Going beyond geometry with F-theory

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Non-geometry as a generalization of geometry

We are used to constructing manifolds by patching. That is, we specify the global structure of our theory by defining the theory on a set of open patches with topology $\mathbb{R}^n$, and specify a gluing action on the fields when the different patches overlap.

The patching should be by a physical equivalence of the object begin patched:

- For a geometric object (such as a fiber in a fibration) we can patch by diffeomorphisms.
- Or more generally, for a charged field, we can patch by gauge transformations.

Non-geometry as a generalization of geometry

These patching transition functions appear naturally when we have *monodromy* around a certain object. A familiar example is the $D7$ brane in IIB: as we go around the core of the $D7$ we have a monodromy $C_0 \to C_0 + 1$.

We often think of this in the language of F-theory: geometrize the axio-dilaton $\tau = C_0 + i/g_s$ as the complex structure of a T^2, and view the $C_0 \to C_1 + 1$ as a geometric monodromy, coming from a non-trivial fibration of a torus.

One can try to generalize this viewpoint: we view the patching as a geometric transformation in some auxiliary space.
[Candelas,Constantin,Damian,Larfors,Morales] [Martucci,Morales,Pacifici] [Braun,Fucito,Morales]

Non-geometry as a generalization of geometry

If we have a monodromy in F-theory, we can read off the light degrees of freedom (let us stay in 8d, for simplicity).

For example, if the monodromy is $\tau \to \tau + 1$ (and we have a supersymmetric solution), we know that we have a $D7$ brane, with field content $\mathcal{N} = 1$ $U(1)$ SYM in 8d, at least locally.

More complicated setups are understood similarly: if we have

$$M = \begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix} \tag{1}$$

we have a $SO(8+2n)$ stack, etc.[1]

[1] We are ignoring discrete fluxes here.

Non-geometry as a generalization of geometry

More formally, we have a **Kodaira classification**: we can classify all the possible ways in which we can have a degeneration of the fiber while keeping supersymmetry, or equivalently all the local monodromies using $SL(2,\mathbb{Z})$ patching.

By a number of techniques (for example, by going to the M-theory dual and resolving) we can read off the gauge dynamics for each configuration in the Kodaira list.

Non-geometry as a generalization of geometry

We are not restricted to $SL(2,\mathbb{Z})$, any symmetry of string theory will do! It could be geometric, or it may be something like T-duality: as we surround the defect the size of the fibered dimensions gets inverted. (Interesting because in this case there is no global notion of geometry.)

A basic question is how to read the light degrees of freedom living on the soliton.

Heterotic T-folds

We will be focusing on a particularly interesting class of non-geometric theories: non-geometric heterotic backgrounds.

In particular, if we take the $(E_8 \times E_8)$ heterotic string on T^2 without Wilson lines, it has a U-duality group given by $\mathfrak{D} = (SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho) \rtimes \mathbb{Z}_2^2$. Here τ is the complex structure of the torus, and $\rho = B + iJ$ is the complexified Kähler form.

We can construct non-trivial backgrounds by fibering over a base ($\mathbb{P}^1$ is the minimal compact case) with patching by elements in $\mathfrak{D}$.

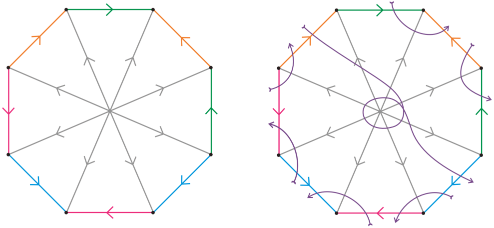
Genus two fibrations

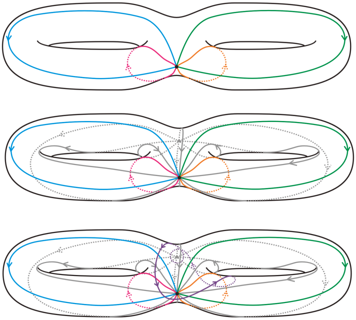
In the spirit of F-theory, we could think of the duality group $\mathfrak{D} = (SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho) \rtimes \mathbb{Z}_2^2$ as the natural action on the union of two T^2.

