4 SUGRA in SSP II. Higher N and higher D (on-shell)

4.1. WARM-UP: SYM IN FLAT SSP.

SPACETIME

$D=3,4,\dots,10$

- GAUGE FIELD $\Longleftrightarrow$ 1-FORM

$$\mathbb{F}_2 = \tfrac{1}{2} dx^\mu \wedge dx^\nu \, \mathbb{F}_{\nu\mu} \equiv \tfrac{1}{2} e^b \wedge e^a \, \mathbb{F}_{ab} = d\mathbb{A} - \mathbb{A}\wedge\mathbb{A}$$

$$\delta \mathbb{F}_2 = D\delta\mathbb{A}, \qquad \mathbb{A} = dx^\mu \mathbb{A}_\mu = e^a \mathbb{A}_a$$

IN FLAT SPACE $e^a(x) = dx^\mu \delta_\mu{}^a$.

- FERMIONIC FIELD IN Adj of gauge group, χ^α

$$D\chi^\alpha = d\chi^\alpha + [\mathbb{A}, \chi^\alpha]$$

$$DD\chi^\alpha = [\mathbb{F}_2, \chi^\alpha] = dD\chi^\alpha - \mathbb{A}\wedge D\chi^\alpha - D\chi^\alpha \wedge \mathbb{A}$$

be careful with diff. form! (or work with components)

$$d[\mathbb{A},\chi^\alpha] = d(\mathbb{A}\chi^\alpha - \chi^\alpha\mathbb{A}) = \mathbb{A}\wedge d\chi^\alpha + d\chi^\alpha\wedge\mathbb{A} + [d\mathbb{A}, \chi^\alpha]$$

Spacetime action in 10D

$$S = \int d^{10}x \left(\frac{1}{4g^2} \operatorname{tr} \mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu} + i \operatorname{tr} \chi^\alpha \sigma^\mu_{\alpha\beta} \underbrace{D_\mu \chi^\beta}_{\partial_\mu \chi^\beta + [\mathbb{A}_\mu, \chi^\beta]} \right)$$

$\mu = 0,1,\dots,9;\quad \alpha = 1,\dots,16$

Lower D actions may be obtained by dimensional reduction.

What we need to know about 10D KG coeffs (generalized Pauli matrices)

$$\boxed{\begin{gathered} \sigma^a_{\alpha\beta} = \sigma^a_{\beta\alpha}, \qquad \tilde\sigma^{b\,\alpha\beta} = \tilde\sigma^{b\,\beta\alpha} \qquad 16\times16 \\ (\sigma^a\tilde\sigma^b)_\alpha{}^\beta = \eta^{ab}\delta_\alpha{}^\beta + (\sigma^{ab})_\alpha{}^\beta = (\tilde\sigma^b\sigma^a)^\beta{}_\alpha \\ \sigma_{a(\alpha\beta}\sigma^a_{\gamma)\delta} \equiv 0; \qquad \alpha,\beta,\gamma = 1,\dots,16 \end{gathered}}$$

No $C^{\alpha\beta}$, $C_{\alpha\beta}$! (in MW repr.)

$$\delta S = \int d^{10}x \, \operatorname{tr}\Big(\big(-\tfrac{1}{g^2} D_\mu \mathbb{F}^{\mu\nu} - i\sigma^\nu_{\alpha\beta}\{\chi^\alpha,\chi^\beta\}\big)\delta\mathbb{A}_\nu + 2i\,\delta\chi^\alpha \sigma^\mu_{\alpha\beta} D_\mu \chi^\beta + \partial_\mu(\dots)\Big)$$

$\partial_\mu(\dots) = \frac{1}{g^2}\operatorname{tr}(\mathbb{F}^{\mu\nu}\delta\mathbb{A}_\nu) + i\operatorname{tr}\chi\sigma^\mu\delta\chi$

SUSY (ex: to obtain $\delta_\epsilon S = 0$)

$$\boxed{\begin{aligned} \delta\mathbb{A}_\mu &= \epsilon\sigma_\mu\chi \\ \delta\chi^\alpha &= -\tfrac{i}{4g^2}(\epsilon\sigma^{\mu\nu})^\alpha \mathbb{F}_{\mu\nu} \end{aligned}}$$

Notice $\operatorname{tr} \sigma^\nu_{\alpha\beta}\{\chi^\alpha,\chi^\beta\}\delta_\epsilon\mathbb{A}_\nu = 0$ due to $\sigma_{a(\alpha\beta}\sigma^a_{\gamma)\delta} = 0$

4. 1. - (1/3)

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SYM Lagrangian 10D.

$$S = \int d^{10}x \, \frac{1}{4g^2} \mathrm{tr}\, \mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu} + i \mathrm{tr}(\chi^\alpha \sigma^\mu_{\alpha\beta} D_\mu \chi^\beta)$$

$$D_\mu\chi^\beta = \partial_\mu \chi^\beta + [A_\mu, \chi^\beta]$$

$$\mathrm{tr}\, \chi^\alpha \sigma^\mu_{\alpha\beta} [\delta A_\mu, \chi^\beta] = -\mathrm{tr}(\chi^\alpha \sigma^\mu_{\alpha\beta}\chi^\beta + \chi^\beta\chi^\alpha\sigma^\mu_{\alpha\beta}) - \mathrm{tr} \ldots \{\chi^\beta\} \sigma^\mu_{\alpha\beta}\delta A_\mu$$

