D=11 SUGRA

MAX. SUGRA IN D=4 : $\mathcal{N}=8$. [E. Cremmer, B. Julia NPB '78]
It can be obtained by dim. reduction from D=11 $\mathcal{N}=1$ SUGRA, which is much simpler.

[E. Cremmer, B. Julia & J. Scherk, PLB 1978]

$$e^a(x) = dx^\mu e_\mu{}^a(x) \quad , \quad \psi^\alpha(x) = dx^\mu \psi_\mu{}^\alpha(x)$$

$$\mu = 0,1,\dots,10 \quad , \quad a = 0,1,\dots,10 \quad , \quad \alpha = 1,\dots,32$$

GRAVITON ⊕ gravitino are not sufficient to close SUSY algebra.

$$A_3(x) = \frac{1}{3!} dx^\rho \wedge dx^\nu \wedge dx^\mu A_{\mu\nu\rho}(x)$$

3rd rank antisymm. tensor (3-form) gauge field

$$\# e_\mu{}^a = 44 = \frac{(D-3)D}{2}$$
$$\# \psi_\mu{}^\alpha = 128 = \frac{(D-3)2^{5}}{2}$$
$$\# A_{\mu\nu\rho} = 84 = \frac{(D-2)(D-3)(D-4)}{3!}$$

massless fields, on-shell

$$\boxed{\delta_\epsilon e^a = -2i\, \epsilon^\alpha \Gamma^a_{\alpha\beta} \psi^\beta}$$

$$\Gamma^a_{\alpha\beta} = \Gamma^a_\alpha{}^\gamma C_{\gamma\beta} = \Gamma^a_{\beta\alpha} \leftarrow \text{real}$$

$$\tilde{\Gamma}^{a\alpha\beta} = C^{\alpha\gamma} \Gamma^a_\gamma{}^\beta = \tilde{\Gamma}^{a\beta\alpha},$$

imaginary in our $(+-,\dots,-)$ notation

$$\Gamma^a \Gamma^b + \Gamma^b \Gamma^a = 2\eta^{ab}$$

$$\Gamma^a_{\alpha\gamma} \tilde{\Gamma}^{b\gamma\beta} = \Gamma^a_\alpha{}^\gamma \Gamma^b_\gamma{}^\beta = \eta^{ab} \delta_\alpha{}^\beta + \Gamma^{ab}{}_\alpha{}^\beta$$

Important Fierz identities

$$\boxed{\Gamma_{a(\alpha\beta} \Gamma^{ab}{}_{\gamma\delta)} \equiv 0}$$

$$\boxed{\Gamma_{a(\alpha\beta} \Gamma^{abcde}{}_{\gamma\delta)} = 3\, \Gamma^{bc}_{(\alpha\beta} \Gamma^{de}{}_{\gamma\delta)}}$$

$\Gamma^a_{\alpha\beta}$, $\Gamma^{ab}_{\alpha\beta}$, $\Gamma^{a_1\dots a_5}_{\alpha\beta}$ ← symmetric

$C_{\alpha\beta}$, $\Gamma^{abc}_{\alpha\beta}$, $\Gamma^{abcd}_{\alpha\beta}$ ← antisymmetric

SUSY

$$\delta \psi_a^{\alpha} = \mathcal{D}_a \epsilon^{\alpha} - \frac{i}{18}\left(F_{a[3]}\, \epsilon \Gamma^{[3]} + \frac{1}{8} F^{[4]} \epsilon \Gamma_{a[4]}\right)^{\alpha}$$

$$\Updownarrow$$

$$\delta e_{\mu}^{a} = -2i\, \psi_{\mu} \sigma^{a} \epsilon \,,$$

$$\delta \psi_{\mu}^{\alpha} = \mathcal{D}_{\mu} \epsilon^{\alpha} - \frac{i}{18}\left(F_{\mu a_1 a_2 a_3} (\epsilon \Gamma^{a_1 a_2 a_3})^{\alpha} + \frac{1}{8} F^{b_1 b_2 b_3 b_4} e_{\mu}^{a} (\epsilon \Gamma_{a b_1 \dots b_4})^{\alpha}\right)$$

