.)

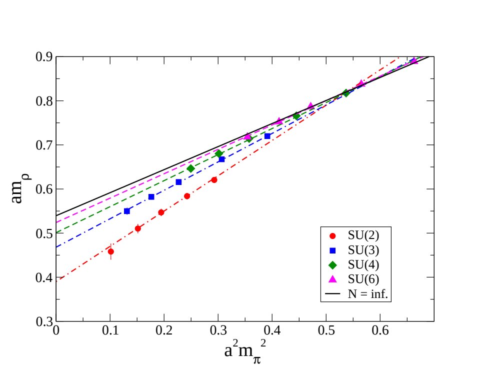
Large N limit on the lattice

The lattice approach allows us to go beyond perturbative and diagrammatic arguments. For a given observable

1. Continuum extrapolation
   - Determine its value at fixed a and N
   - Extrapolate to the continuum limit
   - Extrapolate to $N \to \infty$ using a power series in $1/N^2$
2. Fixed lattice spacing
   - Choose a in such a way that its value in physical units is common to the various N
   - Determine the value of the observable for that a at any N
   - Extrapolate to $N \to \infty$ using a power series in $1/N^2$

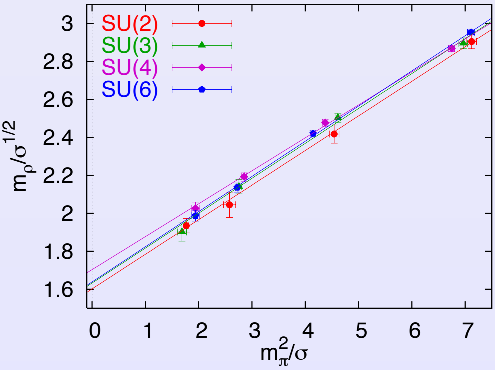
Study performed for various observables both at zero and finite temperature for $2 \leq N \leq 8$ (and $N = 17$!)

m_ρ vs. m_π^2 at $N = \infty$



[L. Del Debbio, B. Lucini, A. Patella and C. Pica, JHEP 0803 (2008) 062]

m_π vs. m_ρ - fixing σ



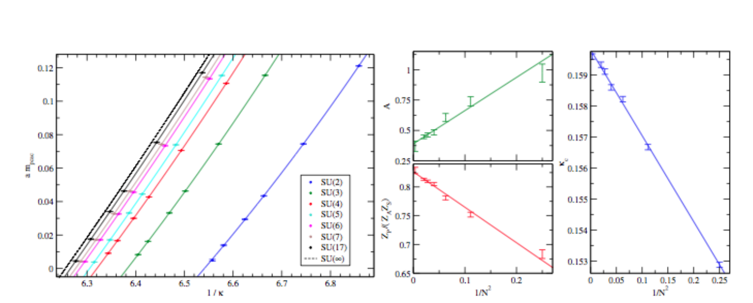
[G. Bali and F. Bursa, JHEP 0809 (2008) 110]

Computing the lowest-lying isotriplet spectrum

1. Aims: determining the mesonic spectrum (including some excitations) and decay constants
2. Calculations performed for $2 \leq N \leq 7$ and $N = 17$
3. β fixed across the various n by imposing $a\sqrt{\sigma} = 0.2093$, implying $a \simeq 0.093$fm (or $a^{-1} \simeq 2.1$ GeV)
4. Range of κ down to $m_\pi \simeq 0.5\sqrt{\sigma}$ for $N \geq 5$ and $m_\pi \simeq 0.75\sqrt{\sigma}$ for $N \leq 4$
5. Size $24^3 \times 48$ for $N \neq 17$, $12^3 \times 24$ for $N = 17$ (finite size effects negligible at large N)
6. 200 configurations (80 configurations for $N = 17$)

[G. Bali, F. Bursa, L. Castagnini, S. Collins, L. Del Debbio, B. Lucini and M. Panero, arXiv:1304.4437]

