# Multiple fibrations in Calabi-Yau constructions

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Based on work with:

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1405.2073
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# CICYS of various types:

- An example of a configuration matrix (CICY four-fold 244)

$$\left[\begin{array}{c|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^2 & 1 & 2 \\ \mathbb{P}^3 & 0 & 4 \end{array}\right]$$

- This represents a family of Calabi-Yau four-folds defined by the solutions to the polynomials

$$p_1 = \sum_{i,a} c_{i,a} x^i y^a \quad p_2 = \sum_{i,\ldots,\delta} d_{iab\alpha\beta\gamma\delta} x^i y^a y^b z^\alpha z^\beta z^\gamma z^\delta$$

- In the ambient space $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$
- Calabi-Yau property is determined by a condition on the sum of the rows.

- Three-folds first studied and classified in 1980s:
    - Hübsch, Commun.Math.Phys. 108 (1987) 291
    - Green et al, Commun.Math.Phys. 109 (1987) 99
    - Candelas et al, Nucl.Phys. B 298 (1988) 493
    - Candelas et al, Nucl.Phys. B 306 (1988) 113
  - 7890 configuration matrices in the data set.
- Four-folds studied and classified in 2013-14:
    - Brunner, Lynker and Schimmrigk, Nucl. Phys. B 498 (1997) 156.
    - JG, Haupt and Lukas, JHEP 1307 (2013) 017
    - JG, Haupt and Lukas, JHEP 1409 (2014) 093
  - 921,497 configuration matrices in the data set.
- Generalized CICYs: see talk by Seung-Joo Lee.
- These data sets have proven to be very useful in a variety of contexts…

# Elliptic Fibrations

- Consider configuration matrices which can be put in the form:

$$\left[\ \mathcal{A}_1 \ \middle|\ \mathcal{F}\ \right] = T^2$$

$$\text{Base: } \left[\ \mathcal{A}_2 \ \middle|\ \mathcal{B}\ \right] \qquad \left[\begin{array}{c|cc} \mathcal{A}_1 & 0 & \mathcal{F} \\ \mathcal{A}_2 & \mathcal{B} & \mathcal{T} \end{array}\right]$$

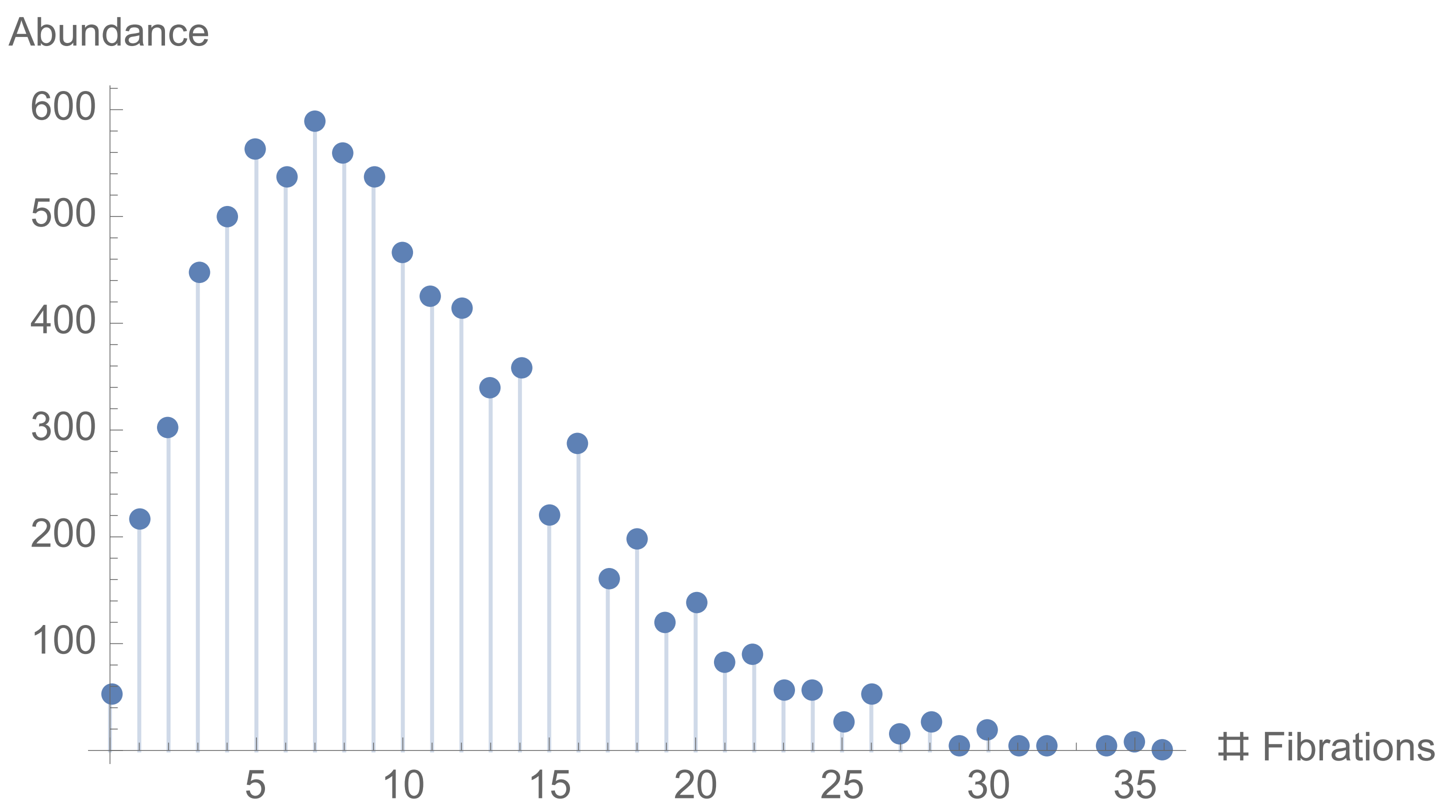
- This is an elliptically fibered Calabi-Yau!
- Essentially all CICYs are fibered in this manner. For example 7837 out of 7890 threefolds (99.3%)