($\delta A_\mu \chi^\beta - \chi^\beta \delta A_\mu$)

$$\mathbb{F}_2 = \tfrac{1}{2} dx^\nu \wedge dx^\mu \mathbb{F}_{\mu\nu} = dA - A\wedge A$$

$$\delta \mathbb{F}_2 = D\,\delta A$$

$$D\chi^\alpha = d\chi^\alpha + [A, \chi^\alpha]$$

$$DD\chi^\alpha = [\mathbb{F}, \chi^\alpha] = d\,D\chi^\alpha - [A, D\chi^\alpha] = -[A, d\chi^\alpha] - [A, [A,\chi^\alpha]]$$

$$= d\,d\chi^\alpha + A\wedge D\chi^\alpha \ldots = 0 + [A, d\chi^\alpha] + [dA, \chi^\alpha]$$

$$= -A\wedge A\,\chi^\alpha + A\chi A + \ldots - A\chi A + \chi A\wedge A$$

$$\delta S = \int d^{10}x \left(\underline{\frac{1}{g^2}\, \mathrm{tr}\, \mathbb{F}^{\mu\nu} D_\mu \delta A_\nu} + 2i\, \mathrm{tr}\, \delta\chi^\alpha \sigma^\mu_{\alpha\beta} D_\mu\chi^\beta + i\partial_\mu \mathrm{tr}(\chi^\alpha\sigma^\mu_{\alpha\beta}\delta\chi^\beta) - i\sigma^\mu_{\alpha\beta}\mathrm{tr}\{\chi^\alpha, \chi^\beta\}\delta A_\mu \right)$$

$$= \partial_\mu \mathrm{tr}\left(\frac{1}{g^2}\mathbb{F}^{\mu\nu}\delta A_\nu\right) - \frac{1}{g^2}\mathrm{tr}\, D_\mu \mathbb{F}^{\mu\nu}\cdot \delta A_\nu$$

$$\mathbb{F} = E^b \wedge E^\alpha \sigma_{b\alpha\beta} W^\beta + \tfrac{1}{2} E^b \wedge E^a \mathbb{F}_{ab}$$

$$\boxed{\begin{aligned}\delta A_\mu &= \epsilon\sigma_\mu \chi \\ \delta\chi^\beta &= -\frac{i}{4g^2}(\epsilon\sigma^{\mu\nu})^\alpha \cdot \mathbb{F}_{\mu\nu}\end{aligned}}$$

$$\delta \frac{1}{4g^2}\mathrm{tr}\, \mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu} = -\frac{1}{g^2}\mathrm{tr}\, \mathbb{F}^{\mu\nu}\epsilon\sigma_\mu D_\nu \chi$$

cancel

$$\delta\, i\,\mathrm{tr}(\chi^\alpha\sigma^\mu_{\alpha\beta} D_\mu \chi^\beta) = 2\cdot\frac{1}{4g^2}\mathrm{tr}\,\epsilon\sigma^{ab}\sigma^c D_c\chi \cdot \mathbb{F}_{ab} = \frac{2}{4g^2}\mathrm{tr}\,\epsilon\sigma_a D_b\chi\, \mathbb{F}^{ab} + \frac{1}{2g^2}\mathrm{tr}\,\epsilon\sigma^{abc} D_c\chi\cdot\mathbb{F}_{ab}$$

$$= \partial(\ldots) - \frac{1}{2g^2}\mathrm{tr}\,\epsilon\sigma^{abc}\chi \cdot D_{[c}\mathbb{F}_{ab]}$$

$$= 0$$

Bianch.

$$\mathrm{tr}\{\chi^\alpha, \chi^\beta\}\epsilon\sigma_{\gamma\delta}\chi^\gamma \sigma^\mu_{\alpha\beta} = \epsilon\,\mathrm{tr}\,\underline{\{\chi^\alpha,\chi^\beta\}\chi^\gamma} \quad \sigma_{\mu\alpha\beta}\sigma^\mu_{\gamma\delta}$$

$$= \mathrm{tr}(\chi^\alpha\chi^\beta\chi^\gamma + \chi^\beta\chi^\alpha\chi^\gamma)$$

$$= \mathrm{tr}\{\chi^\beta, \chi^\gamma\}\chi^\alpha$$

$$= \mathrm{tr}$$

Bosonic eq.

$$0 = \sigma^a_{\beta\alpha} D_a W^\alpha$$

$$0 = \sigma^a_{\beta\alpha} \mathcal{D}_\gamma D_a W^\alpha = \sigma^a_{\beta\alpha} D_a \mathcal{D}_\gamma W^\alpha + \sigma^a_{\beta\alpha}\{\mathbb{F}_{\gamma a}, W^\alpha\} =$$

$$\mathcal{D}_\gamma W^\alpha = \tfrac{i}{2}\mathbb{F}_{bc}\sigma^{bc}{}_\gamma{}^\alpha, \qquad \mathbb{F}_{\gamma a} = \sigma_{a\gamma\delta} W^\delta$$

$$\delta_\epsilon E^a$$

$$= \frac{i}{2}(\sigma^{bc}\sigma^a)_{\gamma\beta} D_a \mathbb{F}_{bc} + \sigma^a_{\beta\alpha}\sigma_{a\gamma\delta}\{W^\alpha, W^\delta\}$$

$$= \frac{i}{2}\sigma^{abc} D_{[a}\mathbb{F}_{bc]} - i\sigma^b_{\beta\gamma} D^a \mathbb{F}_{ab}, \qquad = -2\sigma^a_{\beta\gamma}\sigma_{a\alpha\delta}\{W^\alpha, W^\delta\}$$

$$= -2i(-i)$$

$$D^a\mathbb{F}_{ab} = 2i\,\sigma_{b\alpha\beta}\{W^\alpha, W^\beta\} = 0$$

SSP

$$A = dZ^M A_M(Z) = E^a A_a + E^\alpha A_\alpha$$

10D $\quad E^a = dX^a - i d\theta\sigma^a\theta \,;\quad E^\alpha = d\theta^\alpha \quad \alpha = 1, \ldots, 16$

$$T^a = dE^a = -i d\theta\sigma^a \wedge d\theta = -i E^\alpha \wedge E^\beta \sigma^a_{\alpha\beta}$$

$$\mathbb{F}_{\alpha\beta} = 0$$

$$\mathbb{F} = E^b \wedge E^\alpha F_{\alpha b} + \tfrac{1}{2}E^b \wedge E^a F_{ab} = -(\mathbb{F})^\dagger$$

