Supergravity in spacetime

Local susy

$$e_\mu{}^a\,,\ \psi_\mu{}^\alpha$$

$$\longrightarrow\ e^a = dx^\mu e_\mu{}^a \qquad \leftarrow \text{bosonic 1-form}$$

$$\psi^\alpha = dx^\mu \psi_\mu{}^\alpha \qquad \leftarrow \text{fermionic 1-form}$$

Diff. forms (bosonic)

$$\Omega_q = \frac{1}{q!} dx^{\mu_q} \wedge \ldots \wedge dx^{\mu_1} \, \Omega_{\mu_1 \ldots \mu_q}(x)$$

<u>Exterior derivative</u> $\equiv$ differential

$$d = dx^\mu \partial_\mu \; ; \qquad d\,dx^\mu \equiv 0 \, , \qquad dd \equiv 0$$

Acts from the right:

$$d\Omega_q = \frac{1}{(q+1)!} dx^{\mu_{q+1}} \wedge \ldots \wedge dx^{\mu_2} \wedge dx^{\mu_1} \, \underline{(p+1)\, \partial_{[\mu_1} \Omega_{\mu_2 \ldots \mu_{q+1}]}(x)}$$

$$\equiv \frac{1}{q!} dx^{\mu_q} \wedge \ldots \wedge dx^{\mu_1} \wedge d\Omega_{\mu_1 \ldots \mu_q}(x)$$

$$\text{„} \partial_{\mu_1} \Omega_{\mu_2 \ldots \mu_{q+1}} + (-)'' \ldots$$

cyclyc perm. $(-)^{\text{parity}}$

Exterior product

$$\Omega_q \wedge \Omega_p = (-)^{pq} \, \Omega_p \wedge \Omega_q \, ,$$

$$dx^\mu \wedge dx^\nu = - dx^\nu \wedge dx^\mu$$

$$d(\Omega_q \wedge \Omega_p) = \Omega_q \wedge d\Omega_p + (-)^p \, d\Omega_q \wedge \Omega_p$$

Contruction with variation symbol (formal)

$$i_\delta(\Omega_q \wedge \Omega_p) = \Omega_q \wedge i_\delta \Omega_p + (-)^p \, i_\delta \Omega_q \wedge \Omega_p$$

(Generalized) Lie derivative formula

$$\delta \Omega_q = i_\delta d\Omega_q + d\, i_\delta \Omega_q$$

If we treat it literally:

$$i_\delta dx^\mu = \delta x^\mu$$

$$i_\delta \Omega_q = \frac{1}{(q-1)!} dx^{\mu_q} \wedge \ldots \wedge dx^{\mu_2} \delta x^{\mu_1} \Omega_{\mu_1 \ldots \mu_q}(x)$$

$$\delta \Omega_q = i_\delta d\Omega_q + d i_\delta \Omega_q$$

describes general coordinate or diffeomorphism transformation.

But generalized Lie derivative can be used in a wider context

$$\delta \Omega_q = i_\delta d\Omega_q + d i_\delta \Omega_q$$

e.g. if Ω_q is a q-form gauge field and the theory is inv under $\delta \Omega_q = d\, \alpha_{q-1}$ (gauge symm.), $i_\delta \Omega_q$ can be identified with parameter of gauge symm.

We will also use covariant Lie derivative formula for variation of covariant objects

$$\delta \Omega_q^A = i_\delta D\Omega_q^A + D i_\delta \Omega_q^A$$

where $D\Omega_q^A = d\Omega_q^A - \Omega_q^B \wedge \underbrace{A_B{}^A}_{\text{connection (1-form gauge field)}}$

in the cases when $i_\delta A_B{}^A$ can be identified with parameter of (manifest) gauge symm. of the theory

(2.) SUGRA in spacetime & diff. forms.

mostly minus $\eta^{ab} = (+,-,\dots,-)$

$(e_\mu{}^a, \psi_\mu{}^\alpha) \leftrightarrow dx^\mu e_\mu{}^a = e^a$

$dx^\mu \psi_\mu{}^\alpha = \psi^\alpha$

2.1. GRAVITY AND DIFF. FORMS

Cristoffel and spin connection: $0 = \hat{\mathcal{D}}_\mu e_\nu{}^a = \mathcal{D}_{[\mu} e_{\nu]}{}^a + \Gamma^\rho_{\mu\nu} e_\rho{}^a$

$$S_{EH} = \frac{1}{2\kappa^2}\int \epsilon_{abcd}\, R^{ab}\wedge e^c \wedge e^d = \frac{1}{\kappa^2}\int d^4x \underbrace{\sqrt{|g|}}_{e}\, R(e,\omega)$$