- Example:

$$\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^3 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \end{array} \left(\begin{array}{cc:cccc} 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ \hdashline 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \end{array}\right)$$

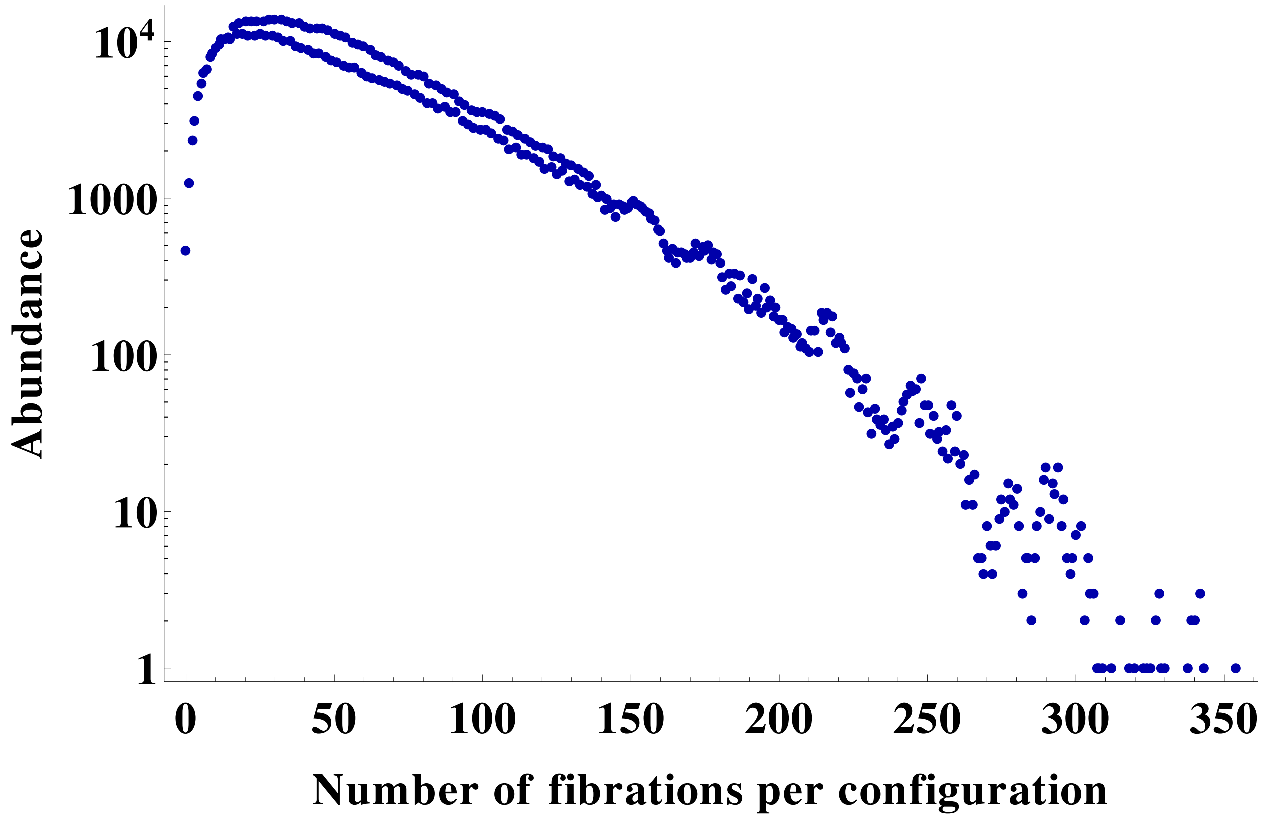
- This is not an artifact of the threefolds. For fourfolds 921,020 out of 921,497 configuration matrices are obviously elliptically fibered in this way (99.9%)
- See also related work for other constructions by Wati Taylor and collaborators (e.g. arXiv:1406.0514 by S. Johnson and W. Taylor)
- A given manifold/configuration matrix may admit many obvious elliptic fibrations…

