# OFF-SHELL FORMULATION OF SIMPLE ($D=4$, $N=1$) SUGRA

## MINIMAL SET OF AUX. FIELDS

WE WILL BEGIN FROM superspace formulation:
IMPOSE A WEAKER CONSTRAINTS WHICH DO NOT RESULT IN EQS. OF MOTION.

(SIMILAR TO $D=4$ $N=1$ SYM CONSTRAINTS $\mathbb{F}_{\underline{\alpha}\underline{\beta}}=0$)

THIS WILL ALLOW US TO WRITE THE SUPERGRAVITY ACTION IN TERMS OF SUPERFIELDS. SUSY (OF THIS ACTION) IS CLOSED OFF-SHELL (WITHOUT THE USE OF EQS. OF MOTION)

THE SPACETIME COMPONENT SUGRA ACTION WITH AUXILIARY FIELDS CAN BE OBTAINED FROM THIS SUPERFIELD SUGRA ACTION; AND WE WILL DERIVE IT.

# Superspace CONSTRAINTS AND SOLUTION OF BIANCHI IDENTITIES.

LET US IMPOSE WEAKER CONSTRAINTS

BUT KEEPING $\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} = +2i\sigma^a_{\alpha\dot\alpha}\mathcal{D}_a$ (no curvature contribution)

$$R_{\alpha\dot\beta}{}^{ab} = 0 \quad \text{and} \quad T_{\alpha\dot\beta}{}^a = -2i\sigma^a_{\alpha\dot\beta}$$

AS WELL AS $T_{\alpha\beta}{}^A = 0\,, \quad T_{\dot\alpha\dot\beta}{}^A = 0$

[ LATTER WE WILL DISCUSS THE ORIGIN OF THE CONSTRAINTS. NOW ONLY COMMENTS ]

Determines $E_{\alpha\dot\alpha}{}^M = \sigma^a_{\alpha\dot\alpha}E_a{}^M$ and $\omega_a{}^{cd}$ in terms of $E_\alpha^M, \bar{E}_{\dot\alpha}^M, \omega_\alpha^{ab}, \omega_{\dot\alpha}^{ab}$

TYPICAL CONVENTIONAL CONSTRAINTS, LIKE $T_{ab}{}^c = 0$ in GR.

THUS WE CANNOT IMPOSE $T_{ab}{}^c = 0$ as we have already fixed $\omega_a{}^{cd}$

LET US STUDY BIs WITH

$$T^a = -2i\sigma^a_{\alpha\dot\alpha}E^\alpha \wedge \bar{E}^{\dot\alpha} + \tfrac{1}{2}E^c \wedge E^b\, T_{bc}{}^a$$

$$T^{\underline{\alpha}} = E^c \wedge E^{\underline{\beta}}\, T_{\underline{\beta} c}{}^{\underline{\alpha}} + \tfrac{1}{2}E^c \wedge E^b\, T_{bc}{}^{\underline{\alpha}}$$

$$R^{ab} = +\tfrac{1}{2}E^\alpha \wedge E^\beta R_{\alpha\beta}{}^{ab} + \tfrac{1}{2}\bar{E}^{\dot\alpha}\wedge\bar{E}^{\dot\beta} R_{\dot\alpha\dot\beta}{}^{ab} + E^c \wedge E^{\underline{\gamma}} R_{\underline{\gamma} c}{}^{ab} + \tfrac{1}{2}E^d \wedge E^c R_{cd}{}^{ab}$$

$$T_{\beta\dot\beta\,\gamma\dot\gamma\,\alpha} := \epsilon_{\beta\gamma}\,t_{\alpha\dot\beta\dot\gamma} - \tfrac{1}{8}\epsilon_{\dot\beta\dot\gamma}\left(W_{\alpha\beta\gamma} + \epsilon_{\alpha(\beta}t_{\gamma)}\right)$$

$$T_{\beta\dot\beta\,\gamma\dot\gamma\,\dot\alpha} := \epsilon_{\dot\beta\dot\gamma}\,\bar{t}_{\dot\alpha\beta\gamma} - \tfrac{1}{8}\epsilon_{\beta\gamma}\left(\bar{W}_{\dot\alpha\dot\beta\dot\gamma} + \epsilon_{\dot\alpha(\dot\beta}\bar{t}_{\dot\gamma)}\right)$$

## BIs for torsion

$$\boxed{0 = I_3{}^a := \mathcal{D}T^a + E^b \wedge R_b{}^a}$$

$$E^c\wedge E^b\wedge E^{\underline{\beta}}\, R_{\underline{\beta}[bc]}{}^a + \tfrac{1}{2}E^b\wedge E^\beta \wedge E^\alpha R_{\alpha\beta b}{}^a + \tfrac{1}{2}E^b\wedge\bar{E}^{\dot\beta}\wedge\bar{E}^{\dot\alpha}R_{\dot\alpha\dot\beta b}{}^a + \tfrac{1}{2}E^b\wedge E^d \wedge E^c R_{[cd b]}{}^a$$

$$2i\sigma^a_{\alpha\dot\alpha}T^\alpha \wedge \bar{E}^{\dot\alpha} + 2i\sigma^a_{\alpha\dot\alpha}\bar{T}^{\dot\alpha}\wedge E^\alpha - 2iE^c\wedge E^\alpha \wedge \bar{E}^{\dot\alpha}\sigma^b_{\alpha\dot\alpha}T_{bc}{}^a + \tfrac{1}{2}E^c \wedge E^b \wedge E^{\underline{\alpha}}\mathcal{D}_{\underline{\alpha}}T_{bc}{}^a +$$

