First order form of the CJS action [Julia & Silva, JHEP 2000]

$$S = \int_{M^{11}} \mathcal{L}_{11}[E^a, \psi^\alpha, \omega^{ab}, A_3, F_{a_1a_2a_3a_4}]$$

$$\begin{aligned}
\mathcal{L}_{11} &= \frac{1}{4} R^{ab} \wedge E^{\wedge 9}_{ab} - D\psi^\alpha \wedge \psi^\beta \wedge \bar{\Gamma}^{(8)}_{\alpha\beta} + \\
&+ \frac{1}{4}\psi^\alpha \wedge \psi^\beta \wedge (T^a + i/2\, \psi \wedge \psi\, \Gamma^a) \wedge E_a \wedge \bar{\Gamma}^{(6)}_{\alpha\beta} + \\
&+ (dA_3 - a_4) \wedge (*F_4 + b_7) + \frac{1}{2} a_4 \wedge b_7 - \\
&- \frac{1}{2} F_4 \wedge *F_4 - \frac{1}{3} A_3 \wedge dA_3 \wedge dA_3 \,,
\end{aligned}$$

$$\begin{aligned}
\textit{where} \quad a_4 &:= \frac{1}{2}\psi^\alpha \wedge \psi^\beta \wedge \bar{\Gamma}^{(2)}_{\alpha\beta} := \\
&:= -\frac{1}{4}\psi^\alpha \wedge \psi^\beta \wedge E^a \wedge E^b \Gamma_{ab\,\alpha\beta} \,, \\
b_7 &:= \frac{i}{2}\psi^\alpha \wedge \psi^\beta \wedge \bar{\Gamma}^{(5)}_{\alpha\beta} := \\
&:= \frac{i}{2 \cdot 5!}\psi^\alpha \wedge \psi^\beta \wedge E^{a_1} \wedge \ldots \wedge E^{a_5} \Gamma_{a_1 \ldots a_5 \alpha\beta} \,,
\end{aligned}$$

and $\boxed{F_4 = \frac{1}{4!} E^{a_4} \wedge \ldots \wedge E^{a_1} F_{a_1 \ldots a_4}}$ with independent $F_{a_1 \ldots a_4}(Z)$,

$$\begin{aligned}
*F_4 &:= -\frac{1}{4!} E^{\wedge 7}_{a_1 \ldots a_4} F^{a_1 \ldots a_4} \\
&\equiv \frac{1}{7!4!} E^{b_7} \wedge \ldots \wedge E^{b_1}\, \varepsilon_{b_1 \ldots b_7 a_1 \ldots a_4} F^{a_1 \ldots a_4} \,.
\end{aligned}$$

We have also used the compact notation

$$\begin{aligned}
\bar{\Gamma}^{(k)}_{\alpha\beta} &:= \frac{1}{k!} E^{a_k} \wedge \ldots \wedge E^{a_1} \Gamma_{a_1 \ldots a_k \alpha\beta} \qquad \textit{and} \\
E^{\wedge(11-k)}_{a_1 \ldots a_k} &:= \frac{1}{k!} \varepsilon_{a_1 \ldots a_k b_1 \ldots b_{11-k}} E^{b_1} \wedge \ldots E^{b_{11-k}} \,.
\end{aligned}$$

Ex: to obtain eqs of motion from the first order action

$$\delta_A S = \int \mathcal{G}_8 \wedge \delta A_3 \; , \qquad \frac{\delta S}{\delta A_3} := \mathcal{G}_8 = d(*F_4 + b_7 - A_3 \wedge dA_3) = 0 \; ,$$