- Threefolds:




- Total of 77,744 different elliptic fibrations in data set.
- Average of 9.85 fibrations per manifold…

# Fourfolds:




- Total of 50,114,908 different elliptic fibrations in data set.
- Average of 54.4 fibrations per manifold.

- In our simple example we also have:

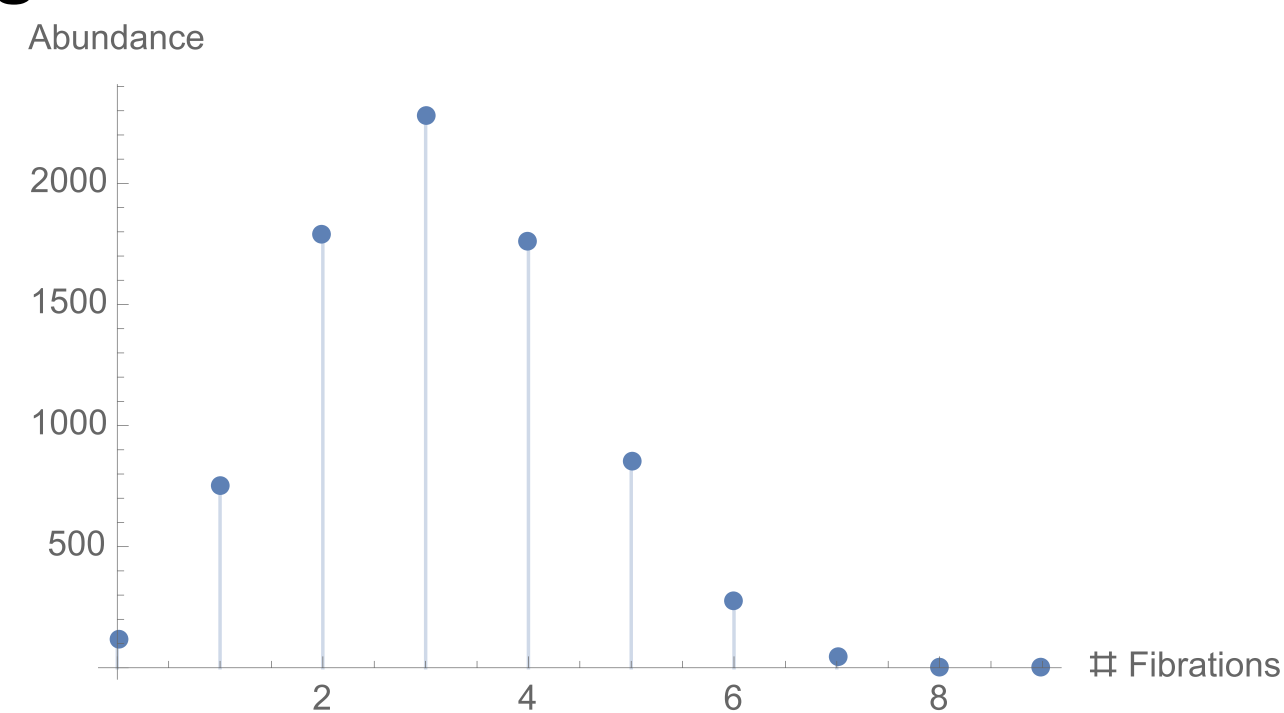
$$\left(\begin{array}{c|c:ccccc} \mathbb{P}^1 & 0 & 1 & 1 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^3 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^1 & 1 & 0 & 0 & 1 & 0 & 0 \\ \mathbb{P}^2 & 2 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{c|c:ccccc} \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^3 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^2 & 1 & 0 & 2 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{c|:cccccc} \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^2 & 1 & 0 & 2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \left(\begin{array}{c|:cccccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^3 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^2 & 1 & 0 & 2 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c|:cccccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \mathbb{P}^2 & 1 & 0 & 2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \end{array}\right)$$