$$\delta A_3 = + \; \psi^{\alpha} \wedge \bar{\Gamma}^{(2)}_{\alpha\beta} \epsilon^{\beta} \quad \Leftrightarrow \quad \delta A_{\mu\nu\rho} = 3 \psi_{[\mu} \Gamma_{\nu\rho]} \epsilon$$

$$\bar{\Gamma}^{(2)}_{\alpha\beta} := \frac{1}{2} E^{b} \wedge E^{a}\, \Gamma_{ab\,\alpha\beta}$$

$$\bar{\Gamma}^{(5)}_{\alpha\beta} := \frac{1}{5!} E^{a_5} \wedge \dots \wedge E^{a_1} (\Gamma_{a_1 \dots a_5})_{\alpha\beta}$$

$$\Gamma^{(p)}{}_{\alpha}{}^{\beta} = \frac{(-1)^{\frac{p(p-1)}{2}}}{p!} \underbrace{\Gamma^{(1)}{}_{\alpha}{}^{\gamma} \wedge \dots \wedge \Gamma^{(1)}{}_{\delta}{}^{\beta}}_{p} \,, \quad \Gamma^{(p)}_{\alpha\beta} = \Gamma^{(p)}{}_{\alpha}{}^{\gamma} C_{\gamma\beta} \qquad \Gamma_{\mu\nu} \equiv e_{\mu}{}^{a} e_{\nu}{}^{b}\, \Gamma_{ab}$$

CJS action (schematically)

$$S'_{CJS} = \int d^{11}x\; e R + \sim e \psi_{\mu} \Gamma^{\mu\nu\rho} D_{\nu} \psi_{\rho} + \sim e F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} +$$

$$+ \sim \int A_3 \wedge F_4 \wedge F_4 + \int \sim \psi\psi \cdot \text{bosonic} + \sim \psi\psi\psi\psi$$

$\uparrow$ CS term. Suggests to write all the action in diff. forms

Below we give the explicit form of the 1st order action.

AS SUSY CAN BE OBTAINED FROM SUPERSPACE ...

$$\delta_\epsilon e^a = i_\epsilon T^a|_0 , \quad \delta_\epsilon \psi^\alpha = i_\epsilon T^\alpha|_0 , \quad \delta_\epsilon A_3 = i_\epsilon F_4|_0 ,$$

we can guess the form of superspace constraints and their consequences [obtained in Cremmer & Ferrara 1980, Brink & Howe 1980]

Constraints are imposed on torsion and curvature

$$T^a := DE^a = dE^a - E^b \wedge \omega_b{}^a , \qquad a = 0,1,\ldots,9,10$$

$$T^\alpha := DE^\alpha = dE^\alpha - E^\beta \wedge \omega_\beta{}^\alpha , \qquad \alpha = 1,\ldots,32$$

$$R^{ab} := d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b ,$$

of supervielbein 1-forms $E^A = (E^a, E^\alpha) = dZ^M E_M^a$

$$Z^M = (x^\mu, \theta^{\check\alpha}) , \quad \mu = 0,1,\ldots,9,10 , \ \check\alpha = 1,\ldots,32$$

and spin connection $\omega^{ab} = -\omega^{ba} = dZ^M \omega_M^{ab}$

$$\omega_\alpha{}^\beta = \frac{1}{4} \omega^{ab} \Gamma_{ab\,\alpha}{}^\beta$$

THE ESSENTIAL CONSTRAINTS ARE

$$\boxed{T_{\alpha\beta}{}^a = -2i\Gamma^a_{\alpha\beta}}$$

OTHER CONSTRAINTS ARE CONVENTIONAL ($T_{cb}{}^a = 0$, etc.).