$$+2iE^c\wedge E^\alpha \wedge \bar{E}^{\dot\alpha}\left(T_{\alpha c}{}^\beta \sigma^a_{\beta\dot\alpha} + \sigma^a_{\alpha\dot\beta}T_{\dot\alpha c}{}^{\dot\beta}\right) + 2iE^b\wedge E^\alpha \wedge E^\beta\, T_{(\alpha|b}{}^{\dot\beta}\sigma^a_{|\beta)\dot\beta} + 2iE^b\wedge\bar{E}^{\dot\alpha}\wedge\bar{E}^{\dot\beta}\, T_{(\dot\alpha|b}{}^{\alpha}\sigma^a_{\alpha|\dot\beta)}$$

$$-iE^b\wedge E^c\wedge E^\alpha \sigma^a_{\alpha\dot\alpha}T_{bc}{}^{\dot\alpha} - iE^b\wedge E^c\wedge\bar{E}^{\dot\alpha}\,T_{bc}{}^\alpha\sigma^a_{\alpha\dot\alpha} + \tfrac{1}{2}E^d\wedge E^c\wedge E^b\,\mathcal{D}_{[b}T_{cd]}{}^a$$

TO SIMPLIFY: WE JUST CHECK THAT THIS IS SOLUTION

dim 2:

$\sim E^b\wedge\bar{E}^{\dot\alpha}\wedge\bar{E}^{\dot\beta}$ : $0 = 2i\,T_{(\dot\alpha|b}{}^\alpha\sigma^a_{\alpha|\dot\beta)} + \tfrac{1}{2}R_{\dot\alpha\dot\beta b}{}^a \Rightarrow \boxed{T_{\dot\alpha b}{}^\alpha = -\tfrac{i}{8}\sigma_{b\dot\alpha}{}^\alpha R}\,,$

$$\boxed{R_{\dot\alpha\dot\beta}{}^{\dot\gamma\dot\delta} = \delta_{(\dot\alpha}^{\dot\gamma}\delta_{\dot\beta)}^{\dot\delta}R} \Leftarrow \boxed{R_{\dot\alpha\dot\beta}{}^{ab} = -\tfrac{1}{2}\bar{\sigma}^{ab}_{\dot\alpha\dot\beta}R}$$

$\sim E^b\wedge E^\alpha\wedge E^\beta$ : $0 = 2i\,T_{(\alpha|b}{}^{\dot\alpha}\sigma^a_{|\beta)\dot\alpha} + \tfrac{1}{2}R_{\alpha\beta b}{}^a \Rightarrow \boxed{T_{\alpha b}{}^{\dot\alpha} = +\tfrac{i}{8}\sigma_{b\alpha}{}^{\dot\alpha}\bar{R}}\,,$

$$R_{\alpha\beta}{}^{\gamma\delta} = -\delta_{(\alpha}^\gamma\delta_{\beta)}^\delta \Leftarrow \boxed{R_{\alpha\beta}{}^{ab} = -\tfrac{1}{2}\sigma^{ab}_{\alpha\beta}\bar{R}}$$

$\sim E^b\wedge E^\alpha\wedge\bar{E}^{\dot\alpha}$ : $0 = 2i\,T_{\alpha c}{}^\beta\sigma^a_{\beta\dot\alpha} + 2i\sigma^a_{\alpha\dot\beta}T_{\dot\alpha c}{}^{\dot\beta} - 2i\sigma^b_{\alpha\dot\alpha}T_{bc}{}^a \Rightarrow$

$$\boxed{T_{\alpha c}{}^\beta = \tfrac{i}{8}(\sigma_c\bar{\sigma}_b)_\alpha{}^\beta G^b}$$

$$\boxed{T_{\dot\alpha c}{}^{\dot\beta} = -\tfrac{i}{8}(\bar{\sigma}_b\sigma_c)^{\dot\beta}{}_{\dot\alpha}G^b}$$

$$\boxed{T_{bc}{}^a = +\tfrac{1}{4}\epsilon^a{}_{bcd}G^d}$$

Resuming: $I_{\underline{\alpha}\beta c}{}^{a} = 0 \Rightarrow$

$$T^{a} = -2i\sigma^{a}_{\alpha\dot\alpha} E^{\alpha}\wedge \bar E^{\dot\alpha} + \frac{1}{8} E^{c}\wedge E^{b}\, \epsilon^{a}{}_{bcd}\, G^{d}$$

$$T^{\alpha} = \frac{i}{8} E^{c}\wedge E^{\beta}(\sigma_c\hat\sigma_\beta)_{\beta}{}^{\alpha} G^{b} - \frac{i}{8} E^{c}\wedge \bar E^{\dot\beta}\sigma_{c\beta}{}^{\alpha} R + \frac{1}{2} E^{c}\wedge E^{b} T_{bc}{}^{\alpha}$$

$$T^{\dot\alpha} = \frac{i}{8} E^{c}\wedge E^{\beta}\sigma_{c\beta}{}^{\dot\alpha}\bar R - \frac{i}{8} E^{c}\wedge \bar E^{\dot\beta}(\sigma_b\hat\sigma_c)^{\dot\alpha}{}_{\dot\beta} G^{b} + \frac{1}{2} E^{c}\wedge E^{b} T_{bc}{}^{\dot\alpha}$$

$$T_{\beta\dot\beta\,\gamma\dot\gamma\,\alpha} := \epsilon_{\beta\gamma}\, t_{\alpha\underline{\dot\beta\dot\gamma}} - \frac{1}{8}\epsilon_{\dot\beta\dot\gamma}\left(W_{\alpha\beta\gamma} + \epsilon_{\alpha(\beta} t_{\gamma)}\right)$$