$$0 = D\mathbb{F} = -i E^\alpha \wedge E^\beta \wedge E^\gamma \sigma^b_{\alpha\beta} F_{\gamma b} + E^b \wedge E^\alpha \wedge E^\beta D_\alpha F_{\beta b} - i E^b \wedge E^\alpha \wedge E^\beta \sigma^c_{\alpha\beta} F_{cb} + \tfrac{1}{2}E^b \wedge E^a \wedge E^\gamma (D_\gamma \mathbb{F}_{ab} - 2 D_{[a} F_{\gamma b]}) + \tfrac{1}{2}E^c \wedge E^b \wedge E^a D_{[a} F_{bc]}$$

$$0 = \underline{D_\alpha F_{\beta b}} - i\sigma^c_{\alpha\beta}\mathbb{F}_{cb}$$

$$= \sigma_{b(\alpha|\gamma} D_{|\beta)} W^\gamma$$

$$D_\alpha \mathbb{F}_{ab} = -2\sigma_{[a|\alpha\beta} D_{|b]} W^\beta$$

$$\boxed{D_\alpha W^\beta = \frac{i}{2}\sigma^{ab}{}_\alpha{}^\beta \mathbb{F}_{ab}}$$

$$\mathcal{D}_\beta \mathcal{D}_\gamma W^\alpha = i\sigma^a_{\beta\gamma} D_a W^\alpha = \frac{i}{2}\sigma^{ab}{}_{(\beta}{}^\alpha D_{\gamma)} \mathbb{F}_{ab} =$$

$$\delta_\gamma$$

$$= -i\,\sigma^{ab}{}_{(\beta}{}^\alpha \sigma_{a|\gamma)\delta} D_b W^\delta$$

$$i\sigma^a_{\beta\alpha} D_a W^\alpha = -\frac{i}{2}(\sigma^{ab}\sigma_a)_{\beta\delta} D_b W^\delta = +\frac{9i}{2}\sigma^b_{\beta\delta} D_b W^\delta$$

$$M = D\mathbb{F} = 0 \qquad M_{\alpha\beta\gamma} = 0,\ M_{\alpha\beta\gamma} = 0$$

$$I = DM = \tfrac{1}{2}E^b \wedge E^a \wedge E^\alpha \wedge E^\beta (i\sigma^c_{\alpha\beta} M_{abc} + \tfrac{1}{2} D_\alpha M_{\beta ab}) + \tfrac{1}{2}E^b \wedge E^\alpha \wedge E^\beta \wedge E^\gamma (\sigma^a_{\alpha\beta} M_{\gamma ab})$$

$$+ \wedge E^a \wedge E^b \wedge E^\beta \wedge E^\delta$$

$$\sigma^a_{(\alpha\beta} M_{\gamma)ab} = 0 \Rightarrow M_\gamma$$

$$\sigma^{\beta\gamma}_c$$

$$\sigma^a_{\alpha\beta}\sigma_{a\gamma\delta} = 0 - \sigma^a_{(\alpha\beta}\delta_\gamma{}^\delta$$

$$0 = 3\sigma^a_{(\alpha\beta} M_{\gamma)ab} = \sigma^a_{\alpha\beta} M_{\gamma ab} + 2\sigma^a_{\gamma(\alpha} M_{\beta)ab}$$

$$\tfrac{1}{2}\tilde\sigma_a{}^{\alpha\beta}$$

$$M_{\gamma ab} = \ldots(\sigma\sigma_a)_\gamma{}^\beta M_{\beta cb} = -2M_{\gamma ab} + \sigma_{a\gamma\delta}\tilde\sigma^{cde}M_{ecb}$$

$$3M_{\gamma ab} = \sigma_{a\gamma\delta}\tilde\sigma^{cd\delta}M_{ecb} \Rightarrow 3\tilde\sigma^{a\beta\gamma}M_{\gamma ab} = 10\tilde\sigma^{ade}M_{eab}$$

10D SYM IN SSP $\Sigma_0^{(10|16)}$ $\left(Z^M = (x^\mu, \theta^\alpha) \; \begin{smallmatrix}\mu=0,1,\dots,9\\ \alpha=1,\dots,16\end{smallmatrix}\right)$

$$\mathbb{A} = E^a \mathbb{A}_a + E^\alpha \mathbb{A}_\alpha = dZ^M \mathbb{A}_M(Z)$$

$$E^a = dx^a - i d\theta^\alpha \sigma^a_{\alpha\beta}\theta^\beta , \qquad E^\alpha = d\theta^\alpha$$

SSP CONSTRAINTS / STRUCTURE EQS OF $\Sigma_0^{(10|16)}$

$$\boxed{\begin{aligned} T^a &= dE^a = -iE^\alpha\wedge E^\beta \sigma^a_{\alpha\beta} \\ T^\alpha &= dE^\alpha = 0 \end{aligned}}$$

and $R^{ab}=0$ so we gauge fix $\omega^{ab}=0$

WHAT CONSTRAINTS WE HAVE TO IMPOSE ON $\mathbb{F}=d\mathbb{A}$?