Consequences of the constraints can be found by investigating the BIs

$$I_3^a = DT^a + E^b \wedge R_b{}^a = 0 , \qquad DT^\alpha + E^\beta \wedge R_\beta{}^\alpha = 0 , \qquad DR^{ab} = 0 .$$

where $R_\beta{}^\alpha = +\frac{1}{4} R^{ab} \Gamma_{ab\,\beta}{}^\alpha$.

THE RESULTING EXPRESSIONS FOR TORSIONS AND CURVATURE ARE

$$T^a = -i E^\alpha \wedge E^\beta \Gamma^a_{\alpha\beta} ,$$

$$T^\alpha = -\frac{i}{18} E^a \wedge E^\beta \left(F_{ab_1b_2b_3} \Gamma^{b_1b_2b_3} + \frac{1}{8} F^{c_1c_2c_3c_4} \Gamma_{ac_1c_2c_3c_4} \right)_\beta{}^\alpha + \frac{1}{2} E^b \wedge E^a T_{ab}{}^\alpha(Z) ,$$

$$R^{ab} = E^\alpha \wedge E^\beta \left(-\frac{1}{3} F^{abc_1c_2} \Gamma_{c_1c_2} + \frac{i}{3\cdot 5!} *F^{abc_1\ldots c_5} \Gamma_{c_1\ldots c_5} \right)_{\alpha\beta} +$$

$$+ E^c \wedge E^\alpha \left(-i T^{ab\beta} \Gamma_{c\beta\alpha} + 2i T_c{}^{[a|\beta} \Gamma^{|b]}{}_{\beta\alpha} \right) + \frac{1}{2} E^d \wedge E^c R_{cd}{}^{ab}(Z)$$

where F_{abcd} obeys $D_{[a} F_{bcde]} = 0$ ($\Rightarrow$ F_{abcd} is the field strength of A_{abc}) AS WELL AS EQS OF MOTION. OTHER SG EQS ALSO FOLLOW FROM THE CONSTRAINTS.

CONSTRAINTS ARE ON-SHELL ONES

$$\Rightarrow \left\{ T_{bc}{}^{\beta} \Gamma^{abc}{}_{\beta\alpha} = 0 \iff T_{ab}{}^{\beta} \Gamma^{b}{}_{\beta\alpha} = 0 \right. \quad (*)$$

EX: TO PROVE (USING THE CLIFFORD ALGEBRA)

$$R_{acb}{}^{c} + \frac{1}{3} F_{a[3]} F_{b}{}^{[3]} - \frac{1}{36} F_{[4]} F^{[4]} = 0 \quad (**)$$

$$D_{[a_1}(*F_4)_{a_2 \ldots a_8]} - \frac{7!}{4!\,4!} F_{[a_1 \ldots a_4} F_{a_5 \ldots a_8]} = 0 \quad (***)$$

THE HIGHER COMPONENTS (in θ-expansion of the) (SUPER-)FIELD STRENGTHS ARE EXPRESSED IN TERMS OF THE SAME SUPERFIELDS.

IN PARTICULAR

$$\mathcal{D}_\alpha T_{ab}{}^{\beta} = -\frac{1}{4} R_{ab}{}^{cd} \Gamma_{cd\,\alpha}{}^{\beta} - 2\left(\mathcal{D}_{[a} T_{b]\alpha}{}^{\beta} + T_{[a|\alpha}{}^{\gamma} T_{b]\gamma}{}^{\beta}\right)$$

$$T_{b\alpha}{}^{\beta} = \frac{i}{16}\left(F_{b[3]}\Gamma^{[3]} + \frac{1}{8} F^{[4]} \Gamma_{a[4]}\right)_\alpha{}^{\beta}$$

which suggests that the Einstein eq. (**) appears as next-to leading ($\sim\theta$) component of the RS eq. (*)

$$\mathcal{D}_\alpha F_{abcd} = -6\, T_{[ab}{}^{\beta} \Gamma_{cd]\beta\alpha}, \quad \ldots$$

