Sections and Multiple fibration in CICY 3-Folds

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VirginiaTech Invent the Future®

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Outline

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F-theory Compactification

F-theory: geometric reinterpretation of orientifold IIB theory with (p,q) 7-brane and varying axion-dilaton τ (auxilary 2-torus).

Geometric: Elliptic fibration on Calabi-Yau 3-fold (or 4-folds).

$$\pi : X_3 \xrightarrow{\mathbb{E}} B_2$$

If the fibration has a section, X_3 can be written in Weierstrass:

$$\mathbb{E} : y^2 = x^3 + f(u)xz^4 + g(u)z^6$$

Singular elliptic fibers over discriminant $\Delta = 4f^3 + 27g^2 = 0$.

7-brane wrapping on the divisors $D \in B_2$ and their intersection $\Rightarrow$ Gauge Group, Matter, Yukawa ...

Today, focus on 6d theory (Compactified on $\mathbb{E}CY_3$). Parallel for $\mathbb{E}CY_4$.

Motivations I

The Section is still mysterious.

What we know:

- Holomorphic $s : B_2 \hookrightarrow X_3$. Rational $s' : B_2 \dashrightarrow B_2 \hookrightarrow X_3$
- The sections form an Abelian group called Mordell-Weil group

$$\# \, U(1)s = \mathrm{rk}(MW) \leq h^{1,1}(X_3) - h^{1,1}(B_2) - 1$$

- ...

Questions 1: Given a fibration structure, does it have sections ?
Questions 2: If yes, can we construct all these sections explicitly ?

Motivations II

The fibration structure plays a important role in F-theory duality:

- F-theory on K3 fibration n+1-folds $\Leftrightarrow$ Heterotic on elliptic fibration n-folds
- F-theory on elliptic fibration n+1-folds $\Leftrightarrow$ Type IIB on n-folds
- . . .

Moreover, many manifolds not only have one fibrations, but many.

$$\mathrm{Multiplefibration} \Leftrightarrow \mathrm{F-theory\ duality} \ \Leftrightarrow \mathrm{String\ duality\ webs}$$

Questions 3: What are these geometry and the string dualities?
Questions 4: How many of them?

Outline

How to Identify the Section S Topologically

Divisor of the section s by S, linebunlde by $[S] = \mathcal{O}_{X_3}(S)$.
Some criteria for section $s \in \Gamma(X_3, \mathcal{O}_{X_3}(S))$:

- Oguiso criteria: $\forall \text{pt.}\, p \in B_2$, $S \cdot F_p = 1$
- Intersection criteria: $S^2 \cdot D_\alpha = -[c_1(B_2)] \cdot S \cdot D_\alpha$ for $D_\alpha \in B_2$
- Euler number of Section: $\chi(S) \geq \chi(B_2)$, "=" for holomorphic
- Cohomology criteria: Non-trivial linebundle cohomology $h^0(X_3, \mathcal{O}_{X_3}(S)) > 0$ which count the number of global sections defined by linebundle $\mathcal{O}_{X_3}(S)$.

Anderson, Antoniadis, Bizet, Borchmann, Braun, Braun, Choi, Collinucci, Cvetic, Donagi, Etxebarria, Grassi, Gray, Grimm, Hayashi, Keitel, Klevers, Kuntzler, Krippendorf, Klemm, Leontaris, Lopes, Mayrhofer, Mayorga, Morrison, Oehlmann, Park, Palti, Piragua, Pena, Piragua, Ruhle, S-Nameki, Song, Taylor, Valandro,Weigand . . .

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Is this sufficient to determine the section? May not.

A 2-fold Example: K3

$$X_2 = \left[\begin{array}{c|cc} \mathbb{P}^1_x & 1 & 1 \\ \mathbb{P}^2_y & 1 & 2 \\ \mathbb{P}^1_z & 1 & 1 \end{array} \right] .$$

defined by coordinate ring: $[P^{(1,1,1)}_{x,y,z} = 0,\ Q^{(1,2,1)}_{x,y,z} = 0]$

Linebundle of the putative section: $[S] = \mathcal{O}_X(\alpha[1], \alpha[2], \alpha[3])$

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- Oguiso: $S \cdot F_p = 1 :\ 2\alpha[1] + 3\alpha[2] = 1$
- Intersection: $2\alpha[1] + 3\alpha[2] = -\alpha[1](3\alpha[2] + 2\alpha[3]) - \alpha[2](\alpha[2] + 3\alpha[3])$
- Euler: $-6\alpha[1]\alpha[2] - 2\alpha[2]^2 - 4\alpha[1]\alpha[3] - 6\alpha[2]\alpha[3] \geq \chi[B_1] = 2$
- Cohomology: $h^0(X_2, \mathcal{O}(S)) > 0$

$\Longrightarrow$ 2 putative holomorphic section and 0 rational section:

$$[S] = \mathcal{O}_X(-1,1,1),\ \mathcal{O}_X(2,-1,4) \text{ with } h^*(X_2, \mathcal{O}_X(S)) = (1,0,0)$$

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Q: Are they really sections? Can we construct them explicitly?