SUGGESTION: we expect to have $\delta_\epsilon \mathbb{A} = i_\epsilon \mathbb{F} = E^a \epsilon \sigma_a \chi = i_\epsilon \left(E^a\wedge E^\alpha (\sigma_a \chi)_\alpha\right)$

(THIS IS JUST SUGGESTION: WE ARE WITH RIGID SUSY, WHERE $i_\epsilon E^a \neq 0$ but $\delta_\epsilon E^a = 0$)

SO WE IMPOSE $\boxed{\mathbb{F}_{\alpha\beta} = 0}$ $\qquad \boxed{\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 2i\sigma^a_{\alpha\beta}\mathcal{D}_a}$

$\mathbb{F}_{\alpha\beta}\tilde{\sigma}^{\alpha\beta}_{a_1\dots a_5}=0$ essential

$\mathbb{F}_{\alpha\beta}\tilde{\sigma}^{\alpha\beta}_{a}=0$ CONVENTIONAL & DETERMINES A_a IN TERMS OF A_α

ACTUALLY, AS WE WILL SEE IN A MOMENT, ALSO $\mathbb{F}_{\alpha a} = \sigma_{a\alpha\beta} W^\beta$ WORKS!

$$\mathbb{F} = d\mathbb{A} - \mathbb{A}\wedge\mathbb{A} = E^b\wedge E^\alpha \mathbb{F}_{\alpha b} + \tfrac{1}{2} E^b\wedge E^a \mathbb{F}_{ab} = -(\mathbb{F})^\dagger$$

$\Downarrow$ $\qquad dE^a = -iE^\alpha\wedge E^\beta \sigma^a_{\alpha\beta}$

$$0 = \mathcal{D}\mathbb{F} = -iE^\alpha\wedge E^\beta\wedge E^\gamma \sigma^b_{(\alpha\beta}\mathbb{F}_{\gamma)b} + E^b\wedge E^\alpha\wedge E^\beta\left(\mathcal{D}_{(\alpha}\mathbb{F}_{\beta)b} - i\sigma^a_{\alpha\beta}\mathbb{F}_{ab}\right) +$$
$$+\tfrac{1}{2}E^b\wedge E^a\wedge E^\alpha\left(\mathcal{D}_\alpha \mathbb{F}_{ab} - 2\mathcal{D}_{[a|}\mathbb{F}_{\alpha|b]}\right) + \tfrac{1}{2}E^c\wedge E^b\wedge E^a \mathcal{D}_{[a}\mathbb{F}_{bc]}$$

dim 3/2: $\sim E^\alpha\wedge E^\beta\wedge E^\gamma$ $(M_{\alpha\beta\gamma}=0)$: $\quad \sigma^b_{(\alpha\beta}\mathbb{F}_{\gamma)b}=0$ $\quad$ THE GENERAL SOLUTION $\quad \boxed{\mathbb{F}_{\alpha b} = \sigma_{b\alpha\beta} W^\beta}$

REMEMBER: $\sigma^b_{(\alpha\beta|}\sigma_{b|\gamma)\delta}=0$ in D=10! (3, 4, 6)

dim 2: $\sim E^b\wedge E^\alpha\wedge E^\beta$ $\qquad \underbrace{\mathcal{D}_{(\alpha}\mathbb{F}_{\beta)b}}_{=\sigma_{b(\alpha|\gamma}\mathcal{D}_{|\beta)}W^\gamma = \mathcal{D}_{(\alpha|}W^\gamma\cdot\sigma_{b\gamma|\beta)}} - i\sigma^a_{\alpha\beta}\mathbb{F}_{ab} = 0$

To solve this eqs we should appreciate that $\sigma^a_{\alpha\beta}, \sigma^{a_1a_2a_3a_4a_5}_{\alpha\beta}$ - symm, $\sigma^{abc}_{\alpha\beta}$ - asymm.

decompose $\mathcal{D}_\beta W^\gamma = a_0\delta_\beta{}^\gamma + a_{ab}\sigma^{ab}{}_\beta{}^\gamma + a_{a_1\dots a_4}\sigma^{a_1\dots a_4}{}_\beta{}^\gamma$

and find that eq. implies $a_0=0$, $a_{a_1\dots a_4}=0$ $\qquad a_{cd}(\sigma^{cd}\sigma_b)_{\alpha\beta} = +2a_{ab}\sigma^a_{\alpha\beta}$

$$\boxed{\mathcal{D}_\alpha W^\beta = \frac{i}{2}\sigma^{ab}{}_\alpha{}^\beta \mathbb{F}_{ab}}$$

ACTUALLY THESE ARE ALL THE INDEPENDENT RELATIONS FOR MAIN SUPERFIELDS

AS CAN BE PROVED BY STUDYING IDENTITIES FOR IDENTITIES $\boxed{\mathcal{D}M_3=0}$

where $M_3 = \mathcal{D}\mathbb{F}$ ARE L.H.S.-s OF BI's

Ex: TO PROVE THE DEPENDENCE OF HIGHER DIM. BIs $M_{abc} \doteq 0$ and $M_{ab\alpha} \equiv 0$

<u>PROOF:</u> WRITE "IDENTITY FOR IDENTITY" $0 = DM_3$ setting $M_{\alpha\beta\gamma} = 0$, $M_{a\beta\gamma} = 0$:

$$0 = \frac{1}{2} E^b \wedge E^a \wedge E^\alpha \wedge E^\beta \left(-i\sigma^c_{\alpha\beta} M_{abc} + \frac{i}{2} D_\alpha M_{\beta ab}\right) + \frac{i}{2} E^b \wedge E^\alpha \wedge E^\beta \wedge E^\gamma \, \sigma^a_{\alpha\beta} M_{ab\gamma} + \sim E^{\wedge 3}$$

- If we set $M_{\alpha ab} = 0$, this gives $\sigma^c_{\alpha\beta} M_{abc} = 0 \Rightarrow M_{abc} = 0$
  Thus dim 3 id. is dependent provided dim 5/2 id. is.
- For $M_{ab\alpha}$ we have $\sigma^a_{(\alpha\beta} M_{\gamma)ab} = 0$
  contracting with $\frac{3}{2}\tilde{\sigma}_{a'}^{\beta\gamma}$ we find $M_{\alpha ab} = -(\sigma^c \tilde{\sigma}_a)_\alpha{}^\beta M_{\beta cb} =$
  $= -2M_{\alpha ab} + \sigma_{a\alpha\beta} \tilde{\sigma}^{c\beta\gamma} M_{\gamma cb}$
  $3M_{\alpha ab} = \sigma_{a\alpha\beta} \tilde{\sigma}^{c\beta\gamma} M_{\gamma cb}$
  contracting with $\tilde{\sigma}^{a\delta\alpha}$
  $\tilde{\sigma}^{c\beta\gamma} M_{\gamma cb} = 0 \Rightarrow M_{\alpha ab} = 0$

BUT IN OUR SIMPLE CASE IT IS NOT DIFFICULT (AND CONVENIENT) TO STUDY ALL $M_{ab\alpha}$, M_{abc}

dim 5/2: $\boxed{D_\alpha F_{ab} = -2\sigma_{[a|\alpha\beta} D_{|b]} W^\beta}$ dim 3: $\boxed{D_{[a} F_{bc]} = 0}$.