Ex: To show that $T_{\beta a}{}^{\alpha} = -\frac{i}{18}\left(F_{a[3]}\Gamma^{[3]} + \frac{1}{8}F^{[4]}\Gamma_{a[4]}\right)_{\beta}{}^{\alpha}$

and $R_{\alpha\beta}{}^{ab} = -\frac{2}{3}F^{ab[2]}\Gamma_{[2]\alpha\beta} + \frac{i}{3\cdot 5!}\,{*F}^{ab[5]}\,\Gamma_{[5]\alpha\beta}$

Solve dim 2 component of $I_3^a = DT^a + E^b \wedge R_b{}^a = 0$

$I_{\alpha\beta b}{}^{a} \equiv 0$

and dim 3/2 component of $I_3{}^{\alpha} = DT^{\alpha} + E^{\beta}\wedge R_{\beta}{}^{\alpha} = 0$

$I_{\delta\gamma\beta}{}^{\alpha} \equiv 0$.

Ex: To show that $I_{\alpha cb}{}^{a} = 0 \Rightarrow R_{\alpha c}{}^{ab} = -i\,T^{ab\beta}\Gamma_{c\beta\alpha} + 2i\,T_c{}^{[a|\beta}\Gamma^{|b]}{}_{\beta\alpha}$

• (see D=4 case)

SUGGESTIONS:

• THE proof uses Clifford algebra

$\Rightarrow \quad \Gamma^{c_1\ldots c_p}\Gamma_a = \Gamma^{c_1\ldots c_p}{}_a + p\,\Gamma^{[c_1\ldots c_{p-1}}\delta^{c_p]}{}_a$,

and $\quad \Gamma_{a_1\ldots a_6} = -\frac{i}{5!}\epsilon_{a_1\ldots a_6 b_1\ldots b_5}\Gamma^{b_1\ldots b_5} \quad (\Leftarrow \Gamma_{01\ldots 9\#} = -iI)$, $\# \equiv 10$

• IN DIFF FORMS AND WITH $T^a = -iE^{\alpha}\wedge E^{\beta}\Gamma^a_{\alpha\beta}$, $T^{\alpha} = E^b \wedge E^{\beta}T_{\beta b}{}^{\alpha} + \frac{1}{2}E^c\wedge E^b T_{bc}{}^{\alpha}$

$0 = DT^a + E^b\wedge R_b{}^a = 2i\,T^{\gamma}\wedge E^{\beta}\Gamma^a_{\beta\gamma} + \frac{1}{2}E^b\wedge E^{\alpha}\wedge E^{\beta}R_{\alpha\beta b}{}^{a} + E^b\wedge E^c\wedge E^{\alpha}R_{\alpha[cb]}{}^{a} + \frac{1}{2}E^d\wedge E^c\wedge E^b R_{[bcd]}{}^{a}$

where $2i\,T^{\gamma}\wedge E^{\beta}\Gamma^a_{\beta\gamma} = 2i\,E^b\wedge E^{\alpha}\wedge E^{\beta}\,T_{(\alpha|b}{}^{\gamma}\Gamma^a_{\gamma|\beta)} - i\,E^c\wedge E^b\wedge E^{\beta}\,T_{bc}{}^{\gamma}\Gamma^a_{\gamma\beta}$

$0 = DT^{\alpha} + E^{\beta}\wedge R_{\beta}{}^{\alpha} = i\,E^{\gamma}\wedge E^{\delta}\wedge E^{\beta}\,\Gamma^b_{(\gamma\delta}T_{\beta)b}{}^{\alpha} + E^b\wedge T^{\beta}\,T_{\beta b}{}^{\alpha} + E^{\beta}\wedge R_{\beta}{}^{\alpha}$