The PCAC mass



PCAC mass and chiral limit

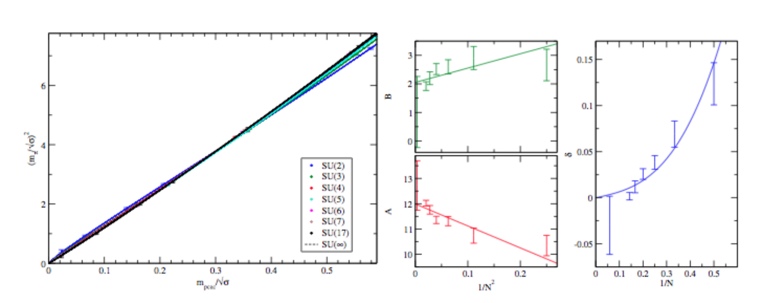
Putting together large-N and χPT predictions

$$a\, m_{\text{PCAC}} = \frac{Z_P}{Z_A Z_S} (1 + A\, a\, m_{\text{PCAC}}) \frac{1}{2} \left(\frac{1}{\kappa} - \frac{1}{\kappa_c} \right),$$

with A, K and $\frac{Z_P}{Z_A Z_S}$ expected to have $1/N^2$ corrections in the large-N limit:

$$\begin{aligned} \frac{Z_P}{Z_A Z_S} &= 0.8291(20) - \frac{0.699(45)}{N^2}, \\ A &= 0.390(13) + \frac{2.73(26)}{N^2}, \\ \kappa_c &= 0.1598555(33)(447) - \frac{0.028242(68)(394)}{N^2} \end{aligned}$$

The pseudoscalar



Fit results for the pseudoscalar

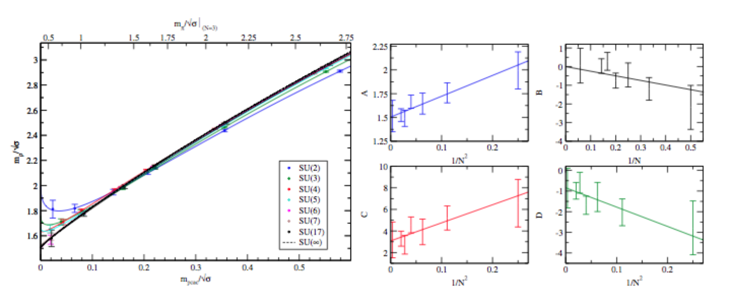
Ansatz:

$$(am_\pi)^2 = A\,(am_q)^{\frac{1}{1+\delta}} + B\,(am_q)^2$$

Results:

$$\begin{aligned} A &= 11.99(0.10) - \frac{8.7(1.6)}{N^2} \\ B &= 2.05(0.13) + \frac{5.0(2.2)}{N^2} \\ \delta &= \frac{0.056(19)}{N} + \frac{0.94(21)}{N^3} \end{aligned}$$

The vector


$m_\pi/\sqrt{\sigma}|_{(N=3)}$
$m_\rho/\sqrt{\sigma}$
$m_{pcac}/\sqrt{\sigma}$
SU(2)
SU(3)
SU(4)
SU(5)
SU(6)
SU(7)
SU(17)
SU(∞)
A
B
C
D
$1/N^2$
1/N

Fit results for the vector

Ansatz:

$$m_\rho = A + B m_q^{1/2} + C m_q + D m_q^{3/2}$$

Results:

$$\begin{aligned} A &= 1.504(51) + \frac{2.19(75)}{N^2} \\ B &= -\frac{2.47(94)}{N} \\ C &= 3.08(53) + \frac{16.8(8.2)}{N^2} \\ D &= -0.84(31) - \frac{9.4(4.8)}{N^2} \end{aligned}$$