This picture has been recently sharpened by generalizing to the case where there is a single non-trivial Wilson line on the T^2, breaking $E_8 \times E_8 \to E_7 \times E_8$. [Clingher,Doran] [Morrison,Malmendier] [Jockers,Gu]. It turns out that the set of vacua of this theory is isomorphic to set of complex structures for a genus 2 surface.

The formalism is isomorphic to that appearing in "G-theory". [Candelas,Constantin,Damian,Larfors,Morales] [Martucci,Morales,Pacifici] [Braun,Fucito,Morales]

Genus two fibrations





Genus two fibrations

More precisely, the physical data of the T^2 compactification with a Wilson line in $SU(2) \subset E_8$ is given by the Narain space

$$\frac{O(2,3,\mathbb{R})}{O(2,\mathbb{R}) \times O(3,\mathbb{R})} \tag{2}$$

modulo the U-duality group $O(2,3,\mathbb{Z})$ (some subtleties with order two subgroups, but I will ignore them here [Malmendier,Morrison]).

This data turns out to be isomorphic to the complex structure of a genus two curve, given as an element of the upper Siegel half-plane:

$$\mathbb{H}_2 = \left\{ \underline{\tau} = \begin{pmatrix} \tau & \beta \\ \beta & \rho \end{pmatrix} \middle| \; \tau, \rho, \beta \in \mathbb{C}, \mathrm{Im}(\underline{\tau}) > 0 \right\} \tag{3}$$

The space of such curves is $\mathbb{H}_2 / USp(4,\mathbb{Z})$. In the limit $\beta \to 0$ one has a pinching of the torus into two tori, with independent τ and ρ.

Non-geometric heterotic fibrations

We can fiber the genus 2 fiber over a base, and if the transition functions are general elements of $USp(4, \mathbb{Z})$ we have a non-geometric background (only the shift $\rho \to \rho + 1$ is geometric).

To read the low energy physics one can resort to heterotic-F-theory duality. The map has been understood recently. [Malmendier,Morrison][Jockers,Gu] It is easiest to define the physics starting from the genus two fibration.

Schematically, given a genus two surface, it is easy to construct some modular forms of $\underline{\tau}$. ($\sim f, g$) Given these modular forms, one can easily construct the dual $K3$.

A quick reminder of the genus one case

The branching cubic

Recall that a genus one curve can be specified by a double cover of $\mathbb{P}^1$ branched at four points, one of which we can fix at ∞:

$$y^2 = x^3 + fx + g = \prod_{i=1}^{3}(x - e_i)\,. \tag{4}$$

The roots are connected pairwise by branch cuts, so

$$g = \frac{d}{2} - 1 = 1\,. \tag{5}$$

The complex structure τ of the torus is determined by f, g, or equivalently the positions of the roots. One has

$$f = e_1e_2 + e_1e_3 + e_2e_3 \quad ; \quad g = -e_1e_2e_3\,. \tag{6}$$

A quick reminder of the genus one case

Modular forms

f, g can also be constructed as modular forms of τ:

$$f = -\frac{1}{3}E_4(\tau)\,, \quad g = -\frac{2}{27}E_6(\tau)\,, \tag{7}$$

with

$$E_{2k}(\tau) = \frac{1}{2\zeta(2k)} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{(m\tau+n)^{2k}}\,. \tag{8}$$

These modular forms transform with weights $4, 6$ under $SL(2,\mathbb{Z})$, so we have that a rescaling $(f, g) \to (\lambda^2 f, \lambda^3 g)$ for $\lambda \in \mathbb{C}^*$ defines a torus with the same complex structure.

One can construct a modular and rescaling invariant that gives a one-to-one map from $\mathbb{C}$ to the fundamental domain:

$$j(\tau) \propto \frac{f^3}{4f^3 + 27g^2} \tag{9}$$

A quick reminder of the genus one case

Duality with F-theory

We can view a $E_8 \times E_8$ heterotic compactification without Wilson line as being specified by two complex numbers τ, ρ. The F-theory dual is given by an elliptically fibered K3 manifold with two E_8 singularities (and generically, otherwise smooth). This can be parameterized by

$$y^2 = x^3 + \alpha z^4 x + (z^5 + \beta z^6 + z^7)\,. \tag{10}$$

(Singularities at $z = 0, \infty$.)