$+ E^b\wedge E^{\beta}\wedge E^{\gamma}\,D_{(\gamma}T_{\beta)b}{}^{\alpha} - E^b\wedge E^a\wedge E^{\beta}\,D_{[a|}T_{\beta|b]}{}^{\alpha} - \ldots$

where $E^{\beta}\wedge R_{\beta}{}^{\alpha} = \frac{1}{2}E^{\gamma}\wedge E^{\delta}\wedge E^{\beta}R_{(\gamma\delta\beta)}{}^{\alpha} - E^b\wedge E^{\beta}\wedge E^{\gamma}R_{(\gamma|b|\beta)}{}^{\alpha} + \frac{1}{2}E^b\wedge E^a\wedge E^{\beta}R_{ab\beta}{}^{\alpha}$

COMMENTS:

• dim 3 component $I_{dcb}{}^{a} \equiv 0$ gives the usual BI of GR $R_{[bcd]}{}^{a} \equiv 0$.

The SSP of 11D SUGRA allows for the existence of the super 3-form gauge superfield

$$A_3 = \frac{1}{3!} dZ^K \wedge dZ^N \wedge dZ^M A_{MNK} = \frac{1}{3!} E^C \wedge E^B \wedge E^A A_{ABC}(Z)$$

WITH THE FIELD STRENGTH

$$\mathcal{F}_4 = dA_3 = \frac{1}{4!} E^D \wedge E^C \wedge E^B \wedge E^A F_{ABCD}(Z)$$

obeying the constraints [Cremmer & Ferrara 80; Brink & Howe 80]

$$\mathcal{F}_4 := dA_3 = \frac{1}{2} E^\alpha \wedge E^\beta \wedge \bar{\Gamma}^{(2)}_{\alpha\beta} + \frac{1}{4!} E^d \wedge E^c \wedge E^b \wedge E^a F_{abcd}$$

$$\frac{1}{4} E^b \wedge E^a \wedge E^\alpha \wedge E^\beta (\Gamma_{ab})_{\alpha\beta}$$

SUSY $\delta_\epsilon A_3 = \frac{1}{2} e^b \wedge e^a \wedge \psi \Gamma_{ab} \epsilon = i_\epsilon \mathcal{F}_4$ | CAN BE OBTAINED FROM THESE CONSTRAINTS)

ACTUALLY IT IS CONVENIENT TO STUDY BI $d\mathcal{F}_4 = 0$ together with BIs for torsion (and curvature) $I_3^a = 0,\ I_3^\alpha = 0\ (DR^{ab} = 0)$.

$$0 = d\mathcal{F}_4 = -\frac{1}{2} E^b \wedge E^\alpha \wedge E^\beta \wedge E^\gamma \wedge E^\delta \Gamma^a_{(\alpha\beta|} \Gamma_{ab|\gamma\delta)} - \frac{1}{2} E^b \wedge E^a \wedge T^\alpha \wedge E^\beta \Gamma_{ab\,\alpha\beta} -$$

($\Gamma^a_{(\alpha\beta|}\Gamma_{ab|\gamma\delta)} = 0 \to 0$; $T^\alpha \ni E^c \wedge E^\gamma T_{\gamma c}{}^\alpha + \frac{1}{2} E^d \wedge E^c T_{cd}{}^\alpha$)

$$- \frac{1}{3!} E^d \wedge E^c \wedge E^b \wedge E^\alpha \wedge E^\beta \Gamma^a_{\alpha\beta} F_{abcd} + \frac{1}{4!} E^d \wedge E^c \wedge E^b \wedge E^a \wedge E^\alpha D_\alpha F_{abcd} + \frac{1}{4!} E^e \wedge E^d \wedge E^c \wedge E^b \wedge E^a \partial F_{[abcde]}$$

Ex: to show that our form of $T_{\beta c}{}^\alpha$ solves dim $\frac{1}{2}$ identity in $d\mathcal{F}_4 = 0$