Other states

$$\frac{m_{a_1}}{\sqrt{\sigma}} = \left(2.860(21) + \frac{0.84(36)}{N^2}\right) + \left(2.289(35) - \frac{2.02(61)}{N^2}\right)\frac{m_{\mathrm{PCAC}}}{\sqrt{\sigma}}$$

$$\frac{m_{b_1}}{\sqrt{\sigma}} = \left(2.901(23) + \frac{1.07(40)}{N^2}\right) + \left(2.273(38) - \frac{2.83(72)}{N^2}\right)\frac{m_{\mathrm{PCAC}}}{\sqrt{\sigma}}$$

$$\frac{m_{a_0}}{\sqrt{\sigma}} = \left(2.402(34) + \frac{4.25(62)}{N^2}\right) + \left(2.721(53) - \frac{6.84(96)}{N^2}\right)\frac{m_{\mathrm{PCAC}}}{\sqrt{\sigma}}$$

Excited states

$$\frac{m_{\pi^\star}}{\sqrt{\sigma}} = \left(3.392(57) + \frac{1.0(1.1)}{N^2}\right) + \left(2.044(80) - \frac{1.2(1.6)}{N^2}\right)\frac{m_{\mathrm{PCAC}}}{\sqrt{\sigma}}$$

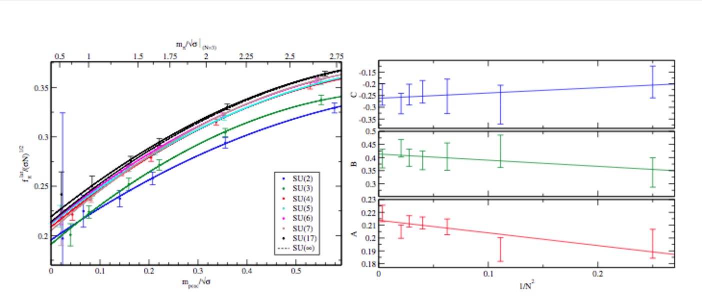
$$\frac{m_{\rho^\star}}{\sqrt{\sigma}} = \left(3.696(54) + \frac{0.23(55)}{N^2}\right) + \left(1.782(67) - \frac{1.30(54)}{N^2}\right)\frac{m_{\mathrm{PCAC}}}{\sqrt{\sigma}}$$

$$\frac{m_{a_0^\star}}{\sqrt{\sigma}} = \left(4.356(65) + \frac{1.8(1.4)}{N^2}\right) + \left(1.902(98) - \frac{2.9(2.1)}{N^2}\right)\frac{m_{\mathrm{PCAC}}}{\sqrt{\sigma}}$$

$$\frac{m_{a_1^\star}}{\sqrt{\sigma}} = \left(4.587(75) + \frac{1.2(1.2)}{N^2}\right) + \left(1.76(12) - \frac{2.1(19)}{N^2}\right)\frac{m_{\mathrm{PCAC}}}{\sqrt{\sigma}}$$

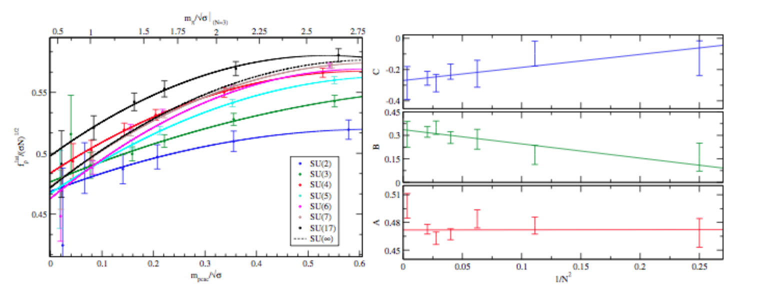
$$\frac{m_{b_1^\star}}{\sqrt{\sigma}} = \left(4.609(99) + \frac{1.7(1.5)}{N^2}\right) + \left(1.87(15) - \frac{2.5(2.2)}{N^2}\right)\frac{m_{\mathrm{PCAC}}}{\sqrt{\sigma}}$$