$\wedge \equiv$ EXTERIOR PRODUCT

$d = dx^\mu \partial_\mu$ = EXTERIOR DERIV. ↳ ACTS FROM THE RIGHT

CURVATURE $\quad R^{ab} := \frac{1}{2} dx^\nu \wedge dx^\mu R_{\mu\nu}{}^{ab} = \frac{1}{2} e^d \wedge e^c R_{cd}{}^{ab} = d\omega^{ab} - \omega^{ac}\wedge \omega_c{}^b$,

TORSION $\quad T^a := \frac{1}{2} dx^\nu \wedge dx^\mu T_{\mu\nu}{}^a = \frac{1}{2} e^c \wedge e^b T_{bc}{}^a = De^a = de^a - e^b \wedge \omega_b{}^a$;

BIs: $DR^{ab} = 0$, $DT^a = -e^b \wedge R_b{}^a$

$dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = d^4x\, \epsilon^{\mu\nu\rho\sigma}$ $\qquad \epsilon^{0123} = 1 = -\epsilon_{0123}$

FORMAL EXTERIOR DERIVATIVES

$S_{EH} = \frac{1}{\kappa^2}\int_{M^4} \mathcal{L}_4^{EH}$; $\boxed{\mathcal{L}_4^{EH} = \frac{1}{2\kappa}\, \epsilon_{abcd} R^{ab}\wedge e^c \wedge e^d}$, $\boxed{d\mathcal{L}_4^{EH} = \epsilon_{abcd} R^{ab} \wedge e^c \wedge T^d}$

2.3. Variation of the Einstein action and equivalent forms of Einstein eqs.

in tensor notation, but starting from $R^{ab} = d\omega^{ab} - \omega^{ac}\wedge\omega_c{}^b \Rightarrow DR^{ab} = 0$

Lie derivative prescriptions: $\quad$ – calculate d $\quad$ – use $i_\xi d = \delta$

Covariant Lie derivative prescription: $\quad$ – calculate $\mathcal{D}$ $\quad$ – use $i_\xi \mathcal{D} = \delta$,

Notice: $i_\xi e^a = 0$, $i_\xi \omega^{ab} = 0$ for nontrivial variations (gauge symm. → next), $i_\xi R^{ab} = \delta\omega$

$\boxed{\delta R^{ab} = D\delta\omega^{ab}} \Rightarrow \boxed{\delta R_{\mu\nu}{}^{ab} = 2 D_{[\mu}\delta\omega_{\nu]}{}^{ab}} = 2(\partial_{[\mu}\delta\omega_{\nu]}{}^{ab} + 2\,\delta\omega_{[\mu}{}^{[a}{}_c\, \omega_{\nu]}{}^{cb]})$

↗ $p=2$, $A = ab$, $i_\xi R^{ab} = \delta\omega^{ab}$ $\qquad$ ↖ Lorentz covariant derivative

$\boxed{\delta\Omega_p^A = i_\xi \mathcal{D}\Omega_p^A + \mathcal{D} i_\xi \Omega_p^A}$

$\Rightarrow \delta e^a = i_\xi T^a + D i_\xi e^a$ ← (target space copy of diff. gauge symm.)

↑ formal expression for variation of e^a

Using identities (to prove:)

(Ex. 1:) $\quad e D_a \xi^a = \partial_\mu(e\xi^\mu) - e\, T_{ba}{}^b \xi^a$

(Ex. 2:) $\quad D_\mu(e e_a{}^\mu) = e\, T_{ba}{}^b$ $\qquad T_{\mu\nu}{}^c = D_{[\mu} e_{\nu]}{}^c$

we can show (Ex. 3:)