$$T_{\beta\dot\beta\,\gamma\dot\gamma\,\dot\alpha} := \epsilon_{\dot\beta\dot\gamma}\, \bar t_{\dot\alpha\underline{\beta\gamma}} - \frac{1}{8}\epsilon_{\beta\gamma}\left(\bar W_{\dot\alpha\dot\beta\dot\gamma} + \epsilon_{\dot\alpha(\dot\beta} \bar t_{\dot\gamma)}\right)$$

$$R^{ab} = -\frac{1}{4} E^{\alpha}\wedge E^{\beta}\sigma^{ab}_{\alpha\beta}\bar R - \frac{1}{4}\bar E^{\dot\alpha}\wedge \bar E^{\dot\beta}\hat\sigma^{ab}_{\dot\alpha\dot\beta} R + E^{c}\wedge E^{\underline{\beta}} R_{\underline{\beta}c}{}^{ab} + \frac{1}{2} E^{d}\wedge E^{c} R_{cd}{}^{ab}$$

$$\Leftrightarrow\ R^{\alpha\beta} = -\frac{1}{2} E^{\alpha}\wedge E^{\beta}\bar R + \sim E^{c}, \qquad R^{\dot\alpha\dot\beta} = +\frac{1}{2}\bar E^{\dot\alpha}\wedge \bar E^{\dot\beta} R + \sim E^{c}$$

FURTHER STUDY OF BI'S SHOW THAT ALL THE COMPONENTS OF $T^{A}$ AND $R^{ab}$ ARE EXPRESSED THROUGH THE MAIN SUPERFIELDS: $R$, $\bar R$, $G_a$, $\bar W_{\dot\alpha\dot\beta\dot\gamma}$, $W_{\alpha\beta\gamma}$ AND THEIR DERIVATIVES

THESE WE HAVE SEEN IN THE "ON-SHELL" $T^{A}$ AND $R^{ab}$

dim 5/2: $I_{\underline{\alpha}cb}{}^{a} = 0$

$\sim E^{c}\wedge E^{b}\wedge E^{\underline{\beta}}$

$$0 = R_{\alpha[bc]a} + \underbrace{\frac{1}{2} D_\alpha T_{bca}}_{\frac{1}{8}\epsilon_{abcd} D_\alpha G^{d}} - i\sigma_{a\alpha\dot\alpha} T_{bc}{}^{\dot\alpha} \Rightarrow$$

$$\Rightarrow\ \boxed{R_{\gamma c\, ab} = 2i\sigma_{[a|\gamma\dot\gamma} T_{|b]c}{}^{\dot\gamma} - i\sigma_{c\gamma\dot\gamma} T_{ab}{}^{\dot\gamma} - \frac{1}{8}\epsilon_{abcd} D_\gamma G^{d}}$$

$\frac{1}{4}\sigma^{ab}_{\alpha\beta}\sigma^{c}_{\delta\dot\delta}$

Ex: TO OBTAIN

$\sim E^{c}\wedge E^{b}\wedge \bar E^{\dot\beta}$

$$R_{\gamma\,\delta\dot\delta\,\alpha\beta} = -i\epsilon_{\gamma\delta}\bar t_{\dot\delta\underline{\alpha\beta}} - i\epsilon_{\gamma(\alpha|}\bar t_{\dot\delta\,\delta|\beta)} - \frac{3i}{16}\epsilon_{\gamma(\alpha}\epsilon_{\beta)\delta}\bar t_{\dot\delta} + \frac{i}{8}\epsilon_{\delta(\alpha|}\bar D G_{\gamma|\beta)\dot\delta}$$

$- \text{c.c.}$

$$R_{\dot\gamma\,\delta\dot\delta\,\dot\alpha\dot\beta} = -i\epsilon_{\dot\gamma\dot\delta} t_{\delta\underline{\dot\alpha\dot\beta}} - i\epsilon_{\dot\gamma(\dot\alpha|} t_{\delta\dot\delta|\dot\beta)} - \frac{3i}{16}\epsilon_{\dot\gamma(\dot\alpha}\epsilon_{\dot\beta)\dot\delta} t_{\delta} - \frac{i}{8}\epsilon_{\dot\delta(\dot\alpha|}\bar D_{\dot\gamma} G_{\delta|\dot\beta)}$$

$-\frac{1}{4}\hat\sigma^{ab}_{\dot\alpha\dot\beta}\sigma^{c}_{\delta\dot\delta}$

$$R_{\gamma\,\delta\dot\delta\,\dot\alpha\dot\beta} = i\epsilon_{\dot\delta(\dot\beta}\bar t_{\dot\alpha)\underline{\gamma\delta}} + \frac{i}{4}\epsilon_{\gamma\delta}\bar W_{\dot\alpha\dot\beta\dot\gamma} + \frac{i}{16}\epsilon_{\gamma\delta}\epsilon_{\dot\delta(\dot\alpha}\bar t_{\dot\beta)} - \frac{i}{8}\epsilon_{\dot\delta(\dot\alpha|} D_{\gamma} G_{\delta|\dot\beta)}$$

$- \text{c.c.}$

$$R_{\dot\gamma\,\delta\dot\delta\,\alpha\beta} = i\epsilon_{\delta(\alpha} t_{\beta)\underline{\dot\gamma\dot\delta}} + \frac{i}{4}\epsilon_{\dot\gamma\dot\delta} W_{\alpha\beta\gamma} + \frac{i}{16}\epsilon_{\dot\gamma\dot\delta}\epsilon_{\delta(\alpha} t_{\beta)} + \frac{i}{8}\epsilon_{\delta(\alpha|}\bar D_{\dot\gamma} G_{|\beta)\dot\delta}$$

Ex: TO OBTAIN

dim 3.