THESE CONSTRAINTS DESCRIBE 10D SYM MODEL ON THE MASS SHELL (ON-SHELL)

- THE FERMIONIC EQS CAN BE EXTRACTED FROM

$$\boxed{D_\beta W^\alpha = \frac{i}{2} \sigma^{ab}{}_\beta{}^\alpha F_{ab}}$$

Take D_γ derivative, symmetrize $(\beta\gamma)$
use dim 5/2 (BI)

$$i\sigma^a_{\beta\gamma} D_a W^\alpha = \frac{i}{2} \sigma^{bc}{}_{(\beta}{}^\alpha \underbrace{D_{\gamma)} F_{bc}}_{-2\sigma_{[b|(\gamma)\delta} D_{c]} W^\delta}$$

THE RESULTING EQ IS SOLVED BY $\boxed{i\sigma^a_{\beta\gamma} D_a W^\gamma = 0}$
(SIMPLEST WAY TO SEE THIS: TO CONTRACT WITH δ_α^γ)

- THE BOSONIC EQS CAN BE OBTAINED FROM D_α OF THE FERMIONIC EQS.

$$0 = \sigma^a_{\alpha\beta} D_\gamma D_a W^\alpha = \sigma^a_{\alpha\beta} D_a \underbrace{D_\gamma W^\alpha}_{\frac{i}{2} F_{bc} (\sigma^{bc})_\gamma{}^\alpha} + \sigma^a_{\alpha\beta} \underbrace{\{F_{\gamma a}, W^\alpha\}}_{\sigma_{a\gamma\delta}\{W^\delta, W^\alpha\}}$$

$$\frac{i}{2} \sigma^{abc}{}_{\beta\gamma} D_{[a} F_{bc]} - i\sigma^b_{\beta\gamma} D^a F_{ab} \qquad \sigma^a_{(\alpha\beta}\sigma_{a\gamma\delta)} = 0 \qquad -2\sigma^a_{\beta\gamma}\sigma_{a\alpha\delta}\{W^\alpha, W^\delta\}$$

$$\Rightarrow \boxed{D^a F_{ab} = 2i\sigma_{b\alpha\beta}\{W^\alpha, W^\beta\}}$$

SYM IN FLAT $D=4$ $\mathcal{N}=1$ SSP

BY ANALOGY WITH 10D SYM, THE CONSTRAINTS ARE

$$\mathbb{F}_{\alpha\beta}=0\,,\qquad \mathbb{F}_{\dot\alpha\dot\beta}=0\,,\qquad \mathbb{F}_{\alpha\dot\beta}=0$$

$\Updownarrow$ $\boxed{\{\mathcal{D}_\alpha,\mathcal{D}_\beta\}=0}$ ESSENTIAL $\qquad$ $\Updownarrow$ $\{\bar{\mathcal{D}}_{\dot\alpha},\bar{\mathcal{D}}_{\dot\beta}\}=0$ $\qquad$ $\Updownarrow$ CONVENTIONAL $\{\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot\beta}\}=2i\sigma^a_{\alpha\dot\beta}\mathcal{D}_a$

ALLOW FOR THE EXISTENCE OF CHIRAL SUPERFIELDS WITH VALUES IN A NONTRIVIAL REPRESENTATION OF THE GAUGE GROUP

GRASSMANN ANALITICITY

BIs: for $\mathbb{F} = E^b\wedge E^\alpha \mathbb{F}_{\alpha b}+E^b\wedge\bar{E}^{\dot\alpha}\mathbb{F}_{\dot\alpha b}+\frac12 E^b\wedge E^a\mathbb{F}_{ab}$

$$\mathbb{F}_{(\alpha\beta)\dot\beta}=\sigma^b_{(\alpha|\dot\beta}\mathbb{F}_{|\beta)b}$$

$$0=\mathcal{D}\mathbb{F}=2iE^\alpha\wedge E^\beta\wedge\bar{E}^{\dot\beta}\,\mathbb{F}_{(\alpha\beta)\dot\beta}+2iE^\alpha\wedge\bar{E}^{\dot\alpha}\wedge\bar{E}^{\dot\beta}\,\mathbb{F}_{(\dot\alpha|\alpha|\dot\beta)}+$$

$$+E^b\wedge E^\alpha\wedge E^\beta\,\mathcal{D}_{(\alpha}\mathbb{F}_{\beta)b}+E^b\wedge\bar{E}^{\dot\alpha}\wedge\bar{E}^{\dot\beta}\,\bar{\mathcal{D}}_{(\dot\alpha}\mathbb{F}_{\dot\beta)b}+E^b\wedge E^\alpha\wedge\bar{E}^{\dot\alpha}\left(\mathcal{D}_\alpha\mathbb{F}_{\dot\alpha b}+\bar{\mathcal{D}}_{\dot\alpha}\mathbb{F}_{\alpha b}-2i\sigma^a_{\alpha\dot\alpha}\mathbb{F}_{ab}\right)$$

$$+\frac12 E^c\wedge E^b\wedge E^\alpha\left(\mathcal{D}_\alpha\mathbb{F}_{bc}-2\mathcal{D}_{[b|}\mathbb{F}_{\alpha|c]}\right)+\frac12 E^c\wedge E^b\wedge\bar{E}^{\dot\alpha}\left(\bar{\mathcal{D}}_{\dot\alpha}\mathbb{F}_{bc}-2\mathcal{D}_{[b|}\mathbb{F}_{\dot\alpha|c]}\right)+\frac{1}{3!}E^c\wedge E^b\wedge E^a\,\mathcal{D}_{[a}\mathbb{F}_{bc]}$$