Pion decay constant



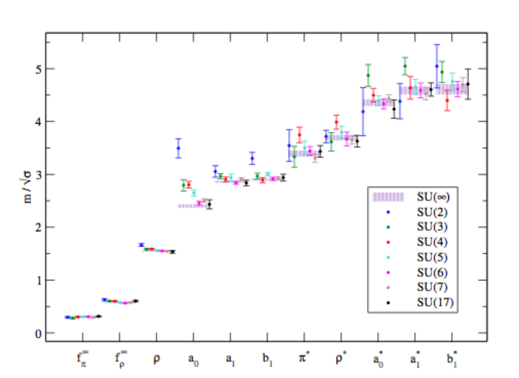
Fit $f_\pi^{\mathrm{lat}}/\sqrt{N\sigma} = A + B \cdot m_{\mathrm{PCAC}}/\sqrt{\sigma} + C \cdot m_{\mathrm{PCAC}}^2/\sigma$

ρ decay constant

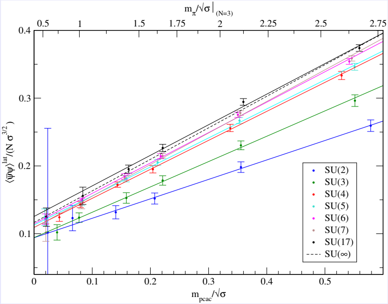


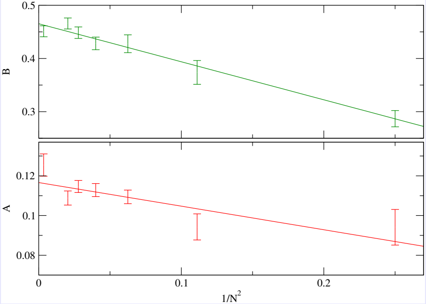
Fit $f_\rho^{\text{lat}}/\sqrt{N\sigma} = A + B \cdot m_{\text{PCAC}}/\sqrt{\sigma} + C \cdot m_{\text{PCAC}}^2/\sigma$

Approaching $N = \infty$



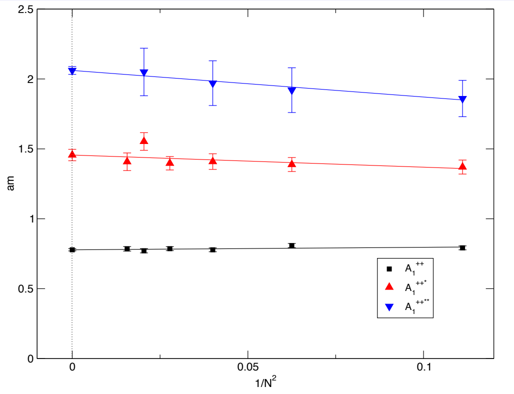
Chiral condensate





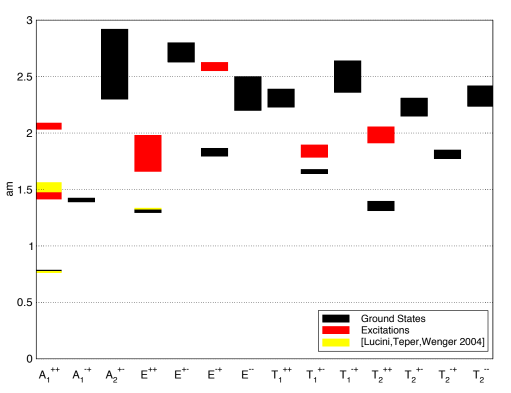
Fit $\langle\overline{\psi}\psi\rangle^{lat}/(N\sigma^{3/2}) = A + B \cdot m_{\rm PCAC}/\sqrt{\sigma}$