$\sim E^{d}\wedge E^{c}\wedge E^{b}$

$$0 = \underbrace{D_{[b} T_{cd]}{}^{a}}_{\frac{1}{4}\epsilon^{a}{}_{[cd]e} G^{e}} + R_{[bcd]}{}^{a}$$

$$\Leftarrow\ \frac{1}{12}\epsilon^{a}{}_{bcd}\, \mathcal{D}_e G^{e} - \frac{1}{12}\mathcal{D}^{a} G^{f}\epsilon_{fbcd}$$

BIs for the fermionic torsion:

$0 \equiv I_3{}^{\alpha} = \mathcal{D}T^{\alpha} + E^{\beta} \wedge R_{\beta}{}^{\alpha}$ and c.c.

$\sim E^{c} \wedge \bar{E}^{\dot\beta} \wedge \bar{E}^{\dot\gamma}$ $\quad \boxed{\bar{\mathcal{D}}_{\dot\gamma} R = 0}$ $\quad$ c.c. $\quad \boxed{\mathcal{D}_{\gamma} \bar{R} = 0}$

$\sim E^{c} \wedge E^{\beta} \wedge E^{\gamma}$ $\quad 0 = - R_{(\gamma|c|\beta)\alpha} + \frac{i}{8}(\sigma_c \tilde{\sigma}_b)_{(\beta|\alpha}\mathcal{D}_{|\gamma)} G^{b}$

$\Leftarrow \dots$

Ex: to obtain

$$0 = \frac{3i}{2}\epsilon_{\alpha|(\beta}\left(t_{\dot\delta\,\underline{\gamma)|\delta)}} - \frac{1}{8}\cdot\frac{1}{2}\left(\mathcal{D}_{\gamma)}G_{|\delta)\dot\delta} + \mathcal{D}_{|\delta)}G_{|\gamma)\dot\delta}\right)\right) - \frac{i}{4}\epsilon_{\alpha\delta}\left(t_{\dot\delta\beta\gamma} - \frac{1}{8}\mathcal{D}_{(\beta}G_{\gamma)\dot\delta}\right) + \frac{3i}{32}\epsilon_{\alpha(\beta}\epsilon_{\gamma)\delta}\left(t_{\dot\delta} - \mathcal{D}^{\varepsilon}G_{\varepsilon\dot\delta}\right)$$

$\Downarrow$

$\boxed{t_{\dot\alpha\beta\gamma} = \frac{1}{8}\mathcal{D}_{(\beta}G_{\gamma)\dot\alpha}}$ $\quad$ c.c. $\quad \boxed{t_{\alpha\dot\beta\dot\gamma} = -\frac{1}{8}\bar{\mathcal{D}}_{(\dot\beta|}G_{\alpha|\dot\gamma)}}$

$\boxed{t_{\dot\alpha} = \mathcal{D}^{\varepsilon}G_{\varepsilon\dot\alpha}}$ $\quad$ c.c. $\quad \boxed{t_{\alpha} = -\bar{\mathcal{D}}^{\dot\varepsilon}G_{\alpha\dot\varepsilon}}$

$\Rightarrow$ $\boxed{R_{\dot\beta\dot\gamma\alpha\beta} = \frac{i}{4}\epsilon_{\dot\beta\dot\gamma}W_{\alpha\beta\gamma}}$ $\Leftrightarrow$ $\boxed{R_{\dot\gamma c}{}^{\alpha\beta} = -\frac{i}{8}\sigma_{c\gamma\dot\gamma}W^{\alpha\beta\gamma}}$

– c.c. $\boxed{R_{\beta\gamma\dot\alpha\dot\beta} = \frac{i}{4}\epsilon_{\beta\gamma}\bar{W}_{\dot\alpha\dot\beta\dot\gamma}}$ $\Leftrightarrow$ $\boxed{R_{\gamma c}{}^{\dot\alpha\dot\beta} = -\frac{i}{8}\sigma_{c\gamma\dot\gamma}\bar{W}^{\dot\alpha\dot\beta\dot\gamma}}$

$$\boxed{R_{\gamma\delta\dot\delta\alpha\beta} = -\frac{i}{4}\epsilon_{\gamma\delta}\mathcal{D}_{(\alpha}G_{\beta)\dot\delta} - \frac{i}{4}\epsilon_{\gamma(\alpha}\epsilon_{\beta)\delta}\,\underbrace{\mathcal{D}^{\varepsilon}G_{\varepsilon\dot\delta}}_{-\bar{\mathcal{D}}_{\dot\delta}\bar{R}}}$$

$\Updownarrow$

$$\boxed{R_{\gamma c}{}^{\alpha\beta} = \frac{i}{8}(\sigma_c\tilde{\sigma}_d)_{\gamma}{}^{(\alpha}\mathcal{D}^{\beta)}G^{d} - \frac{i}{8}\bar{\mathcal{D}}_{\dot\alpha}\bar{R}\,\tilde{\sigma}_c{}^{\dot\alpha(\alpha}\delta_{\gamma}{}^{\beta)}}$$

$\sim E^{c} \wedge E^{\beta} \wedge \bar{E}^{\dot\beta}$ $\quad 0 = \underbrace{-R_{\dot\beta\gamma\dot\gamma\beta\alpha}}_{\Leftarrow\dots} + \frac{i}{8}\sigma^{c}_{\gamma\dot\gamma}\sigma_{c\beta\dot\varepsilon}\bar{\mathcal{D}}_{\dot\beta}G^{\dot\varepsilon}_{\alpha} - \frac{i}{8}\sigma^{c}_{\gamma\dot\gamma}\sigma_{c\alpha\dot\beta}\mathcal{D}_{\beta}R - 2i\underbrace{T_{\dot\beta\gamma\dot\gamma\alpha}}_{\Leftarrow\dots}$