dim 3/2: $\mathbb{F}_{(\alpha\beta)\dot\beta}=0 \Rightarrow \mathbb{F}_{\alpha b}=i\sigma_{b\alpha\dot\alpha}\bar{W}^{\dot\alpha}$, and c.c. $\mathbb{F}_{\dot\alpha b}=-i\sigma_{b\alpha\dot\alpha}W^\alpha$

$$\mathbb{F}=iE^b\wedge E^\alpha(\sigma_b\bar{W})_\alpha-iE^b\wedge\bar{E}^{\dot\alpha}(W\sigma_b)_{\dot\alpha}+\frac12 E^b\wedge E^a\mathbb{F}_{ab}=-\mathbb{F}^\dagger$$

dim 2: $\sim E^b\wedge E^\alpha\wedge E^\beta$ $\quad\boxed{\mathcal{D}_\beta\bar{W}^{\dot\alpha}=0}$, c.c. $\boxed{\bar{\mathcal{D}}_{\dot\beta}W^\alpha=0}$

$\sim E^b\wedge E^\alpha\wedge\bar{E}^{\dot\alpha}$ $\quad 2i\sigma^a_{\alpha\dot\alpha}\mathbb{F}_{ab}=i\sigma_{b\alpha\dot\beta}\bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\beta}-i\mathcal{D}_\alpha W^\beta\sigma_{b\beta\dot\alpha}$

$$\Leftrightarrow\ \boxed{\mathbb{F}_{\alpha\dot\alpha\,\beta\dot\beta}=\epsilon_{\dot\alpha\dot\beta}\mathcal{D}_{(\alpha}W_{\beta)}-\epsilon_{\alpha\beta}\bar{\mathcal{D}}_{(\dot\alpha}\bar{W}_{\dot\beta)}}$$

$$\Leftrightarrow\ \boxed{\mathcal{D}^\alpha W_\alpha=\bar{\mathcal{D}}^{\dot\alpha}\bar{W}_{\dot\alpha}}$$

$$\boxed{\mathbb{F}_{ab}=\frac14\mathcal{D}_\alpha W^\beta(\sigma_{ab})_\beta{}^\alpha+\frac14(\tilde\sigma_{ab})^{\dot\alpha}{}_{\dot\beta}\bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\beta}}$$

Notice: $\boxed{\mathcal{D}^\alpha W_\alpha+\bar{\mathcal{D}}^{\dot\alpha}\bar{W}_{\dot\alpha}=2\mathbb{D}}$ $\quad$ generically! $\neq 0$

ITS LEADING ($\theta=0$) COMPONENT THIS IS THE AUXILIARY FIELD OF SYM

$\Leftrightarrow$

$$\mathcal{D}^\alpha W_\alpha=\mathbb{D}\,,\qquad\bar{\mathcal{D}}^{\dot\alpha}\bar{W}_{\dot\alpha}=\mathbb{D}$$

SUPERFIELD EQS OF MOTION IS $\boxed{\mathbb{D}=0}$

Then $\boxed{\mathcal{D}^\alpha W_\alpha=0 \ \Rightarrow\ \mathcal{D}^{\dot\alpha\alpha}W_\alpha=0}$ (Cov. FERMIONIC EQ.)

Gauge field theory in SSP of min SG

$$\mathbb{F} = dA + A \wedge A = \tfrac{1}{2} dZ^M \wedge dZ^N F_{NM} = \tfrac{1}{2} E^B \wedge E^A F_{AB} = F^I (T_I)_{(\mathrm{ad})}^{(\ell)}$$

$$-dZ^N A_N \wedge dZ^M A_M \;\; {}^{=} (-)^{NM} dZ^N \wedge dZ^M A_N A_M = +\tfrac{1}{2} dZ^M \wedge dZ^N [A_N, A_M]$$

$$\partial_N A_M - (-)^{NM} \partial_M A_N + [A_N, A_M] = \nabla_A A_B - (-)^{AB} \nabla_B A_A + T_{AB}{}^C A_C + [A_A, A_B]$$

$$\boxed{\mathbb{F}^\dagger = -\mathbb{F}}$$

$$\mathcal{D}\mathcal{D} \mathbb{W}^\alpha = -\mathbb{W}^\beta R_\beta{}^\alpha \qquad \mathcal{D}\mathbb{W}^\alpha = d\mathbb{W}^\alpha + [A, \mathbb{W}^\alpha]$$

$$F_{\alpha\beta} = 0 = F_{\dot\alpha\dot\beta}, \quad F_{\alpha\dot\beta} = 0$$

$$F = E^b \wedge E^\alpha F_{\alpha b} + E^b \wedge \bar E^{\dot\alpha} F_{\dot\alpha b} + \tfrac{1}{2} E^c \wedge E^b F_{bc}$$

$$0 = DF = -T^b \wedge E^\alpha F_{\alpha b} - T^b \wedge \bar E^{\dot\alpha} F_{\dot\alpha b} + E^c \wedge T^b F_{bc} + E^b \wedge E^\alpha \wedge E^\beta \mathcal{D}_\alpha F_{\beta| b} + E^b \wedge \bar E^{\dot\alpha} \wedge \bar E^{\dot\beta} \bar{\mathcal{D}}_{\dot\alpha} F_{\dot\beta| b} + E^b \wedge E^\alpha \wedge \bar E^{\dot\alpha} (\mathcal{D}_\alpha F_{\dot\alpha b} + \bar{\mathcal{D}}_{\dot\alpha} F_{\alpha b}) + E^c \wedge \ldots + E^c \wedge E^b \wedge E^\alpha \left(\tfrac{1}{2} \mathcal{D}_\alpha F_{bc} - \mathcal{D}_{[b|} F_{\alpha|c]}\right) +$$

$$\Leftarrow 2i E^\alpha \wedge E^\beta \wedge \bar E^{\dot\beta} F_{(\alpha|\dot\beta\beta)} + 2i E^{\dot\alpha} \wedge \bar E^{\dot\beta} \wedge E^\beta F_{(\dot\alpha|\beta\dot\beta)} = \tfrac{i}{8} E^b \wedge E^\alpha \wedge E^\alpha \epsilon_{abc}{}^{cd} \sigma_d F_{\alpha c} - \tfrac{i}{8} E^b \wedge E^\alpha \wedge \bar E^{\dot\alpha} \epsilon_{ab}{}^{cd} \sigma_d F_{\dot\alpha c} \; \tfrac{i}{8} E^b \wedge E^a \wedge \bar E^{\dot\alpha} \epsilon_{abcd} (\bar{\mathbb{W}} \sigma^c)_{\dot\alpha} \sigma^d$$