The A^{++} glueball channel



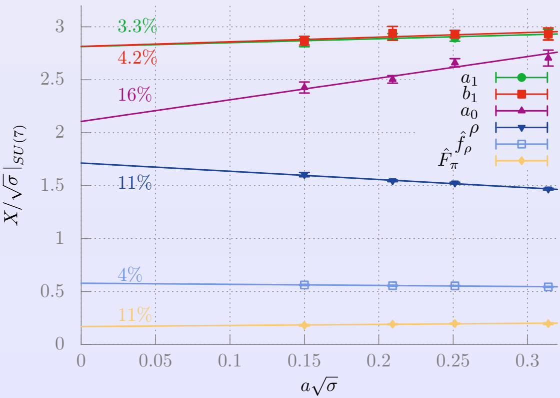
Lattice spacing fixed by requiring $aT_c = 1/6$

The glueball spectrum at $aT_c = 1/6$

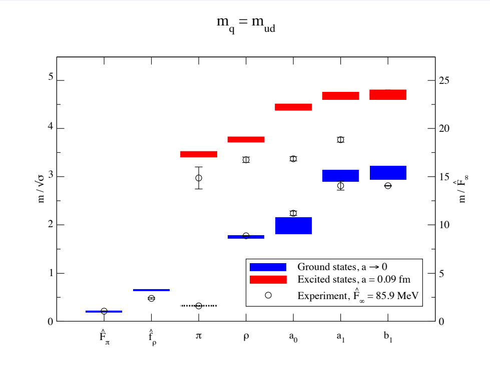


[B. Lucini, A. Rago and E. Rinaldi, JHEP 1008 (2010) 119]

Continuum meson spectrum – SU(7)

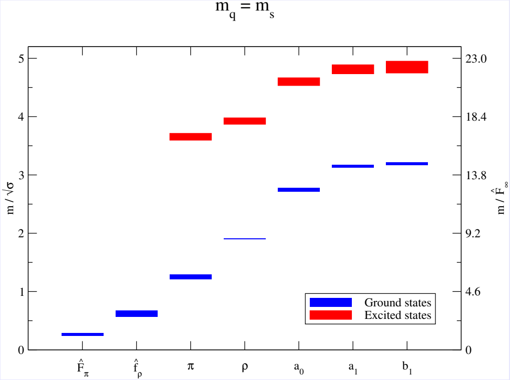


Comparison with QCD



$\sqrt{\sigma}$ fixed from the condition $\hat{F}_\infty = 85.9\ \mathrm{MeV}$, m_{ud} from $m_\pi = 138\ \mathrm{MeV}$

The strange meson spectrum



m_s fixed from $m_\pi(m_s) = (m^2_{K^\pm} + m^2_{k^0} - m^2_{\pi^\pm})^{1/2} = 686.9\ \mathrm{MeV}$

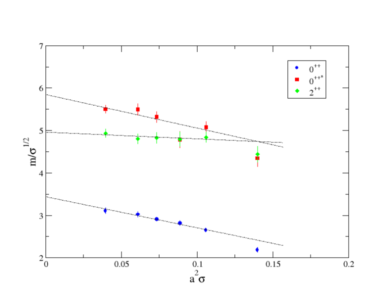
What do we learn?

- Lowest-lying mesons broadly compatible with QCD, excitations off by 20% (however, excitations less controlled in our calculation)
- The calculated large-N masses $m_\rho = 753(14)\ \mathrm{MeV}$ and of the $m_\phi = 981(44)\ \mathrm{MeV}$ are remarkably close to their experimental values $m_\rho = 775$ and $m_\phi = 1019\ \mathrm{MeV}$
- Observed degeneracy of (ρ, a_0) and $(a_1, \pi^\star)$ (predicted by χPT)