Ex: to obtain

$$0 = -\frac{i}{8}\epsilon_{\beta\gamma}\epsilon_{\dot\beta\dot\gamma}\left(\bar{\mathcal{D}}^{\dot\varepsilon}G_{\alpha\dot\varepsilon} + \mathcal{D}_{\alpha}R\right) + \frac{i}{4}\epsilon_{\dot\beta\dot\gamma}\epsilon_{\alpha(\beta}\left(\underbrace{t_{\gamma)}}_{-\bar{\mathcal{D}}^{\dot\varepsilon}G_{\alpha\dot\varepsilon}} - \mathcal{D}_{\gamma)}R\right)$$

$\Leftrightarrow$ $\boxed{\bar{\mathcal{D}}^{\dot\alpha}G_{\alpha\dot\alpha} = -\mathcal{D}_{\alpha}R}$ – c.c. $\boxed{\mathcal{D}^{\alpha}G_{\alpha\dot\alpha} = -\bar{\mathcal{D}}_{\dot\alpha}\bar{R}}$

dim 5/2 : It is easier to obtain their consequences from $\boxed{0 = \mathcal{D}R^{\alpha\beta}}$ $\Leftarrow$ BIs for curvature.

BIs for curvature:

$0 = DR^{\alpha\beta}$

$\sim E^\alpha \wedge E^\beta \wedge E^\gamma$ $\qquad \boxed{\mathcal{D}_\gamma \bar{R} = 0}$

$\sim E^\gamma \wedge E^\delta \wedge \bar{E}^{\dot\gamma}$ $\qquad \boxed{\mathcal{D}^\varepsilon G_{\varepsilon\dot\alpha} = -\bar{\mathcal{D}}_{\dot\alpha}\bar{R}}$

$\sim E^c \wedge \bar{E}^{\dot\gamma} \wedge \bar{E}^{\dot\delta}$ $\qquad \boxed{\bar{\mathcal{D}}_{\dot\gamma} W_{\alpha\beta\gamma} = 0}$

we have already obtained $\sim E^\gamma \wedge \bar{E}^{\dot\gamma} \wedge \bar{E}^{\dot\delta}$: $R_{(\dot\delta|\gamma|\dot\gamma)\alpha\beta} = 0 \Leftrightarrow \boxed{0=0}$

$\sim E^c \wedge E^\gamma \wedge \bar{E}^{\dot\gamma}$ $\qquad 0 = -2i\sigma_{\gamma\dot\gamma}^{b} R_{bc}{}^{\alpha\beta} + \bar{\mathcal{D}}_{\dot\gamma} R_{\gamma c}{}^{\alpha\beta} + \mathcal{D}_\gamma R_{\dot\gamma c}{}^{\alpha\beta} - \frac{i}{8}\sigma_{c\dot\gamma}{}^{(\alpha}\delta_\gamma{}^{\beta)} R\bar{R}$

$$\Downarrow$$

$$\boxed{\mathcal{D}^\gamma W_{\alpha\beta\gamma} = \bar{\mathcal{D}}^{\dot\beta}\mathcal{D}_{(\alpha} G_{\beta)\dot\beta}}$$

$$\mathcal{D}_\delta W_{\alpha\beta\gamma} = C_{\alpha\beta\gamma\delta} + \frac{3}{4}\epsilon_{\delta(\alpha}\mathcal{D}^\varepsilon W_{\beta\gamma)\varepsilon}$$

$$R_{\gamma\dot\gamma\,\delta\dot\delta\,\alpha\beta} = -\frac{1}{8}\epsilon_{\gamma\delta}\bar{\mathcal{D}}_{(\dot\gamma|}\mathcal{D}_{(\alpha}G_{\beta)|\dot\delta)} + \frac{1}{8}\epsilon_{\dot\gamma\dot\delta}\mathcal{D}_{(\delta}W_{\alpha\beta\gamma)} - \frac{1}{16}\epsilon_{\dot\gamma\dot\delta}\epsilon_{\gamma(\alpha}\epsilon_{\beta)\delta}(\bar{\mathcal{D}}\bar{\mathcal{D}}\bar{R} - 2R\bar{R}) +$$
$$+\frac{1}{16}\epsilon_{\dot\gamma\dot\delta}\epsilon_{(\gamma|(\alpha}\mathcal{D}^\varepsilon W_{\beta)|\delta)\varepsilon},$$

$$R_{\gamma\dot\gamma\,\delta\dot\delta\,\alpha\beta} = 2\epsilon_{\gamma\delta}\,\tau_{\dot\gamma\dot\delta\,\alpha\beta} + 2\epsilon_{\dot\gamma\dot\delta}\,\underbrace{\tau_{\gamma\delta\,\alpha\beta}}_{}$$

$$\Leftarrow \tilde{C}_{\alpha\beta\gamma\delta} + \epsilon_{(\alpha|(\gamma}\tau_{\delta)|\beta)} + \epsilon_{\alpha(\gamma}\epsilon_{\delta)\beta}\tau,$$

$$\boxed{\tilde{C}_{\alpha\beta\gamma\delta} = \frac{1}{16}\mathcal{D}_{(\alpha}W_{\beta\gamma\delta)}},$$

$$\boxed{\tau_{\dot\gamma\dot\delta\,\alpha\beta} = -\frac{1}{16}\bar{\mathcal{D}}_{(\dot\gamma|}\mathcal{D}_{(\alpha}G_{\beta)|\dot\delta)}}, \qquad \boxed{\tau_{\alpha\beta} = -\frac{1}{32}\mathcal{D}^\gamma W_{\alpha\beta\gamma}}$$