(Ex.3) $\quad \boxed{\delta(eR) = -2(\overset{\overline{\overline{e}}\, R_{ac}{}^{bc}}{\underbrace{R_a{}^b - \tfrac{1}{2}\delta_a{}^b R}_{G_a{}^b\ =\ \text{Einstein tensor}}})\, e_b{}^\mu \delta e_\mu{}^a - (T_{ab}{}^c e_c{}^\mu + T_{ca}{}^c e_b{}^\mu)\delta\omega_\mu{}^{ab} + \partial_\mu(e e_a{}^\mu e_b{}^\nu \delta\omega_\nu{}^{ab})}$

$\Rightarrow$ Einstein eq. $G_{ab} = R_{ab} - \frac{1}{2}\eta_{ab}R = 0$, "metricity" $T_{ab}{}^c = 0$

But we can proceed further and calculate

$$\delta \mathcal{L}_D = i_\xi d\mathcal{L}_D + d(i_\xi \mathcal{L}_D)$$

THE VARIATION OF THE LAGRANGIAN FORM — complete derivative: not important when $\partial M^4 = \emptyset$

$$\boxed{\delta \mathcal{L}_4^{EH} = \epsilon_{abcd}\, e^c \wedge T^d \wedge \delta\omega^{ab} - \epsilon_{abcd}\, R^{bc} \wedge e^d \wedge \delta e^a + d\left(\tfrac{1}{2}\epsilon_{abcd}\, e^c \wedge e^d \wedge \delta\omega^{ab}\right)}$$

where we set $i_\xi R^{ab} = \delta\omega^{ab}$, $i_\xi T^a = \delta e^a$

and $i_\xi e^a = 0$ (comment see below)

If $\partial M^4 \neq 0$ we need Gibbons-Hawking term

$\delta\omega^{ab}$: $\quad \epsilon_{abcd}\, e^c \wedge T^d = 0 \implies \boxed{T^d = De^d = 0}$ (Ex 4: TO PROVE)

(Ex 5:) to show that $0 = T^a = De^a = de^a - e^b \wedge \omega_b{}^a$ determines the usual $\omega_\mu^a = \omega_\mu^a(e_\rho^c)$

δe^a: $\quad \boxed{\mathcal{G}_{3a} = \epsilon_{abcd}\, R^{bc} \wedge e^d = 0}$

(Ex 6: show that) this is an equivalent form of the Einstein eq. $\boxed{G_{ab} = 0}$

$e^{\wedge 3}_b := \frac{1}{3!}\epsilon_{bc_1c_2c_3}\, e^{c_1} \wedge e^{c_2} \wedge e^{c_3}$ $\quad \boxed{\mathcal{G}_{3a} = 3 e^{\wedge 3}_{[a} R_{bc]}{}^{bc} = +2 e^{\wedge 3}_b \left(R_a{}^b - \tfrac{1}{2}\delta_a{}^b R\right)}$

(2.3) (Tangent space copy) of the diffeomorphism gauge symm.

$i_\xi e^a = \xi^a$ $\;(= \delta x^\mu e_\mu^a)$ $\quad$: $\quad \delta_\xi \mathcal{L}_4^{EH} = \epsilon_{abcd}\, R^{bc} \wedge T^d\, i_\xi e^a$

$\delta\mathcal{L}_4^{EH} / i_\xi e^a = 0$ $\qquad \delta_\xi \mathcal{L}_4^{EH} = 0 \implies \epsilon_{abcd}\, R^{bc} \wedge T^d = 0$

Satisfied identically as a result of $T^a = 0$

THIS IS A NOETHER IDENTITY INDICATING A GAUGE SYMM.

It is DIFF. GAUGE SYMM. $\qquad i_\xi e^a = \delta x^\mu e_\mu^a = \xi^a \iff \delta x^\mu = \xi^\mu(x)$

OR GENERAL COORD. INV. $\qquad \delta e^a = D i_\xi e^a + i_\xi e^c e^b T_{cb}{}^a$

CONCLUSION: $\quad i_\xi e^a$ — GAUGE SYMM. PARAMETER: DIFF. SYMM.

$i_\xi \omega^{ab}$ — " — " — " — ": LOCAL LORENTZ SYMM.

COMPLETELY TRIVIAL IN OUR APPROACH of "covariant" Lie derivatives.

2.2. (2.4) Rarita - Schwinger action in curved spacetime

$\sigma^a_{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\tilde\sigma^{a\dot\beta\beta}$ – Relativistic Pauli matrices.