Divisors of Section $[S] = \mathcal{O}_X(-1,1,1)$

- The divisor of section S can be split as two parts, in terms of
  - divisor of pole $[S_P] = \mathcal{O}(1,0,0)$
  - divisor of zero $[S_Z] = \mathcal{O}(0,1,1)$

$\forall p \in B$, $F_p \cdot S_P = 2$ while $F_p \cdot S_Z = 3$, s.t $F_p \cdot S = 1$.

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- $$s = \frac{N[y,z]}{D[x]}$$

$\forall p \in B$, $D[x]$ has two zeros on the fiber, which should be match with two of the three zeros of $N[y,z]$ on the fiber in the coordinate ring $[P^{(1,1,1)}_{x,y,z} = 0, Q^{(1,2,1)}_{x,y,z} = 0]$. The remaining one point gives the single physical intersection point of $F_p \cap S$.

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$\Longrightarrow$ Matching the zeros of numerator and denominator s.t $s \in \Gamma(X, \mathcal{O}_X(S))$, the free parameter in s count the number of sections, which in this case should be $h^0(X, \mathcal{O}_X(S)) = 1$.

Construct the Section

1. Choose generic complex structure of K_3 :

$$\begin{aligned}P^{(1,1,1)}_{x,y,z} &= 8x_0y_0z_0 + 11x_1y_0z_0 + 17x_0y_1z_0 + 11x_1y_1z_0 + 18x_0y_2z_0 + 6x_1y_2z_0 \\ &+12x_0y_0z_1 + 8x_1y_0z_1 + 19x_0y_1z_1 + 14x_1y_1z_1 + 5x_0y_2z_1 + 3x_1y_2z_1, \\ Q^{(1,2,1)}_{x,y,z} &= x_0y_0^2z_0 + 20x_1y_0^2z_0 + 2x_0y_0^2z_1 + 18x_1y_0^2z_1 + 3x_0y_1y_0z_0 + 7x_1y_1y_0z_0 \\ &+13x_0y_2y_0z_0 + 4x_1y_2y_0z_0 + 5x_0y_1y_0z_1 + 17x_1y_1y_0z_1 + 10x_0y_2y_0 + 8x_1y_2y_0z_1z_1 \\ &+17x_0y_1^2z_0 + 7x_1y_1^2z_0 + 7x_0y_2^2z_0 + 18x_1y_2^2z_0 + 20x_0y_1y_2z_0 + 14x_1y_1y_2z_0 \\ &+18x_0y_1^2z_1 + 16x_1y_1^2z_1 + 4x_0y_2^2z_1 + x_1y_2^2z_1 + 13x_0y_1y_2z_1 + 20x_1y_1y_2z_1\end{aligned}$$

2. Choose random coefficient of denominator $D[x] \in H^0(X, [S_P])$ and let the numerator $N[y,z] \in H^0(X, [S_Z])$ free:

$$\begin{aligned}D[x] &= 19x_0 + 5x_1 \\ N[y,z] &= \mathsf{Sz}_1y_0z_0 + \mathsf{Sz}_2y_0z_1 + \mathsf{Sz}_3y_1z_0 + \mathsf{Sz}_4y_1z_1 + \mathsf{Sz}_5y_2z_0 + \mathsf{Sz}_6y_2z_1\end{aligned}$$