$$\boxed{\tau = -\frac{1}{32}(\bar{\mathcal{D}}\bar{\mathcal{D}}\bar{R} - 2R\bar{R})} \quad \text{c.c.} \quad \boxed{\bar{\tau} = -\frac{1}{32}(\mathcal{D}\mathcal{D}R - 2R\bar{R})}$$

$$\boxed{\mathcal{R} := R_{ab}{}^{ab} = -\frac{3}{16}(\mathcal{D}\mathcal{D}R + \bar{\mathcal{D}}\bar{\mathcal{D}}\bar{R} - 4R\bar{R})}$$

$$\mathcal{R}_{ca} := R_{cba}{}^{b} = +\frac{1}{2}\tilde{\sigma}_c^{\dot\gamma\beta}\tilde{\sigma}_a^{\dot\delta\alpha}(\tau_{\dot\gamma\dot\delta\,\alpha\beta} + \bar{\tau}_{\alpha\beta\,\dot\gamma\dot\delta}) + \frac{1}{2}(\sigma_{ca})^{\alpha\beta}\tau_{\alpha\beta} +$$
$$+\frac{1}{2}\tilde{\sigma}_{ca}{}^{\dot\alpha\dot\beta}\,\bar{\tau}_{\dot\alpha\dot\beta} + \frac{3}{2}(\tau + \bar{\tau})\eta_{ca}$$

SUPERFIELD GENERALIZATION OF THE L.H.S. OF THE RS EQ.:

$$\sigma^b_{\beta\dot\beta}\sigma^c_{\gamma\dot\gamma}T_{bc\alpha} =: T_{\beta\dot\beta\gamma\dot\gamma\alpha} = \epsilon_{\beta\gamma}\underbrace{t_{\alpha\dot\beta\dot\gamma}}_{-\frac{1}{8}\bar{\mathcal{D}}_{(\dot\beta|}G_{\alpha|\dot\gamma)}} - \frac{1}{8}\epsilon_{\dot\beta\dot\gamma}(W_{\alpha\beta\gamma} + \epsilon_{\alpha(\beta}\underbrace{t_{\gamma)}}_{\mathcal{D}_\gamma R = -\bar{\mathcal{D}}^{\dot\gamma}G_{\gamma\dot\gamma}})$$

$$\mathcal{D}_\alpha R = -\bar{\mathcal{D}}^{\dot\alpha}G_{\alpha\dot\alpha} = -\frac{8}{3}\sigma^{bc}{}_\alpha{}^\beta T_{bc\beta}$$

$$\bar{\mathcal{D}}_{(\dot\beta|}G_{\alpha|\dot\gamma)} = -4\,\tilde\sigma^{bc}_{\dot\beta\dot\gamma}T_{bc\alpha}$$

$$\text{c.c.}\quad \mathcal{D}_{(\beta}G_{\gamma)\dot\gamma} = -4\sigma^{bc}_{\beta\gamma}T_{bc\dot\alpha}$$

$$T_{bc\,\alpha} = -\frac{1}{32}\tilde\sigma_{bc}{}^{\dot\beta\dot\gamma}\bar{\mathcal{D}}_{\dot\beta}G_{\alpha\dot\gamma} + \frac{1}{32}\sigma_{bc}{}^{\beta\gamma}W_{\alpha\beta\gamma} + \frac{1}{32}\sigma_{bc\alpha}{}^\beta\mathcal{D}_\beta R$$

$$\Updownarrow$$

$$\boxed{T_{bc}{}^\alpha = -\frac{1}{32}\tilde\sigma_{bc\dot\beta\dot\gamma}\bar{\mathcal{D}}^{\dot\beta}G^{\alpha\dot\gamma} - \frac{1}{32}\mathcal{D}^\beta R(\sigma_{bc})_\beta{}^\alpha + \frac{1}{32}\sigma_{bc\,\beta\gamma}W^{\alpha\beta\gamma}}$$

$$\boxed{\Psi_{\dot\alpha}{}^a := \epsilon^{abcd}T_{bc}{}^\alpha\sigma_{d\alpha\dot\alpha} = -\frac{i}{4}\bar{\mathcal{D}}_{\dot\alpha}G^a + \frac{i}{4}\sigma^a_{\alpha\dot\alpha}\mathcal{D}^\alpha R}$$

$$\tilde\sigma_a{}^{\dot\alpha\alpha}\Psi_{\dot\alpha}{}^a = +\frac{3i}{4}\mathcal{D}^\alpha R$$

$$\sigma_{a\alpha\dot\alpha}\Psi^{\dot\alpha a} = -\frac{3i}{4}\mathcal{D}_\alpha R \iff \mathcal{D}_\alpha R = -\frac{8}{3}\sigma^{bc}{}_\alpha{}^\beta T_{bc\beta}$$

$$\boxed{\mathcal{D}_\alpha R = \frac{4i}{3}\sigma_{a\alpha\dot\alpha}\Psi^{\dot\alpha a}}$$

$$\boxed{\bar{\mathcal{D}}_{\dot\alpha}G^a = 4i\left(\Psi_{\dot\alpha}{}^a - \frac{1}{3}\Psi_{\dot\beta}{}^b(\tilde\sigma_b\sigma^a)^{\dot\beta}{}_{\dot\alpha}\right)}$$