Glueballs in the continuum limit

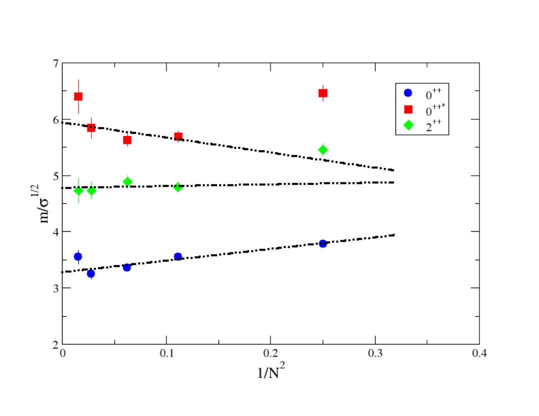
Example

Glueball masses in SU(4)



[B. Lucini and M. Teper, JHEP 0106 (2001) 050]

Large-N extrapolation of glueball masses



[B. Lucini, M. Teper and U. Wenger, JHEP 0406 (2004) 012]

Glueball masses at $N = \infty$

0^{++} $$\frac{m}{\sqrt{\sigma}} = 3.28(8) + \frac{2.1(1.1)}{N^2}$$

0^{++*} $$\frac{m}{\sqrt{\sigma}} = 5.93(17) - \frac{2.7(2.0)}{N^2}$$

2^{++} $$\frac{m}{\sqrt{\sigma}} = 4.78(14) + \frac{0.3(1.7)}{N^2}$$

Accurate $N = \infty$ value, small $\mathcal{O}(1/N^2)$ correction

Topological observables

Topological charge $Q = \frac{1}{32\pi^2} \int \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \, \mathrm{d}^4 x = \int q(x) \, \mathrm{d}^4 x$

Topological susceptibility $\chi_t = \frac{\langle Q^2 \rangle - \langle Q \rangle^2}{V}$

Witten-Veneziano formula (large-N result) $\chi_t = \frac{m^2_{\eta'} f^2_{\eta'}}{2 N_f}$

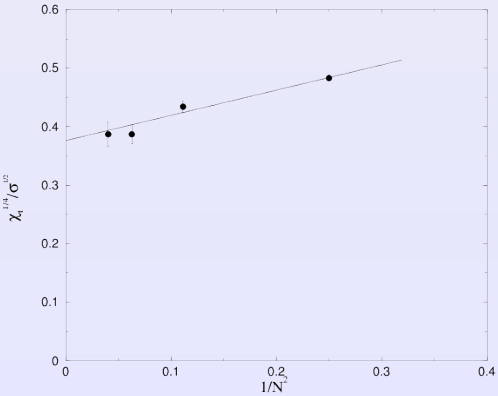
$\hookrightarrow \chi$ finite at large N

From lattice calculations

$$\frac{\chi_t}{\sigma^{1/4}} = 0.397(7) + \frac{0.35(13)}{N^2} - \frac{1.32(41)}{N^4}$$

[BL, M. Teper and U. Wenger, Nucl. Phys. B715 (2005) 461]

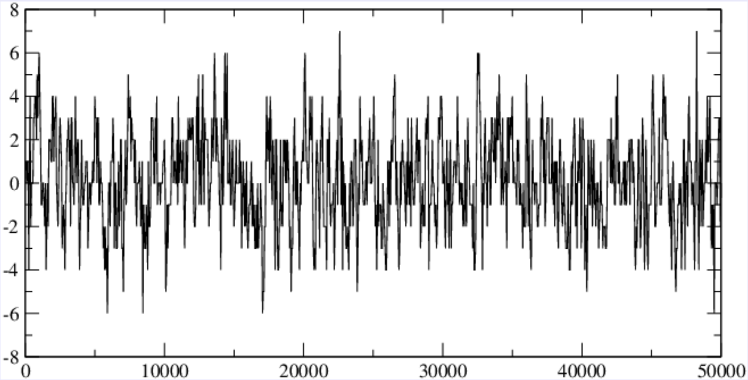
Large-N limit of χ_t – An example



[BL and M. Teper, JHEP 0106 (2001) 050]