This can be written as $D_{[c_1} F_{c_2 \ldots c_8]} - \frac{7!}{4!\,4!} F_{[c_1 \ldots c_4} F_{c_5 \ldots c_8]} = 0$, but includes the (auxiliary) independent field $F_{abcd} = F_{[abcd]}$. However,

$$\delta_F S = \int (dA_3 - a_4 - F_4) \wedge *\delta F_4 \; ,$$

so that $\delta S / \delta F^{a_1 \ldots a_4} = 0$ can be written as $*\frac{\delta S}{\delta F_4} = (dA_3 - a_4 - F_4) = 0$ and identifies

$$F_4 := \frac{1}{4!} E^{a_4} \wedge \ldots \wedge E^{a_1} F_{a_1 \ldots a_4} = dA_3 - \frac{1}{2} \psi^\alpha \wedge \psi^\beta \wedge \bar{\Gamma}^{(2)}_{\alpha\beta} \; .$$

Equations for the spin connection

$$\frac{\delta S_{11}}{\delta \omega^{ab}} = \frac{1}{4} E^{\wedge 8}_{abc} \wedge (T^c + i\psi^\alpha \wedge \psi^\beta \, \Gamma^c_{\alpha\beta}) = 0 \qquad \Rightarrow$$

$$\Rightarrow \quad T^a := DE^a = -i\psi^\alpha \wedge \psi^\beta \, \Gamma^a_{\alpha\beta} \; .$$

Gravitino equations have a compact form

$$\frac{\delta S_{11}}{\delta \psi^\alpha} = 0 \qquad \Rightarrow \qquad \Psi_{10\,\beta} := \hat{\mathcal{D}}\psi^\alpha \wedge \bar{\Gamma}^{(8)}_{\alpha\beta} = 0 \; ,$$

in terms of "connection of generalized holonomy" $w_\beta{}^\alpha := \omega_\beta{}^\alpha + t_{1\beta}{}^\alpha$,

$$\hat{\mathcal{D}}\psi^\alpha := d\psi^\alpha - \psi^\beta \wedge w_\beta{}^\alpha \equiv d\psi^\alpha - \psi^\beta \wedge (\omega_\beta{}^\alpha + t_{1\beta}{}^\alpha) \; ,$$

$$t_{1\beta}{}^\alpha = \frac{i}{18} E^a \left(F_{abcd} \Gamma^{bcd}{}_\beta{}^\alpha + \frac{1}{8} F^{bcde} \Gamma_{abcde\beta}{}^\alpha \right) \; .$$

Finally $\delta e^a \Rightarrow$ the Einstein equation for $D = 11$ supergravity

$$M_{10\,a} := R^{bc} \wedge E^{\wedge 8}_{abc} + 2\,(i_a F_4 \wedge *F_4 + F_4 \wedge i_a(*F_4)) + \mathcal{O}(\psi^{\wedge 2}) = 0$$

$$\Leftrightarrow \; R_{acb}{}^c = -\tfrac{1}{3} F_{ac_1c_2c_3} F_b{}^{c_1c_2c_3} + \tfrac{1}{36} \eta_{ab} F_{c_1c_2c_3c_4} F^{c_1c_2c_3c_4} + \mathcal{O}(\psi^{\wedge 2}).$$

SUSY transformations leaving the action invariant can be read of the supergravity constraints

$$
\begin{aligned}
T^a &= -iE^\alpha \wedge E^\beta \Gamma^a_{\alpha\beta} \,, \\
T^\alpha &= -\frac{i}{18} E^a \wedge E^\beta \left(F_{ac_1c_2c_3} \Gamma^{c_1c_2c_3} + \frac{1}{8} F^{c_1c_2c_3c_4} \Gamma_{ac_1c_2c_3c_4} \right)_\beta^{\ \alpha} + \\
&\quad + \frac{1}{2} E^a \wedge E^b T_{ba}{}^\alpha(Z) \,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_4 &:= dA_3 = \frac{1}{2} E^\alpha \wedge E^\beta \wedge \bar{\Gamma}^{(2)}_{\alpha\beta} + \frac{1}{4!} E^{c_4} \wedge \ldots \wedge E^{c_1} F_{c_1 \ldots c_4}(Z) \,, \\
\mathcal{F}_7 &:= dA_6 + A_3 \wedge dA_3 = \frac{i}{2} E^\alpha \wedge E^\beta \wedge \bar{\Gamma}^{(5)}_{\alpha\beta} + \\
&\quad + \frac{1}{7!} E^{c_7} \wedge \ldots \wedge E^{c_1} F_{c_1 \ldots c_7}(Z) \,.
\end{aligned}
$$