$\sigma^a\tilde\sigma^b = \eta^{ab} + \frac{i}{2}\epsilon^{abcd}\sigma_c\tilde\sigma_d$

$\tilde\sigma^a\sigma^b = \eta^{ab} - \frac{i}{2}\epsilon^{abcd}\tilde\sigma_c\sigma_d$

$\epsilon^{0123} = 1 = -\epsilon_{0123}$

$$R^{ab} = \frac{1}{2}R^{\alpha\beta}(\sigma^a\tilde\sigma^b)_{\alpha\beta} - \frac{1}{2}R^{\dot\alpha\dot\beta}(\tilde\sigma^a\sigma^b)_{\dot\alpha\dot\beta}$$

$$R_\alpha{}^\beta = \frac{1}{4}R^{ab}(\sigma_a\tilde\sigma_b)_\alpha{}^\beta; \quad R^{\dot\beta}_{\dot\alpha} = -\frac{1}{4}R^{ab}(\tilde\sigma_a\sigma_b)^{\dot\beta}{}_{\dot\alpha}$$

$\Leftarrow d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma\wedge\omega_\gamma{}^\beta$; $\Leftarrow d\bar\omega_{\dot\alpha}{}^{\dot\beta} - \bar\omega_{\dot\alpha}{}^{\dot\gamma}\wedge\bar\omega_{\dot\gamma}{}^{\dot\beta}$

$$\boxed{\sigma^{(1)}_{\alpha\dot\alpha} = e^a\sigma_{a\alpha\dot\alpha}}$$

$$\boxed{D\psi^\alpha = d\psi^\alpha - \psi^\beta\omega_\beta{}^\alpha}; \quad \omega_\beta{}^\alpha = \frac{1}{4}\omega^{ab}(\sigma_a\tilde\sigma_b)_\beta{}^\alpha = \pm(\bar\omega_{\dot\beta}{}^{\dot\alpha})^*$$

(?) $\psi^\alpha\wedge\psi^\beta = +\psi^\beta\wedge\psi^\alpha$

$\sigma_{ab}\sigma_c + \sigma_c\tilde\sigma_{ab} = -2i\epsilon_{abcd}\sigma^d$

Ex: 6

$$\mathcal{L}^{RS}_4 = -4\, D\psi^\alpha\wedge\sigma^{(1)}_{\alpha\dot\alpha}\wedge\bar\psi^{\dot\alpha} + 4\,\psi^\alpha\wedge\sigma^{(1)}_{\alpha\dot\alpha}\wedge D\bar\psi^{\dot\alpha}$$
$$= \mp 8\, d^4x\, \epsilon^{\mu\nu\rho\sigma}e_\mu{}^a(\psi_\nu\sigma_a\overleftrightarrow{D}_\rho\bar\psi_\sigma)$$

$$d\mathcal{L}_{RS} = -4\underbrace{(DD\psi^\alpha\wedge\sigma^{(1)}_{\alpha\dot\alpha}\wedge\bar\psi^{\dot\alpha} - \psi^\alpha\wedge\sigma^{(1)}_{\alpha\dot\alpha}\wedge DD\bar\psi^{\dot\alpha})} - 8\, D\psi^\alpha\wedge\sigma^{(1)}_{\alpha\dot\alpha}\wedge D\bar\psi^{\dot\alpha} \mp 4(D\psi^\alpha\wedge\sigma_{a\alpha\dot\alpha}\bar\psi^{\dot\alpha} + \psi\wedge\sigma_a D\bar\psi)\wedge T^a$$

$\Leftarrow -4\,\psi^\alpha\wedge(R_\alpha{}^\beta\sigma^{(1)}_{\beta\dot\alpha} + \sigma^{(1)}_{\alpha\dot\beta}\bar R^{\dot\beta}{}_{\dot\alpha})\wedge\bar\psi^{\dot\alpha}$

$= -\frac{1}{4}R^{ab}\wedge e^c\underbrace{(\sigma_{ab}\sigma_c + \sigma_c\tilde\sigma_{ab})}_{-2i\epsilon_{abcd}\sigma^d}$