- Note that we have a variety of different bases here (Hirzebruchs, $\mathbb{P}^1 \times \mathbb{P}^1$ , $\mathbb{P}^2$ etc in this case).
- It doesn't just have to be *elliptic* fibration structures that exist in a CICY…

# K3 Fibrations in threefolds:

- 7768 of the 7890 threefold configuration matrices are K3 fibered (98.5%)
- There are 24,469 different obvious K3 fibrations in the list for an average of 3.1 K3 fibrations per configuration matrix.



- In our simple example:

$$\left(\begin{array}{c||cccccc} \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^3 & 0 & 0 & 1 & 1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 & 0 & 0 & 0 \\ \mathbb{P}^2 & 1 & 2 & 0 & 0 & 0 & 0 \\ \hdashline \mathbb{P}^1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{c||cccccc} \mathbb{P}^2 & 0 & 0 & 1 & 1 & 0 & 0 \\ \mathbb{P}^3 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^1 & 1 & 0 & 2 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

- Again this example is slightly less rich than the average case…

- One could ask if the K3 fibers are elliptically fibred…

# Fibering K3 Fibers:

- You can ask about compatibility of elliptic and K3 fibrations of a threefold: how many different ways of elliptically fibering the K3 fibrations are there?
  - 103,513 in total for an average of 13.1 such fibered fibrations per configuration matrix
  - Note this is bigger than the number of elliptic fibrations on their own…
  - Example in our case: (there are six in total in the two K3 fibrations)

$$\left(\begin{array}{c|cc|cccc}
\mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\
\mathbb{P}^3 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hdashline
\mathbb{P}^1 & 1 & 0 & 1 & 0 & 0 & 0 \\
\mathbb{P}^2 & 1 & 2 & 0 & 0 & 0 & 0 \\
\hdashline
\mathbb{P}^1 & 0 & 1 & 0 & 1 & 0 & 0
\end{array}\right)$$

# GCICYs have a similar structure:

- Simple example:

$$\left(\begin{array}{c|cccc} \mathbb{P}^5 & 3 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^1 & 1 & 1 & 1 & -1 \\ \hdashline \hdashline \mathbb{P}^1 & 1 & 1 & -1 & 1 \end{array}\right)$$

- This case is a little more complicated as you have to check the K3 and elliptic fibers are themselves well defined.

- Note the listed case is particularly simply due in part to the large ambient space factor!

- GCICYs are not necessarily smooth – could be useful here! (the above example is smooth as an existence proof however)

- … and you can keep going
  - A generic fourfold will be threefold fibered
    - The threefold fibers will be K3 fibered
      - The K3 fibers will be elliptically fibered
- This type of structure is interesting for a variety of reasons. For example:
  - The set of elliptically fibered Calabi-Yau threefolds is finite…
  - Interesting dualities are implied by the admission of multiple fibrations…

# Conclusions:

- Almost all CICYs are fibered!
- Almost all CICYs are multiply fibered
- Almost all of the fibers are fibered too!

- This type of structure may help us gain a deeper understanding of the set of possible Calabi-Yau (perhaps rendering some parts of string phenomenology a finite problem).

- This structure is also useful in studying dualities, something that we are currently looking into. For this we practically need to know sections of the fibrations, which Xin Gao will talk about next!.