3. Choose many random points on the base $\mathbf{z} \in [z_0 : z_1] \in \mathbb{P}^1_z$, solve the coordinate ring to get the two points on each fiber $S_P \cap F_p$:

$$\{P[x,y,\mathbf{z}] = 0,\ Q[x,y,\mathbf{z}] = 0,\ D[x] = 0\}$$

4. Submit all the solutions to numerator $N[y,z]$ get a highly constrained linear system for Sz_i. Solving $N[\mathbf{y},\mathbf{z}] = 0$:

$$\begin{aligned}&\mathsf{Sz}_2 \to (0.5443786 + 0.i)\mathsf{Sz}_1, \quad \mathsf{Sz}_3 \to (0.7337278 + 0.i)\mathsf{Sz}_1, \\ &\mathsf{Sz}_4 \to (1.0118343 + 0.i)\mathsf{Sz}_1, \quad \mathsf{Sz}_5 \to (0.1420118 + 0.i)\mathsf{Sz}_1, \\ &\mathsf{Sz}_6 \to (0.1893491 + 0.i)\mathsf{Sz}_1\end{aligned}$$

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For $[S] = \mathcal{O}_X(2,-1,4)$, has only zero solution $\mathsf{Sz}_i \to 0 \Rightarrow$ No section.

Outline

Multiplicity of Fibration

The geometry of multiple-fibrations was introduced @ James's talk

One family of multiple-fibrations $\mathbb{E}CY_3 \Leftrightarrow$

- Different axion-dilatons/weakly coupled theories in Type IIB.
- The same M-theory limit.
- Different Heterotic dual: weak/weak or strong /weak.
- e.g. One geometry with double K3-fiber $h^{1,1}(X) = 8, h^{1,2}(X) = 28$.

$$\underset{B=\mathbb{P}^1}{\left[\begin{array}{c|ccc} \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^2 & 0 & 1 & 2 \\ \mathbb{P}^1 & 2 & 0 & 0 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{array}\right]} \Leftrightarrow \underset{B=\mathbb{P}^1}{\left[\begin{array}{c|ccc} \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^2 & 1 & 2 & 0 \\ \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \end{array}\right]};$$

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Anderson, Aspinwall, Duff, Ferrara, Gross, Harvey, Hayashi, Kachru, Klemm, Lerche, Louis, Mayr, Morrison, Minasian, Strominger, Tatar, Taylor, Toda, Vafa, Watari, Witten, Yamazaki ...

- e.g. K3 is also elliptic fibered.

$$\left[\begin{array}{c|ccc} \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^2 & 0 & 1 & 2 \\ \mathbb{P}^1 & 2 & 0 & 0 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{array}\right] \Leftrightarrow \left[\begin{array}{c|ccc} \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^2 & 1 & 2 & 0 \\ \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \end{array}\right];$$

$h^{1,1}(B) = 6, \chi(B) = 8.$

$[S] = \mathcal{O}_X(-1,1,0,1,0), \text{ w. } \chi(S) = 24$

$B = \mathbb{P}^1 \times \mathbb{P}^1, h^{1,1}(B) = 2, \chi(B) = 4.$

$[S] = \mathcal{O}_X(-1,0,1,2,1), \text{w.} \chi(S) = 48,$

$\mathcal{O}_X(-1,1,1,0,0), \text{ w. } \chi(S) = 24$

Scan procedure

- Start with obvious Fibered CY_3 CICY geometry (7868 geometry with 77991 configuration matrixes), check whether it contains section by scanning each entrance of putative section linebundle from -5 to 5.
  - Oguiso+intersection (Constrain 1)
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- Find Multiple fibration with section
  - e.g. Elliptic Multiple fibration on different bases
  - e.g. K3 multiple-fibration

Statistic of the Multiple fibration in CICY CY_3 I

Configuration matrixes with # of rows smaller or equal than 5.
1934 geometries with 7444 configuration matrixes.

	# geometry	# configurations	# holomorphic section	# rational section
Constrain 1	1204	2734	31 / 50 / 75	1204 / 2734 / 34827
Constrain 2	1204	2734	0	1204 / 2734 / 10485
Constrain 3	1080	2391	0	5400
K3-fiber	627	815	0	1422
K3-multiplefiber	142	330	0	563

# geometry w/ multiple fibration on diff bases	2	3	4	more
499	423	60	16	0

Statistic of the Multiple fibration in CICY CY_3 II

Configuration matrixes with # of rows equal to 6.
1815 geometries with 13586 configuration matrixes.

	# geometry	# configurations	# holomorphic section	# rational section
Constrain 1	1660	6461	272 / 571 / 1254	1660 / 6461 / 315844
Constrain 2	1660	6461	0	1660 / 6461 / 58413
Constrain 3	1632	6260	0	26370
K3-fiber	1408	2709	0	10004
K3-multiple fiber	746	2047	0	7608

# geometry w/ multiple fibration on diff bases	2	3	4	more
1041	561	394	81	5

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- Parallel to Calabi-Yau 4-folds case.
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