The duality map is given in this case by [Cardoso, Curio, Lüst, Mohaupt]

$$j(\tau)j(\rho) = -1728^2 \frac{\alpha^3}{27}\,, \tag{11}$$

$$(j(\tau) - 1728)(j(\rho) - 1728) = 1728^2 \frac{\beta^2}{4}\,. \tag{12}$$

Genus two curves

Igusa-Clebsch and Siegel invariants

Recall that $g = \frac{d}{2} - 1$. If we want to construct a $g = 2$ curve we need a sextic (or a quintic):

$$y^2 = \sum_{i=0}^{6} c_i x^i = \prod_{i=1}^{6} (x - e_i)\,. \tag{13}$$

We want to define the analogues of f, g. These are given by the **Igusa-Clebsch** and **Siegel** invariants.

Genus two curves

Igusa-Clebsch invariants

The Igusa-Clebsch invariants are

$$I_2 = \frac{1}{48} \sum_{\sigma \in S_6} (12)(34)(56) \tag{14}$$

$$I_4 = \frac{1}{72} \sum_{\sigma \in S_6} (12)(23)(31)(45)(56)(64) \tag{15}$$

$$I_6 = \frac{1}{12} \sum_{\sigma \in S_6} (12)(23)(31)(45)(56)(64)(14)(25)(36) \tag{16}$$

$$I_{10} = \prod_{i<j} (e_i - e_j)^2 \tag{17}$$

with $(ab) = (e_{\sigma(a)} - e_{\sigma(b)})^2$. Notice that I_{10} is nothing but the discriminant, so it vanishes iff the curve is singular.

Genus two curves

Igusa-Clebsch invariants

Despite having a sextic, and the Igusa-Clebsch invariants being defined in terms of the roots, they can also be expressed purely in terms of the coefficients c_i, without having to solve for the roots (luckily!):

$$\begin{aligned}
I_2 &= 6c_3^2 - 16c_2c_4 + 40c_1c_5 - 240c_0c_6 \\
I_4 &= 4c_2^2c_4^2 - 12c_1c_3c_4^2 + 48c_0c_4^3 - 12c_2^2c_3c_5 + 36c_1c_3^2c_5 + 4c_1c_2c_4c_5 \\
&\quad - 180c_0c_3c_4c_5 - 80c_1^2c_5^2 + 300c_0c_2c_5^2 + 48c_2^3c_6 - 180c_1c_2c_3c_6 \\
&\quad + 324c_0c_3^2c_6 + 300c_1^2c_4c_6 - 504c_0c_2c_4c_6 - 540c_0c_1c_5c_6 + 1620c_0^2c_6^2 \\
I_6 &= \dots \\
I_{10} &= \dots
\end{aligned} \tag{18}$$

Genus two curves

Siegel invariants

There are also analogues of f, g when viewed as modular forms. We need the **Siegel modular forms**

$$\psi_{2k}(\underline{\tau}) = \sum_{(C,D)} \frac{1}{\det(C\underline{\tau} + D)^{2k}} \tag{19}$$

with $(C, D) \sim (M \cdot C, M \cdot D)$ for any $M \in SL(2, \mathbb{Z})$. For our purposes we need $(\psi_4, \psi_6, \chi_{10}, \chi_{12})$ with

$$\chi_{10}(\underline{\tau}) \propto \psi_4(\underline{\tau})\, \psi_6(\underline{\tau}) - \psi_{10}(\underline{\tau})\,, \tag{20}$$

$$\chi_{12}(\underline{\tau}) \propto 441\, \psi_4(\underline{\tau})^3 + 250\, \psi_6(\underline{\tau})^2 - 691\, \psi_{12}(\underline{\tau})\,. \tag{21}$$

These naturally have a projective invariance $(\psi_4, \psi_6, \chi_{10}, \chi_{12}) \sim (\lambda^2\psi_4, \lambda^3\psi_6, \lambda^5\chi_{10}, \lambda^6\chi_{12})$ with $\lambda \in \mathbb{C}^*$ (and similarly for the I_k).