$\sim E^c \wedge E^b \wedge E^a \wedge E^\alpha \wedge E^\beta$:

$$0 = -\frac{1}{2} T_{\alpha|[c}{}^\gamma \Gamma_{ab]\gamma|\beta)} + \frac{1}{6} \Gamma^d_{\alpha\beta} F_{abcd}$$

OTHER CONSEQUENCES OF THESE CONSTRAINTS ARE

$$D_\alpha F_{abcd} = -3!\, T_{[ab}{}^\beta \Gamma_{cd]\beta\alpha}$$

$$D_{[a} F_{bcde]} = 0$$

and

$$D_{[a_1} (*F_4)_{a_2 \dots a_8]} - \frac{7!}{4!\,4!} F_{[a_1 \dots a_4} F_{a_5 \dots a_8]} = 0$$

$\Leftarrow$ SUPERFIELD GENERALIZATIONS OF THE EQS OF MOTION

INTRODUCING

$$F_{a_1\dots a_7} = (* F_4)_{a_1\dots a_7} = \frac{1}{4!}\, \epsilon_{a_1\dots a_7 b_1\dots b_4} F^{b_1\dots b_4}$$

we can write the F_4-equations as generalized BI

$$\mathcal{D}_{[a_1} F_{a_2\dots a_8]} - \frac{7!}{4!\,4!} F_{[a_1\dots a_4} F_{a_5\dots a_8]} = 0$$

THIS OBSERVATION SUGGESTS TO INTRODUCE THE DUAL 6-FORM POTENTIAL IN SSP OF 11D SG. The constraints are

[Lechner & Candiello, NPB 1994]

$$\mathcal{F}_7 := dA_6 + A_3 \wedge dA_3 =$$
$$= \frac{i}{2} E^\alpha \wedge E^\beta \wedge \bar{F}^{(5)}_{\alpha\beta} + \frac{1}{7!} E^{c_7} \wedge \dots \wedge E^{c_1} F_{c_1\dots c_7} \quad (2)$$

BIs $\quad d\mathcal{F}_7 - \mathcal{F}_4 \wedge \mathcal{F}_4 = 0$

PRODUCE

$$D_\alpha F_{a_1\dots a_7} = -21 i\, T_{[a_6 a_7}{}^\beta\, \bar{\Gamma}_{a_1\dots a_5]\beta\alpha}$$

$$D_{[c_1} F_{c_2\dots c_8]} - \frac{7!}{4!\,4!} F_{[c_1\dots c_4} F_{c_5\dots c_8]} = 0$$

AND

$$\boxed{F_{c_1\dots c_7} = (* F_4)_{c_1\dots c_7} \equiv \frac{1}{4!}\, \epsilon_{c_1\dots c_7 b_1\dots b_4} F^{b_1\dots b_4}}$$

$$\Rightarrow \quad D_{[c_1} (* F_4)_{c_2\dots c_8]} - \frac{7!}{4!\,4!} F_{[c_1\dots c_4} F_{c_5\dots c_8]} = 0$$

> Ex: to show that dim 6 BI are satisfied identically
>
> Remember: $\Gamma^a_{(\alpha\beta}\, \Gamma_{a b_1\dots b_4 |\gamma\delta)} = 3\, \Gamma_{[b_1 b_2 |(\alpha\beta}\, \Gamma_{|b_3 b_4]\,\gamma\delta)}$

EQS OF MOTION APPEAR IN THE FORM OF DUALITY RELATIONS!

THE ACTION IN TERMS OF $e_\mu{}^a, \psi_\mu{}^\alpha, A_3, A_6$ (duality-symm. 11D SG action) [Bandos, Berkovits, Sorokin NPB 98] Uses the PST (Pasti-Sorokin-Tonin) technique.

It is out of the scope of this course.

Below we present the first order form of the Cremmer-Julia-Scherk 11D SG action

[Julia & Silva, JHEP 2000]