Namely

$$
\begin{aligned}
\delta e^a &= -i\psi^\alpha \Gamma^a_{\alpha\beta} \epsilon^\beta \,, \qquad \delta A_3 = \psi^\alpha \wedge \bar{\Gamma}^{(2)}_{\alpha\beta} \epsilon^\beta \,, \\
\delta\psi^\alpha &= \mathcal{D}\epsilon^\alpha := D\epsilon^\alpha - \epsilon^\beta t_{1\beta}{}^\alpha \,,
\end{aligned}
$$

where

$$
t_{1\beta}{}^\alpha = -\frac{i}{18} E^a \left(F_{ac_1c_2c_3} \Gamma^{c_1c_2c_3}{}_\beta{}^\alpha + \frac{1}{8} F^{c_1c_2c_3c_4} \Gamma_{ac_1c_2c_3c_4\beta}{}^\alpha \right) \,,
$$

plus suitable transformations of the spin connection.

Killing spinor equations

Can purely bosonic solutions be supersymmetric? As

$$\delta\psi_\mu{}^\alpha = \mathcal{D}_\mu\epsilon^\alpha := D_\mu\epsilon^\alpha - \epsilon^\beta t_{\mu\beta}{}^\alpha$$

$$t_{1\beta}{}^\alpha = dx^\mu t_{\mu\beta}{}^\alpha = -\frac{i}{18}e^a\left(F_{a[3]}\Gamma^{[3]}{}_\beta{}^\alpha + \frac{1}{8}F^{[4]}\Gamma_{a[4]\beta}{}^\alpha\right) ,$$

if $\psi_\mu^\alpha = 0$, then ($\delta e^a = 0$ and $\delta A_3 = 0$ are satisfied identically while) $\delta\psi_\mu^\alpha = 0$ gives a nontrivial **Killing spinor equation**

$$\boxed{\mathcal{D}\epsilon^\alpha := D\epsilon^\alpha - \epsilon^\beta t_{1\beta}{}^\alpha = 0} . \qquad (*)$$

- If the bosonic solution is such that (*) has no solution $\Rightarrow$ that bosonic solution is not supersymmetric.
- If the bosonic solution is such that the solution of (*) is nontrivial, than it is supersymmeric or **BPS solution**
- If (*) is solved in terms of arbitrary 32 component constant spinor– the bosonic solution is completely SUSY (solution is 32-parametric); [Figueroa-O'Farrill & Papadopoulos 2002:] all such solutions of 11D SUGRA are locally isomorphic to
  - flat 11D space with $F_4 = 0$,
  - Freund-Rubin $AdS_4 \times S^7$ and $AdS_7 \times S^4$ solutions,
  - Hpp-wave solution
- If the solution of (*) has k arbitrary parameters one says that bosonic solution has k Killing spinors, or that it is $k/32$ SUSY or that it describes $k/32$ BPS state. The most interesting in the String/M-theory perspective are 1/2 BPS M2-brane and M5-brane solutions.

SUSY conditions and equations of motion

Why it is easier to SUSY solutions? Because for them the Killing spinor eq $\boxed{\mathcal{D}\epsilon^\alpha := d\epsilon^\alpha - \epsilon^\beta w_\beta{}^\alpha = D\epsilon^\alpha - \epsilon^\beta t_{1\beta}{}^\alpha = 0}$ plays the role of the 'Lax pair' (associated linear system) for SUGRA equations.