Genus two curves

Relation between Siegel and Igusa-Clebsch invariants

There is a simple relation between the Siegel and Igusa-Clebsch invariants:

$$\begin{aligned} \psi_4 &= 144\, I_4 \\ \psi_6 &= 576\,(3\, I_6 - I_2 I_4) \\ \chi_{10} &= 486\, I_{10} \\ \chi_{12} &= 486\, I_2 I_{10}\,. \end{aligned} \tag{22}$$

Genus two curves

Duality with F-theory

The F-theory dual will be an elliptically fibered $K3$ with E_8 and E_7 singularities, and 5 other isolated I_1 singularities of the fiber. A generic such space is of the form:

$$y^2 = x^3 + (au^4 + bu^3)x + cu^7 + du^6 + eu^5 \tag{23}$$

with (x, y) coordinates on the fiber, and u a coordinate on the $\mathbb{P}^1$ base. This is dual to the heterotic background via the map

$$a = -\frac{1}{48}\psi_4(\underline{\tau})\,, \quad b = -4\chi_{10}(\underline{\tau})\,, \quad d = -\frac{1}{864}\psi_6(\underline{\tau})\,, \quad e = \chi_{12}(\underline{\tau}) \tag{24}$$

and $c = 1$.

Generalizing the Kodaira classification

In the case of fibrations of curves of genus one, we know that we can classify the possible degenerations of the fiber by the Kodaira classification: I_n, II, III, IV and their starred relatives.

We also know how to read the low energy physics of each of these degenerations when they occur in string theory.

How do we read the low energy physics of the non-geometric fibrations associated with genus-two fibrations? (Hint: the title of this talk gives it away.)

Generalizing the Kodaira classification

In the case of fibrations of curves of genus one, we know that we can classify the possible degenerations of the fiber by the Kodaira classification: I_n, II, III, IV and their starred relatives.

We also know how to read the low energy physics of each of these degenerations when they occur in string theory.

How do we read the low energy physics of the non-geometric fibrations associated with genus-two fibrations? (Hint: the title of this talk gives it away.) **Analyse the F-theory dual.**

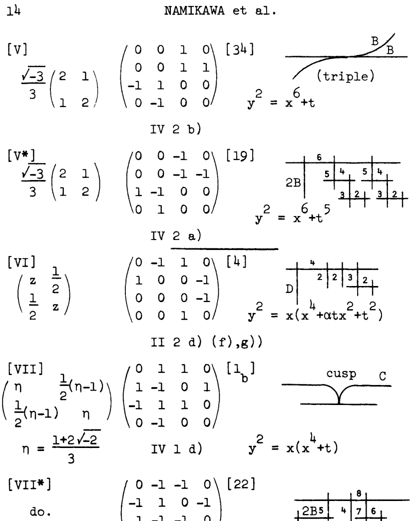
Ogg-Namikawa-Ueno

The generalization of the Kodaira classification to genus 2 was given by Ogg-Namikawa-Ueno.

Many possible cases:

- $K - K' - m$ for K, K' in Kodaira...
- and a bunch of intrinsic ones

for a total of 120 kinds of degenerations.



14 NAMIKAWA et al.

[V] $\frac{\sqrt{-3}}{3}\begin{pmatrix}2&1\\1&2\end{pmatrix}$	$\begin{pmatrix}0&0&1&0\\0&0&1&1\\-1&1&0&0\\0&-1&0&0\end{pmatrix}$ IV 2 b)	[34]	$y^2 = x^6+t$
[V*] $\frac{\sqrt{-3}}{3}\begin{pmatrix}2&1\\1&2\end{pmatrix}$	$\begin{pmatrix}0&0&-1&0\\0&0&-1&-1\\1&-1&0&0\\0&1&0&0\end{pmatrix}$ IV 2 a)	[19]	$y^2 = x^6+t^5$
[VI] $\begin{pmatrix}z&\frac{1}{2}\\\frac{1}{2}&z\end{pmatrix}$	$\begin{pmatrix}0&-1&1&0\\1&0&0&-1\\0&0&0&-1\\0&0&1&0\end{pmatrix}$ II 2 d) (f),g))	[4]	$y^2 = x(x^4+\alpha t x^2+t^2)$
[VII] $\begin{pmatrix}\eta&\frac{1}{2}(\eta-1)\\\frac{1}{2}(\eta-1)&\eta\end{pmatrix}$, $\eta = \frac{1+2\sqrt{-2}}{3}$	$\begin{pmatrix}0&1&1&0\\1&-1&0&1\\-1&1&1&0\\0&-1&0&0\end{pmatrix}$ IV 1 d)	$[1_b]$	$y^2 = x(x^4+t)$
[VII*] do.	$\begin{pmatrix}0&-1&-1&0\\-1&1&0&-1\\1&-1&-1&0\end{pmatrix}$	[22]	