- $\mathcal{D}_\alpha R = -\bar{\mathcal{D}}^{\dot\alpha} G_{\alpha\dot\alpha} \quad \Rightarrow \quad \mathcal{D}_\alpha \mathcal{D}_\beta R = \frac{1}{2}\epsilon_{\alpha\beta}\mathcal{D}\mathcal{D}R = -\mathcal{D}_\alpha \bar{\mathcal{D}}^{\dot\beta} G_{\beta\dot\beta}$

$$\underset{c.c.}{\Rightarrow} \left.\begin{aligned} \mathcal{D}\mathcal{D}R &= -\mathcal{D}^\alpha \bar{\mathcal{D}}^{\dot\alpha} G_{\alpha\dot\alpha} \\ \bar{\mathcal{D}}\bar{\mathcal{D}}\bar{R} &= \bar{\mathcal{D}}^{\dot\alpha}\mathcal{D}^\alpha G_{\alpha\dot\alpha} \end{aligned}\right\} \Rightarrow \boxed{\mathcal{D}\mathcal{D}R - \bar{\mathcal{D}}\bar{\mathcal{D}}\bar{R} = -4i\mathcal{D}_a G^a} \Bigg\} \Rightarrow$$

- $\mathcal{R} := R_{ab}{}^{ab} = -\frac{3}{16}\left(\mathcal{D}\mathcal{D}R + \bar{\mathcal{D}}\bar{\mathcal{D}}\bar{R} - 4R\bar{R}\right) \Rightarrow \mathcal{D}\mathcal{D}R + \bar{\mathcal{D}}\bar{\mathcal{D}}\bar{R} = -\frac{16}{3}R_{ab}{}^{ab} + 4R\bar{R}$

$$\Rightarrow \quad \boxed{\mathcal{D}\mathcal{D}R = -\frac{8}{3}R_{ab}{}^{ab} + 2R\cdot\bar{R} - 2i\mathcal{D}_a G^a}$$

THUS ALL THE COMPONENTS OF CHIRAL SUPERFIELDS R ARE

$R\big|_0$ = aux. complex scalar

$\mathcal{D}_\alpha R\big|_0 = \frac{4i}{3}\, \sigma_{a\alpha\dot\alpha}\, \psi^{\dot\alpha a}\big|_0$ GRAVITINO EQ. OF. MOTION

$$\mathcal{D}\mathcal{D}R\big|_0 = \underbrace{-\frac{8}{3}R_{ab}{}^{ab}\big|_0}_{\text{Riemann curvature scalar (+... see tetra)}} + 2\underbrace{R\cdot\bar{R}\big|_0 - 2i\mathcal{D}_a G^a\big|_0}_{\text{aux. fields}}$$

SUPERFIELD EQS OF MOTION OF 1N 4D SG ARE THUS

$$R = 0 \quad , \quad \bar{R} = 0$$

$$G_a = 0$$

REASON: THIS RESULTS IN (SUPERFIELD GENERALIZATIONS OF) THE Einstein and RS eqs.

$$\Psi_{\dot\alpha}{}^{a} := \epsilon^{abcd} T_{bc}{}^{\alpha} \sigma_{d\alpha\dot\alpha} = 0 \qquad \left(\text{as } \bar{\mathcal{D}}_{\dot\alpha} G^{a} = 4i\left(\Psi_{\dot\alpha}{}^{a} - \tfrac{1}{3}(\Psi^{b}\tilde\sigma_{b}\sigma^{a})_{\dot\alpha}\right)\right)$$

$$R_{ac}{}^{bc} = 0$$

$$\Updownarrow ?$$

$$R_{ac}{}^{bc} - \tfrac{1}{2}\delta_a^b R_{cd}{}^{cd} = 0$$

$$\Big( \text{as } R_{cd\beta}{}^{\alpha} = -\mathcal{D}_\beta T_{cd}{}^{\alpha} + \sim G_a, R, \bar{R}$$

$$\Downarrow$$

$$\mathcal{D}_\alpha \Psi_{\dot\alpha}{}^{a} = -\epsilon^{abcd} R_{bc\alpha}{}^{\beta} \sigma_{d\beta\dot\alpha} + G_a, R, \bar{R}$$

$$= -\epsilon^{abcd} \tfrac{1}{4} R_{bc}{}^{ef} (\sigma_{ef}\sigma_a)_{\alpha\dot\alpha} =$$

$$= -i\left(R_{bc}{}^{ac} - \tfrac{1}{2}\delta_b^a R_{cd}{}^{cd}\right)\sigma^{b}_{\alpha\dot\alpha} + \sim \underbrace{R_{[bcd]e}}_{=0}\sigma^{e}_{\alpha\dot\alpha} \Big)$$

Below we will also discuss the superfield action [Wess & Zumino 77] which reproduces these eqs.

SUSY TRANSFORMATIONS OF THE OFF-SHELL SG.

AUXILIARY FIELDS.

$E^a| = e^a$, $E^\alpha| = \psi^\alpha$

FROM OUR SSP CONSTRAINTS, WITH $i_\epsilon E^a = 0$, $i_\epsilon E^\alpha = \epsilon^\alpha$

$$\delta_\epsilon e^a = i_\epsilon T^a| = -2i\,\psi\sigma^a\bar{\epsilon} + 2i\,\epsilon\sigma^a\bar{\psi}$$

$$\delta_\epsilon \psi^\alpha = i_\epsilon T^\alpha| = D\epsilon^\alpha + \frac{i}{8} e^c_\wedge \left( (\psi\sigma_c\tilde{\sigma}_d)^\alpha \mathcal{G}^d|_0 + (\bar{\psi}\tilde{\sigma}_c)^\alpha R|_0 \right)$$

$$\delta_\epsilon \bar{\psi}^{\dot\alpha} = i_\epsilon \bar{T}^{\dot\alpha}| = D\bar{\epsilon}^{\dot\alpha} - \frac{i}{8} e^c \left( (\tilde{\sigma}_c\psi)^{\dot\alpha} \cdot \bar{R}|_0 + (\tilde{\sigma}_b\sigma_c\bar{\psi})^{\dot\alpha} \mathcal{G}^b|_0 \right)$$

GRAVITINO TRANSFORMATIONS INVOLVE THE CONTRIBUTIONS OF AUXILIARY FIELDS OF THE MIN SUGRA MULTIPLET:

COMPLEX SCALAR $R|_0$ and its c.c. $\bar{R}|_0$

REAL VECTOR $\mathcal{G}_a|_0$.