(1) (2)

$$= -\tfrac{i}{8} E^b \wedge E^a \wedge E^\alpha \epsilon_{abcd} \sigma^d (\sigma^c \bar{\mathbb{W}})_\alpha \quad (1)$$

$$\Leftarrow -2i E^b \wedge E^\alpha \wedge E^\alpha \sigma^a_{\alpha\dot\alpha} F_{ab} + \tfrac{i}{8} E^c \wedge E^b \wedge E^a \epsilon_{[ab|}{}^{ef} \sigma_f F_{|e c]} \quad (3)\ (4)\ (5\,3)$$

$$= E^b \wedge E^\alpha \wedge \bar E^{\dot\alpha} \left(i\sigma_{b\beta\dot\alpha} \mathcal{D}_\alpha \bar{\mathbb{W}}^{\dot\beta} + i \sigma_{b\alpha\dot\beta} \bar{\mathcal{D}}_{\dot\alpha} \mathbb{W}^{\beta}\right) \quad (5\ {}^1\!/_2) \quad (6\ {}^1\!/_2)$$

$$= + i\sigma_{[b|\alpha\dot\alpha} \mathcal{D}_{|c]} \bar{\mathbb{W}}^{\dot\alpha}$$

$$+ E^c \wedge E^b \wedge \bar E^{\dot\alpha} \left(\tfrac{1}{2} \bar{\mathcal{D}}_{\dot\alpha} F_{bc} - \mathcal{D}_{[b|} F_{\dot\alpha|c]}\right) + E^b \wedge T^\alpha F_{\alpha b} + E^b \wedge \bar T^{\dot\alpha} F_{\dot\alpha b} + \tfrac{1}{3!} E^c \wedge E^b \wedge E^a \, 3 D_{[a} F_{bc]}$$

$$\Leftarrow E^b \wedge T^\alpha\, i\sigma_{b\alpha\dot\alpha} \bar{\mathbb{W}}^{\dot\alpha} + E^b \wedge \bar T^{\dot\alpha} (-i\sigma_{b\alpha\dot\alpha} \mathbb{W}^\alpha)$$

$$\left(+ E^c \wedge E^b \wedge E^a \left(\tfrac{1}{2} T_{[ab|}{}^\alpha F_{\alpha|c]} + \tfrac{1}{2} T_{[ab|}{}^{\dot\alpha} F_{\dot\alpha|c]}\right)\right)$$

$$\Leftarrow E^b \wedge E^a \wedge E^\alpha \left(\left(-\tfrac{i}{8}\right) (\sigma_{[a} \tilde\sigma_c \sigma_{b]})_{\alpha\dot\alpha} \bar{\mathbb{W}}^{\dot\alpha} \sigma^c + \tfrac{i}{8} (\sigma_b \tilde\sigma_a)_\alpha{}^\beta R \mathbb{W}_\beta \right) + E^b \wedge E^a \wedge \bar E^{\dot\alpha} \left(-\tfrac{i}{8} (\sigma_b \tilde\sigma_c \sigma_a)_{\alpha\dot\alpha} \mathbb{W}^\alpha \sigma^c + \tfrac{i}{8} (\tilde\sigma_a \sigma_b)_{\dot\alpha\dot\beta} \bar{\mathbb{W}}^{\dot\beta} \bar R\right)$$

$$= -i\epsilon_{abcd} \sigma^d_{\alpha\dot\alpha} \qquad = \tfrac{i}{8} \sigma_{ab\alpha}{}^\beta \mathbb{W}_\beta R$$

$$+ \tfrac{i}{8} \epsilon_{abcd} \sigma^d (\sigma^c \bar{\mathbb{W}})_\alpha \quad (1) \quad (6\,3)$$

dim 3/2
$\sim E^\alpha \wedge \bar E^{\dot\beta} \wedge E^\beta$
c.c.

$F_{(\alpha\,\beta\dot\beta)} = 0 \Rightarrow$
$F_{(\dot\alpha|\beta\dot\beta)} = 0 \Rightarrow$

$$\boxed{\mathbb{F}_{\alpha b} = i \sigma_{b\alpha\dot\alpha} \bar{\mathbb{W}}^{\dot\alpha}} \qquad \bar{\mathbb{W}}^{\dot\alpha} = (\mathbb{W}^\alpha)^\dagger$$

$$\boxed{\mathbb{F}_{\dot\alpha b} = -i \sigma_{b\alpha\dot\alpha} \mathbb{W}^\alpha} = (\mathbb{F}_{\alpha b})^\dagger$$

$\sim E^b \wedge E^\alpha \wedge E^\beta$

$$0 = i\sigma_{b(\alpha|\dot\alpha} \mathcal{D}_{|\beta)} \bar{\mathbb{W}}^{\dot\alpha} \Rightarrow \boxed{\mathcal{D}_\beta \bar{\mathbb{W}}^{\dot\alpha} = 0} \quad \text{c.c.} \quad \sim E^b \wedge \bar E^{\dot\alpha} \wedge \bar E^{\dot\beta} \Rightarrow \boxed{\bar{\mathcal{D}}_{\dot\beta} \mathbb{W}^\alpha = 0} \Rightarrow \boxed{\mathbb{W}_\alpha = -\tfrac{1}{8} (\bar{\mathcal{D}}\bar{\mathcal{D}} - R)(e^{-V} \mathcal{D}_\alpha e^V)}$$

(see p. ...)