Its selfconsistency ('integrability') conditions is

$$0 = \mathcal{D}\mathcal{D}\epsilon^\alpha = -\epsilon^\beta \mathcal{R}_\beta{}^\alpha ,$$

where
$$\begin{aligned}\mathcal{R}_\beta{}^\alpha &:= dw_\beta{}^\alpha - w_\beta{}^\gamma \wedge w_\gamma{}^\alpha \\ &= \frac{1}{4}R^{ab}(\Gamma_{ab})_\alpha{}^\beta + Dt_{1\alpha}{}^\beta - t_{1\alpha}{}^\gamma \wedge t_{1\gamma}{}^\beta\end{aligned}$$

is the so-called 'curvature of generalized holonomy', and
$\boxed{t_{1\beta}{}^\alpha = e^a t_{a\beta}{}^\alpha = \frac{i}{18} e^a \left(F_{ac_1c_2c_3}\Gamma^{c_1c_2c_3} + \frac{1}{8}F^{c_1c_2c_3c_4}\Gamma_{ac_1c_2c_3c_4}\right)_\beta{}^\alpha}$.

For completely SUSY solutions $\epsilon^\beta \mathcal{R}_{ab\beta}{}^\alpha = 0 \quad \forall\, \epsilon^\beta$ and hence

$$32\ susy \Rightarrow \qquad \mathcal{R}_{ab\beta}{}^\alpha = 0 .$$

$\Rightarrow$ completely SUSY configurations of bosonic fields always solve the bosonic equations of motion of D=11 supergravity, which, when $\psi_\mu^\alpha = 0$, can be collected in (see hep-th/0501007=PLB2005)

$$\mathcal{R}_{ab\beta}{}^\gamma \Gamma^b{}_\gamma{}^\alpha = 0 ,$$

and when $\psi_\mu^\alpha \neq 0$ (see hep-th/0501007=PLB2005) in
$*\mathcal{M}_\alpha^{a\beta} = \mathcal{R}_{bc\alpha}{}^\gamma \Gamma^{abc}{}_{\gamma\beta} - 4i((\hat{\mathcal{D}}\psi)_{bc}\Gamma^{[abc})_\beta (\psi_d\Gamma^{d]})_\alpha = 0$ or equiv.

$$\mathcal{M}_{10\,\alpha\beta} := \mathcal{R}_\beta{}^\gamma \wedge E^{\wedge 8}_{abc}\Gamma^{abc}_{\gamma\alpha} + i\hat{\mathcal{D}}\psi^\delta \wedge \psi^\gamma \wedge E^{\wedge 7}_{a_1\ldots a_4}\Gamma^{[a_1a_2a_3}_{\delta\alpha}\Gamma^{a_4]}_{\beta\gamma} = 0 .$$

one can show that

$$\mathcal{M}_{10\,\alpha\beta} = -3i\left(-2i\Gamma^a_{\beta\alpha}M_{10\,a} + \mathcal{G}_8 \wedge \bar{\Gamma}^{(2)}_{\beta\alpha}\right)$$

$\Leftarrow R^{bc} \wedge E^{\wedge 8}_{abc} + \ldots$
Einstein eq. of 11D SUGRA

$\Leftarrow d(*F_7 - A_3 \wedge dA_3 + b_7) = 0$ $\Leftarrow \frac{i}{2}\psi^\alpha \wedge \psi^\beta \wedge \bar{\Gamma}^{(5)}_{\alpha\beta}$
A_3 field eqs

where

$$
\begin{aligned}
M_{10\,a} &:= R^{bc} \wedge E^{\wedge 8}_{abc} + 2\,(i_a F_4 \wedge *F_4 + F_4 \wedge i_a(*F_4)) + \mathcal{O}(\psi^{\wedge 2}) \ = 0 \\
\mathcal{G}_8 &:= d(*F_4 + b_7 - A_3 \wedge dA_3) = 0 \ ,
\end{aligned}
$$

the Einstein equation and the A_3 gauge field equations of 11D SUGRA ($b_7 := \frac{i}{2}\psi^\alpha \wedge \psi^\beta \wedge \bar{\Gamma}^{(5)}_{\alpha\beta}$). In other words:

$$
\begin{aligned}
&M_{10\,a} =\!\propto \mathrm{tr}(\Gamma_a \mathcal{M}_{10}) \ , \\
&\mathcal{G}_8 \wedge E^a \wedge E^b =\!\propto \mathrm{tr}(\Gamma^{ab} \mathcal{M}_{10}) \ .
\end{aligned}
$$