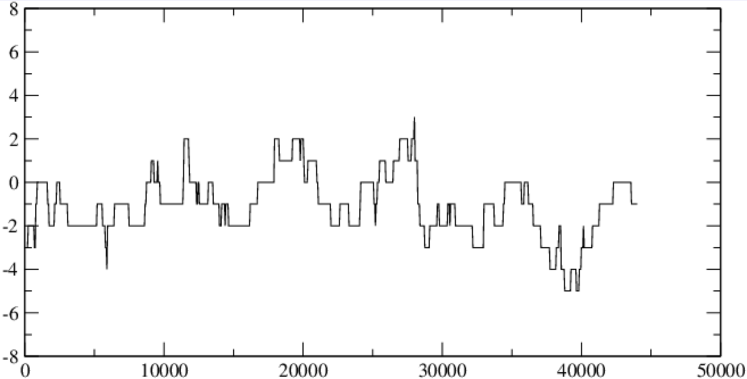
Monte Carlo history of Q – SU(3)

V=16^4, $\beta = 6.0$



Monte Carlo history of Q – SU(5)

V=16^4, $\beta = 17.45$



Decorrelating the topological charge

- At large N small size instantons are suppressed and this explains why the topology does not change
- Open Boundary Conditions (OBC) in time have been proposed as a way to push instantons into the lattice
- OBC have shown to be better than Periodic Boundary Conditions (PBC) in time at decorrelating topology in QCD

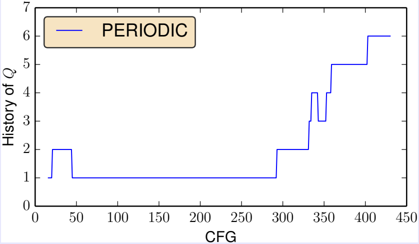
Does this mechanism also work at large N?

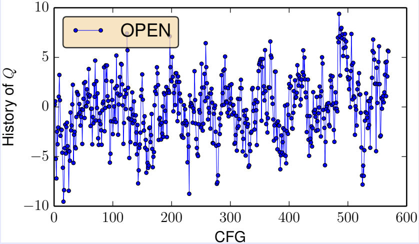
Periodic vs Open BC: a case study

- Gauge group: SU(7)
- Lattice spacing: $a\sqrt{\sigma} \sim 0.21$
- Lattice sizes: $16^3 \times N_t$, $N_t = 32, 48, 64$
- Statistics: $\sim$ 500 configurations, separated by 200 composite sweeps (1 composite sweep is 1 hb + 4 overrelax)
- Purpose: comparing results with PBC and OBC at fixed lattice parameters in a case where there is a severe ergodicity problem
- Observables: Q, gluonic correlators in the 0^{++} and 0^{-+} channels and instantons
- Both cooling and Wilson Flow used to filter UV modes

Monte Carlo history of Q

[A. Amato, G. Bali and B. Lucini, in progress]

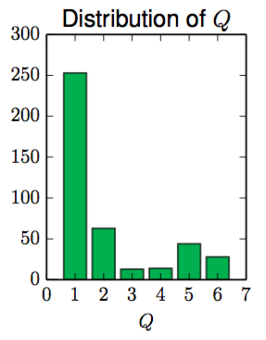




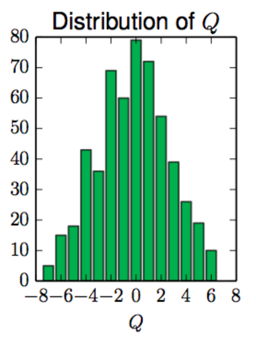
Significantly better decorrelation for OBC

Distribution of Q

PBC



OBC



OBC seem to cure the problem of ergodicity

Conclusions

- Good progress has been made in computing the spectrum of SU(N) gauge theories in the large-N limit, with precise results available at fixed lattice spacing
- Extrapolation to the continuum limit still in progress
- First results show little (and controlled) lattice spacing dependency on most groundstate masses (with noticeable exceptions, e.g. a_0
- Slow topological modes produce a systematic error that is hard to quantify
- OBC are being explored for a more controlled continuum extrapolation