$\sim E^b \wedge E^\alpha \wedge \bar E^{\dot\alpha}$:

$$0 = -i \mathcal{D}_\alpha \mathbb{W}^\beta \sigma_{b\beta\dot\alpha} + i \sigma_{b\alpha\dot\beta} \bar{\mathcal{D}}_{\dot\alpha} \bar{\mathbb{W}}^{\dot\beta} - 2i \sigma^a_{\alpha\dot\alpha} F_{ab} \quad \overset{\tfrac{i}{2}\sigma^b_{\beta\dot\beta}}{\Longleftrightarrow} \quad 0 = -\mathcal{D}_\alpha \mathbb{W}_\beta \epsilon_{\dot\alpha\dot\beta} + (\bar{\mathcal{D}}_{\dot\alpha} \bar{\mathbb{W}}_{\dot\beta}) \epsilon_{\alpha\beta} + \mathbb{F}_{\alpha\dot\alpha\,\beta\dot\beta}$$

$$\tfrac{i}{2} \epsilon^{\dot\alpha\dot\beta} \epsilon^{\alpha\beta}$$

$$-\tfrac{i}{4} \tilde\sigma_a^{\dot\alpha\alpha} : \qquad \mathcal{D}^\alpha \mathbb{W}_\alpha - \bar{\mathcal{D}}^{\dot\alpha} \bar{\mathbb{W}}_{\dot\alpha} = 0$$

$$\mathbb{F}_{ab} = \tfrac{i}{4} \mathcal{D}_\alpha \mathbb{W}^\beta (\sigma_{ab})_\beta{}^\alpha + \tfrac{i}{4} (\tilde\sigma_{ab})^{\dot\alpha}{}_{\dot\beta} \bar{\mathcal{D}}_{\dot\alpha} \bar{\mathbb{W}}^{\dot\beta}$$

$$\Rightarrow \mathbb{F}_{\alpha\dot\alpha\,\beta\dot\beta} = \epsilon_{\dot\alpha\dot\beta} \mathcal{D}_{(\alpha} \mathbb{W}_{\beta)} - \epsilon_{\alpha\beta} \bar{\mathcal{D}}_{(\dot\alpha} \bar{\mathbb{W}}_{\dot\beta)}$$

$$\mathcal{D}^\alpha \mathbb{W}_\alpha + \bar{\mathcal{D}}^{\dot\alpha} \bar{\mathbb{W}}_{\dot\alpha} = 2\mathbb{D} = 2 D^I\, T_I$$

$$\mathcal{D}^\alpha \mathbb{W}_\alpha = \mathbb{D} = \bar{\mathcal{D}}^{\dot\alpha} \bar{\mathbb{W}}_{\dot\alpha}$$

$$-\tfrac{1}{2} \epsilon^{\dot\alpha\dot\beta} \qquad \mathcal{D}_{(\alpha} \mathbb{W}_{\beta)} = -\tfrac{1}{2} \mathbb{F}_{ab} \sigma^{ab}{}_{\alpha\beta}$$

$\sim E^b \wedge E^c \wedge E^\alpha$
$\sim E^b \wedge E^c \wedge E^\alpha$:

$$0 = \tfrac{1}{2} \mathcal{D}_\alpha F_{cb} + i \sigma_{[c|\alpha\dot\alpha} \mathcal{D}_{|b]} \bar{\mathbb{W}}^{\dot\alpha} + \tfrac{i}{2} R \sigma_{cb\alpha}{}^\beta \mathbb{W}_\beta$$

$$\boxed{\mathcal{D}_\alpha \mathbb{W}_\beta = -\tfrac{1}{2} \mathbb{F}_{ab} \sigma^{ab}{}_{\alpha\beta} + \tfrac{1}{2} \epsilon_{\alpha\beta} \mathbb{D}} \; ; \; \mathcal{D}_\alpha \mathbb{W}^\beta = -\tfrac{1}{2} \delta_\alpha^\beta \mathbb{D} - \tfrac{1}{2} \mathbb{F}_{ab} \sigma^{ab}{}_\alpha{}^\beta$$

$$\mathcal{D}_\alpha \mathbb{F}_{cb} = -2i \sigma_{[c|\alpha\dot\alpha} \mathcal{D}_{|b]} \bar{\mathbb{W}}^{\dot\alpha} - R (\sigma_{cb})_\alpha{}^\beta \mathbb{W}_\beta$$

c.c.

$$\bar{\mathcal{D}}_{\dot\alpha} \mathbb{F}_{cb} = 2i \sigma_{[c|\alpha\dot\alpha} \mathcal{D}_{|b]} \mathbb{W}^\alpha + \bar R\, \tilde\sigma_{cb}{}^{\dot\beta}{}_{\dot\alpha} \bar{\mathbb{W}}_{\dot\beta} \quad \Leftarrow \quad \mathcal{D}_{[b}\mathcal{D}_{\alpha]} \mathbb{W}^\beta = T_{\alpha b}{}^{D} \mathcal{D}_D \mathbb{W}^\beta + \mathbb{W}^\gamma R_{\alpha b \gamma}{}^\beta$$

$$\Leftrightarrow \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} \mathbb{F}_{cb} = 2i \sigma^c_{\alpha\dot\alpha} \mathcal{D}_c F_{cb} = -2i \sigma_{[c|\alpha\dot\beta} \bar{\mathcal{D}}_{\dot\alpha} \mathcal{D}_{|b]} \bar{\mathbb{W}}^{\dot\beta} + 2i \sigma_{[c|\beta\dot\alpha} \boxed{\mathcal{D}_\alpha \mathcal{D}_{|b]} \mathbb{W}^\beta} =$$