$AdS_4 \times S^7$ Freund-Rubin solution

The $AdS_4 \times S^7$ metric

$$ds^2 = \left(\frac{r}{R}\right)^4 dx^a \eta_{ab} dx^b - \left(\frac{R}{r}\right)^2 \left(dr^2 + r^2 ds^2_{S^7}\right) .$$

Flux

$$F_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = \frac{3}{R}\varepsilon_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} , \qquad others\ = 0 , \qquad \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} = 0, 1, 2, 3 .$$

The only nonvanishing component of the dual tensor (of the seven form flux) is

$$(F_7)^{\hat{I}_1 \ldots \hat{I}_7} = (*F_4)^{\hat{I}_1 \ldots \hat{I}_7} = -\frac{1}{8R}\, \varepsilon^{\hat{I}_1 \ldots \hat{I}_7} .$$

************************ A BIT MORE DETAILS: *****

Vielbein forms

$$e^a = \left(\frac{r}{R}\right)^2 dx^a , \quad e^r \equiv e^2 = \frac{R}{r}\, dr , \quad e^{\hat{I}} = R\, \Omega^{\hat{I}} ,$$

The natural vielbbein and spin connection on the S^7 sphere is provided by the set of Cartan forms $\Omega^{\hat{I}}$ (such that $ds_{S^7} = \Omega^{\hat{I}} \otimes \Omega^{\hat{I}}$) and $\overset{0}{\omega}{}^{\hat{I}\hat{J}} = -\overset{0}{\omega}{}^{\hat{J}\hat{I}}$ which $\vec{n} := (n_J)$ which obey the Maurer-Cartan equations of the $SO(8)$ group

$$D\Omega^{\hat{I}} = d\Omega^{\hat{I}} - \Omega^{\hat{J}} \wedge \overset{0}{\omega}{}^{\hat{J}\hat{I}} = 0 , \qquad d\overset{0}{\omega}{}^{\hat{I}\hat{J}} + \overset{0}{\omega}{}^{\hat{I}\hat{K}} \wedge \overset{0}{\omega}{}^{\hat{K}\hat{J}} = \Omega^{\hat{I}} \wedge \Omega^{\hat{J}} .$$

can be constructed from the so-called spherical harmonic variables $\vec{u}^{\hat{I}} := (u_J{}^{\hat{I}})$, $J = 1, \ldots, 8$, $\hat{I} = 1, \ldots, 7$

$$\Omega^{\hat{I}} = \vec{n} d\vec{u}^{\hat{I}} , \qquad \omega^{\hat{I}\hat{J}} = -\vec{u}^{\hat{I}} d\vec{u}^{\hat{J}} ,$$

$$(\vec{n}, \vec{u}^{\hat{I}}) := (n_J\, ,\, u_J{}^{\hat{I}}) \in SO(8) \quad \leftrightarrow \quad \begin{cases} \vec{n}\vec{n} = 1 , \\ \vec{n}\vec{u}^{\hat{J}} = 0 , \\ \vec{u}^{\hat{I}}\vec{u}^{\hat{J}} = \delta^{\hat{I}\hat{J}} \end{cases} .$$

1/2 supersymmetric solutions: M2-brane and M5-brane

M2-brane also known as 11D supermembrane

The M2-brane metric $(a, b = 0, 1, 2; I, J = 1, ..., 8)$

$$\begin{aligned} ds^2 &= e^a \otimes e^b \eta_{ab} - e^I \otimes e^I = \\ &= \left(1 + k/r^6\right)^{-2/3} dx^\mu \otimes dx^\nu \eta_{\mu\nu} - \left(1 + k/r^6\right)^{-1/3} dy^m dy^n \delta_{mn} \ , \end{aligned}$$