$(T^a + 2i\psi\sigma^a\bar\psi)$ off $= 0$

$$\boxed{d\mathcal{L}_{RS} = -8\, D\psi^\alpha\wedge\sigma^{(1)}_{\alpha\dot\alpha}\wedge D\bar\psi^{\dot\alpha} + 2i\epsilon_{abcd}\, R^{ab}\wedge e^c\wedge\psi\sigma^d\wedge\bar\psi + 4(D\psi\wedge\sigma_a\bar\psi + \psi\sigma_a\wedge D\bar\psi)\wedge T^a}$$

$$S^{RS} = \frac{1}{\kappa^2}\int\mathcal{L}_{RS} = -\frac{4}{\kappa^2}\int\left(D\psi^\alpha\wedge\sigma^{(1)}_{\alpha\dot\alpha}\wedge\bar\psi^{\dot\alpha} - \psi^\alpha\wedge\sigma^{(1)}_{\alpha\dot\alpha}\wedge D\bar\psi^{\dot\alpha}\right)$$

$= \mp D\bar\psi\wedge\sigma^{(1)}\wedge\psi$

$$= -\frac{8}{\kappa^2}\int d^4x\; \epsilon^{\mu\nu\rho\sigma}\,\psi_\mu\sigma_\nu\overleftrightarrow{D}_\rho\bar\psi_\sigma$$

$\Leftarrow e_\nu{}^a\sigma_{a\alpha\dot\alpha}$

(25) Variation of the free RS action in flat spacetime & pre-susy

$$d\mathcal{L}_4^{RS(0)} = -8\, d\psi^{\alpha} \wedge \sigma^{(1)}_{\alpha\dot\alpha} \wedge d\bar\psi^{\dot\alpha}$$

$$\delta\mathcal{L}_4^{RS(0)} = i_\delta\, d\mathcal{L}_4^{RS(0)} + d\, i_\delta \mathcal{L}_4^{RS(0)} = -8\,(d\psi \wedge \overset{dx^a \sigma_{a\alpha\dot\alpha}}{\overline{\sigma^{(1)}}})_{\dot\alpha} \wedge \delta\bar\psi^{\dot\alpha} + 8\,\delta\psi^{\alpha} \wedge (\sigma^{(1)} \wedge d\bar\psi)_{\alpha} +$$
$$+ d\left(-4\,\delta\psi \wedge \sigma^{(1)} \wedge \bar\psi + 4\,\psi \wedge \sigma^{(1)} \wedge \delta\bar\psi\right)$$

Eqs: $\boxed{(d\psi \wedge \sigma^{(1)})_{\dot\alpha} = 0}$, $\boxed{(\sigma^{(1)} \wedge d\bar\psi)_{\alpha} = 0}$

$$\sim \underbrace{dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho}}_{\sim (dx^{\wedge 3})_{\mu}\, \epsilon^{\mu\nu\rho\sigma}} (\partial_\mu \psi_\nu \cdot \sigma_\rho)_{\dot\alpha}$$
$$\sim (dx^{\wedge 3})_{\mu}\, \epsilon^{\mu\nu\rho\sigma} (\partial_\nu \psi_\rho \sigma_\sigma)_{\dot\alpha}$$

$\Downarrow$

RS eqs. $\boxed{\epsilon^{\mu\nu\rho\sigma} (\partial_\nu \psi_\rho \sigma_\sigma)_{\dot\alpha} = 0}$

$$\sigma^{(1)}_{\alpha\dot\alpha} = dx^a \sigma_{a\alpha\dot\alpha} \implies d\sigma^{(1)}_{\alpha\dot\alpha} = 0$$

$$d\left((d\psi \wedge \sigma^{(1)})_{\dot\alpha}\right) \equiv 0$$

Noether identity for the FERMIONIC gauge symm. ("PRE-SUSY")

$$\delta\psi^{\alpha} = d\epsilon^{\alpha} \quad \text{or, eq.} \quad \delta\psi_\mu^{\alpha} = \partial_\mu \epsilon^{\alpha}$$

IN CURVED SPACETIME:

$$d \to D = d + \omega$$

$$\sigma^{(1)}_{\alpha\dot\alpha} = e^a \sigma_{a\alpha\dot\alpha} \equiv dx^{\mu} e_\mu{}^a \sigma_{a\alpha\dot\alpha}$$