Dualizing Ogg-Namikawa-Ueno

Consider the local model for the sextic with a degeneration

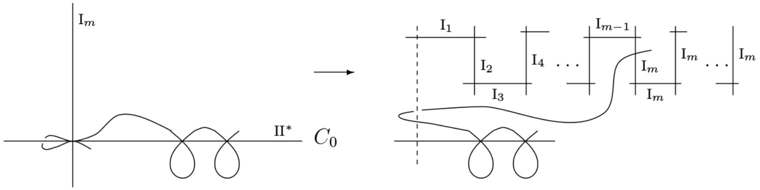
$$y^2 = (x^2 - t^M)((x-\alpha)^2 - t^N)(x-\beta)(x-1) \tag{25}$$

This is the I_M-I_N-0 singularity in the Ogg-Namikawa-Ueno classification. The monodromy can be computed to be (ignoring the Wilson line part)

$$\begin{aligned} \tau &\to \tau + M \\ \rho &\to \rho + N \end{aligned} \tag{26}$$

which we interpret as a N NS5 branes hitting a $\mathbb{C}^2/\mathbb{Z}_M$ singularity in the $E_8 \times E_8$ heterotic string. The dynamics can be computed by dualizing to F-theory [Aspinwall,Morrison, and many recent works]: a $(1,0)$ theory with a number of massless tensors. (From having too high degree of vanishing in the F-theory elliptic fibration.)

The I_n-I_m-0 singularity in F-theory



Result 3 *A collection of k point-like E_8 instantons on a $\mathbb{C}^2/\mathbb{Z}_m$ (that is, type A_{m-1}) quotient singularity, where $k \geq 2m$, yields $n_T' = k$ and local contribution to the gauge algebra*

$$\mathscr{G}_{\text{loc}} \cong \mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \ldots \oplus \mathfrak{su}(m-1) \oplus \mathfrak{su}(m)^{\oplus(k-2m+1)} \oplus \mathfrak{su}(m-1) \oplus \ldots \oplus \mathfrak{su}(2).$$

(From [Aspinwall,Morrison].)

A non-geometric example

The previous example was familiar and with a clear geometric interpretation. There are many genus two fibrations in Ogg-Namikawa-Ueno that are intrinsically non-geometric, for example III-III-m, with monodromy

$$\tau' = \frac{\rho}{\beta^2 - \tau\rho}\,, \rho' = \frac{\tau}{\beta^2 - \tau\rho}\,, \beta' = \frac{\beta}{\beta^2 - \tau\rho}\,. \tag{27}$$

Running through the duality machinery, we find a dual F-theory model with local form:

$$y^2 = x^3 + (au^4)x + u^7 + eu^5 \tag{28}$$

with (turning off the Wilson line for simplicity)

$$a = -36 \cdot (t+3) \cdot t^2 \cdot (t-1)^2 + \ldots \tag{29}$$

$$e = 46656 \cdot t^6 \cdot (t-1)^8 + \ldots \tag{30}$$

so we again have a non-trivial $(1,0)$ CFT.

Conclusions

We find that non-geometric backgrounds in the heterotic string, with one Wilson line at most, can be analyzed by looking to the F-theory dual, and we can read the low-energy physics there.

The whole machinery is conceptually very similar to the genus one fibrations that appear in F-theory, but degenerations of the fiber are associated with non-trivial $(1,0)$ theories in six dimensions, instead of $N=1$ Yang-Mills theories in eight dimensions.

So it seems like studying branes in heterotic non-geometric backgrounds naturally leads us to the study of $(1,0)$ theories, and if we go to four dimensions we are naturally lead to compactifications of $(1,0)$ theories in Riemann surfaces.

Open problems

Many interesting questions!

- Any physical meaning for the genus two curve? Generalizations?
- Can we understand type II in a similar manner? [Candelas,Constantin,Damian,Larfors,Morales] [Martucci,Morales,Pacifici] [Braun,Fucito,Morales]
- Are these theories interesting at all? I.e. what do we get when we compactify to 4d? (Parallel talk by Hayashi.)