$\mu, \nu = 0, 1, 2,\ m, n = 1, ..., 8.$

$$A_{\mu\nu\rho} = \epsilon_{\mu\nu\rho} \left(1 + \frac{k}{r^6}\right)^{-2/3} , \qquad others \ = 0 \ , \tag{1}$$

This solution interpolates between $AdS_4 \times S^7$ and flat M^{11} [G. Gibbons & P. Townsend 1993] This is to say it tends to $AdS_4 \times S^7$ when $r \mapsto 0$ (so that $\frac{k}{r^6} >> 1$) and to 11D Minkowski spacetime when $r \mapsto \infty$ (so that $\frac{k}{r^6} << 1$).

SUSY preserved by M2-brane $(\underline{\alpha}, \underline{\beta} = 1, ..., 32)$

$$\varepsilon^{\underline{\alpha}} = (1 + \bar{\gamma})^{\underline{\alpha}}{}_{\underline{\beta}} \kappa^{\underline{\beta}} \qquad \Leftrightarrow \qquad \varepsilon^{\underline{\alpha}} = \bar{\gamma}^{\underline{\alpha}}{}_{\underline{\beta}} \varepsilon^{\underline{\beta}} \tag{2}$$

is defined with the use of the projector $(1 + \bar{\gamma})$ where

$$\bar{\gamma} := \tfrac{i}{3!}\, \epsilon^{abc} \delta_a{}^{\underline{a}} \delta_b{}^{\underline{b}} \delta_c{}^{\underline{c}} \Gamma_{\underline{abc}}$$

has the properties $\bar{\gamma}^2 = I$, $tr(\bar{\gamma}) = 0$.

We will be back to the origin of this projector...

In a special Lorentz frame 11D $\bar{\gamma} = i\Gamma_1\Gamma_2\Gamma_3$ and, with a certain Γ matrix representation $\bar{\gamma} = \begin{pmatrix} I_{16\times16} & 0 \\ 0 & -I_{16\times16} \end{pmatrix}$ so that the solution the preserved susy is characterized by $\varepsilon^{\underline{\alpha}} = \begin{pmatrix} \epsilon^{\alpha} \\ 0 \end{pmatrix}$ with $\alpha = 1, ..., 16$. Hence $1/2 = 16/32$ part of susy is preserved.

4.2.—(15)*—

M5-brane solution (also known as 5-brane of M-theory)

M2-brane solution is similar to 'electric charge' solution in electrodynamics (although it is not a point-like, but rather extended object).

There is also 1/2 BPS state (=1/2 susy solution of 11D SUGRA) similar to Dirac monopole (also not a point-like but rather extended object). The M2-brane metric ($a, b = 0, 1, ..., 5; I, J = 1, ..., 6$)

$$\begin{aligned} ds^2 &= e^a \otimes e^b \eta_{ab} - e^I \otimes e^I = \\ &= \left(1 + k/r^3\right)^{-1/3} dx^\mu \otimes dx^\nu \eta_{\mu\nu} - \left(1 + k/r^3\right)^{2/3} dy^m dy^n \delta_{mn} \ , \end{aligned}$$

$\mu, \nu = 0, 1, ..., 5, m, n = 1, ..., 6$. As for the Dirac monopole in d=4, the 3-form potential of 11D M5-brane solution is not well defined, so that the solution is characterized by the field strength

$$F_{mnkl} = 3k\epsilon_{mnklp}\frac{y^p}{r^5} \ . \qquad others \ = 0 \ , \tag{3}$$

This solution interpolates between $AdS_7 \times S^4$ and flat M^{11} [G. Gibbons & P. Townsend 1993] This is to say it tends to $AdS_7 \times S^4$ when $r \mapsto 0$ (so that $\frac{k}{r^6} >> 1$) and to 11D Minkowski spacetime when $r \mapsto \infty$ (so that $\frac{k}{r^6} << 1$).