THEIR TRANSFORMATION RULES CAN BE OBTAINED FROM EXPRESSIONS FOR $\mathcal{D}_\alpha R$ and $\mathcal{D}_\alpha \mathcal{G}_a$:

$$\delta_\epsilon R = \frac{4i}{3}\epsilon^\alpha\sigma_{a\alpha\dot\alpha}\Psi^{\dot\alpha a}|_0 \,, \qquad \delta_\epsilon \bar{R} = \frac{4i}{3}\bar{\Psi}^{\alpha a}\sigma_{a\alpha\dot\alpha}\bar{\epsilon}^{\dot\alpha}$$

$$\delta_\epsilon \mathcal{G}^a = 4i\left(\epsilon^\alpha\bar{\Psi}_\alpha{}^a - \Psi_{\dot\alpha}{}^a\bar{\epsilon}^{\dot\alpha}\right)\Big|_0 + \frac{4i}{3}\left(\epsilon\sigma^a\tilde{\sigma}^b\bar{\Psi}_b + \Psi_b\tilde{\sigma}^b\sigma^a\bar{\epsilon}\right)\Big|_0$$

where $\Psi_{\dot\alpha}{}^a := \epsilon^{abcd}\, T_{bc}{}^\alpha\sigma_{d\alpha\dot\alpha}$

THE (SUPER)ALGEBRA OF THESE SUSY TRANSFORMATIONS CLOSES OFF-SHELL (OFF THE MASS SHELL).

THIS IS IN CORRESPONDENCE WITH OFF-SHELL NATURE OF OUR SG CONSTRAINTS (THAT THESE DO NOT RESULT IN EQS. OF MOTION).

## COMMENT ON CONVENTIONAL CONSTRAINTS AND $W^{ab}$ REDEFINITION

IN OUR SET OF CONSTRAINTS INCLUDES $R_{\alpha\beta}{}^{ab}=0$, AND, AS A RESULT, $T_{ab}{}^{c}\neq 0$ $\quad T_{bc}{}^{a}=\frac{1}{8}\epsilon^{a}{}_{bcd}G^{d}$.

OUR LORENTZ CONNECTION $W^{ab}$ IS RELATED TO THE CONNECTION $\overset{old}{W}$ OBEYING $T_{bc}{}^{a}=0$ constraints BY "SHIFT":

$$W^{ab}=\overset{o}{W}{}^{ab}+\Delta W^{ab}, \qquad \boxed{\Delta W^{ab}=-\frac{1}{8}E^{c}\epsilon^{ab}{}_{cd}G^{d}}$$

EX: TO FIND THE RELATION OF $R^{ab}$ TO $\overset{o}{R}{}^{ab}=d\overset{o}{W}{}^{ab}-\overset{o}{W}{}^{ac}\wedge\overset{o}{W}{}_{c}{}^{b}$

$$R^{ab}=\overset{o}{R}{}^{ab}+\mathcal{D}\Delta W^{ab}-\Delta W^{ac}\wedge\Delta W_{c}{}^{b}=$$

$$=\overset{o}{R}{}^{ab}+\frac{i}{4}E^{\alpha}\wedge\bar{E}^{\dot\alpha}\epsilon^{ab}{}_{cd}\sigma^{c}_{\alpha\dot\alpha}G^{d}-\frac{1}{8}E^{c}\wedge\overset{o}{\mathcal{D}}G^{d}\epsilon^{ab}{}_{cd}+$$

$$-\frac{1}{32}E^{d}\wedge E^{c}\delta_{c}^{[a}G^{b]}G_{d}-\frac{1}{64}E^{a}\wedge E^{b}G^{c}G_{c}$$

$$\boxed{R_{\alpha\dot\alpha}{}^{ab}=0=\overset{o}{R}{}_{\alpha\dot\alpha}{}^{ab}+\frac{i}{4}\epsilon^{abcd}\sigma_{c\alpha\dot\alpha}G_{d}}$$

Thus $\overset{o}{R}{}_{\alpha\dot\alpha}{}^{ab}=-\frac{i}{8}\epsilon^{abcd}\sigma_{c\alpha\dot\alpha}G_{d}\neq 0$ !

$$\boxed{R_{\gamma c}{}^{ab}=\overset{o}{R}{}_{\gamma c}{}^{ab}-\frac{1}{8}\epsilon^{ab}{}_{cd}D_{\gamma}G^{d}}$$

$$\boxed{R_{cd}{}^{ab}=\overset{o}{R}{}_{cd}{}^{ab}-\frac{1}{4}\epsilon^{abe}{}_{[c}D_{d]}G_{e}+\frac{1}{16}G^{[a}\delta^{b]}_{[c}G_{d]}+\frac{1}{32}\delta^{a}_{[c}\delta^{b}_{d]}G^{f}G_{f}}$$

$$\boxed{R_{cb}{}^{ab}=\overset{o}{R}{}_{cb}{}^{ab}+\frac{1}{8}\epsilon_{c}{}^{aef}\overset{(o)}{D}_{e}G_{f}-\frac{1}{32}\left(G_{c}G^{a}-\delta_{c}^{a}G_{b}G^{b}\right)}$$

$$\boxed{R_{ab}{}^{ab}=\overset{o}{R}{}_{ab}{}^{ab}+\frac{3}{32}G_{a}G^{a}}$$