Mp-branes and most general 11D SUSY algebra

Most general susy algebra in 11D is [Van Holten & Van Proeyen, J.Phys.A15:3763,1982.]

$$\{Q_{\underline{\alpha}}, Q_{\underline{\beta}}\} = \mathcal{P}_{\underline{\alpha\beta}} = i\Gamma^{\underline{a}}_{\underline{\alpha\beta}} P_{\underline{a}} + \Gamma^{\underline{ab}}_{\underline{\alpha\beta}} Z_{\underline{ab}} + i\Gamma^{\underline{a}_1\ldots\underline{a}_5}_{\underline{\alpha\beta}} Z_{\underline{a}_1\ldots\underline{a}_5}$$

where $Z_{\underline{ab}} = -Z_{\underline{ba}} = Z_{[\underline{ab}]}$ and $Z_{\underline{a}_1\ldots\underline{a}_5} = Z_{[\underline{a}_1\ldots\underline{a}_5]}$ are so–called tensorial central charges.

They are called central because $[Z, P] = 0 = [Z, Q]$, $[Z, Z] = 0$. 'Central' are because they transform nontrivially under the $SO(1, 10)$.

These tensorial central charge can be associated with the existence of supersymmetric extended objects - M2-brane and M5-brane in the case of 11D. Schematically, in the presence of M2-brane

$$Z^{\underline{ab}} =\propto \int_{R^2} d\xi^1 d\xi^2 \epsilon^{0mn} \partial_m \hat{X}^{\underline{a}} \partial_n \hat{X}^{\underline{b}} \,, \qquad \hat{X}^{\underline{a}} = \hat{X}^{\underline{a}}(\tau, \xi^1, \xi^2)$$

and in the presence of M5-brane ($\hat{X}^{\underline{a}} = \hat{X}^{\underline{a}}(\tau, \vec{\xi})$)

$$Z^{\underline{a}_1\ldots\underline{a}_5} =\propto \int_{R^5} d\xi^1 d\xi^2 d\xi^3 d\xi^4 d\xi^5 \epsilon^{0m_1\ldots m_5} \partial_{m_1} \hat{X}^{\underline{a}_1} \ldots \partial_{m_5} \hat{X}^{\underline{a}_5} \,,$$

Preserved susy implies that $\boxed{det(\mathcal{P}_{\underline{\alpha\beta}}) = 0}$ (otherwise $\epsilon^{\underline{\alpha}} \mathcal{P}_{\underline{\alpha\beta}} = 0$ has only trivial solution).

Preserved 1/2 of susy: $rank(\mathcal{P}_{\underline{\alpha\beta}}) = 16$ which implies that $C\mathcal{P}$ is a projector ($C\mathcal{P}C\mathcal{P} =\propto C\mathcal{P}$).

Membrane (supermembrane or M2-brane) ground state:
$P^{\underline{a}} =\propto \delta_0^{[\underline{a}} T$ $Z^{\underline{ab}} =\propto T\delta_1^{[\underline{a}} \delta_2^{\underline{b}]}$

$$\boxed{\{Q_{\underline{\alpha}}, Q_{\underline{\beta}}\} = iT\Gamma^0_{(\underline{\alpha\gamma}}(\delta + i\Gamma^0\Gamma^1\Gamma^2)^{\underline{\gamma}}{}_{\underline{\beta})} = iT(\Gamma^0(I + \bar{\gamma}))_{\underline{\alpha\beta}}}$$

Thus $\epsilon^{\underline{\alpha}} Q_{\underline{\alpha}} | M2 >= 0$ iff $\boxed{(I + \bar{\gamma})^{\underline{\alpha}}{}_{\underline{\beta}} \epsilon^{\underline{\beta}} = 0}$ (the same relation as follows from Killing spinor equation).