The Arithmetic of Elliptic Curves in F-theory

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arXiv:1507.xxxxx (T. Grimm, AK, D. Klevers)

arXiv:1502.05398 (T. Grimm, AK)

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Max-Planck-Institut für Physik

Outline

1. Introduction
2. Elliptic curve arithmetics in F-theory EFT:
   - Mordell-Weil group
   - Group of exceptional divisors
3. Gauge anomaly cancellation from elliptic curve arithmetics
4. Conclusions

Introduction

Study the manifestation of elliptic curve arithmetics in F-theory effective field theory!

Motivation:

- Properties of effective field theories in F-theory compactifications on elliptic fibrations
  (e.g. anomaly cancellation)
- (Implications for the mathematical theory of elliptic curves)

Setup

F-theory on singular elliptically fibered Calabi-Yau three-/fourfolds with multiple rational sections

From elliptic curve arithmetics to effective field theory

F-theory effective action via M-theory:

M-theory

M-theory on (resolved) Calabi-Yau threefold

$\Longleftrightarrow$

F-theory

6D, $\mathcal{N} = (1,0)$ supergravity on S^1, on the Coulomb branch

$NS \otimes \mathbb{Q}$
(Nerón-Severi group)

← duality →

Gauge Theory
(vector fields)

expansion of M-theory 3-form:
$C_3 = A^i[D_i] + \dots$

vector fields A^i in the S^1-reduced supergravity on the Coulomb branch

From elliptic curve arithmetics to effective field theory

F-theory effective action via M-theory:

M-theory		F-theory
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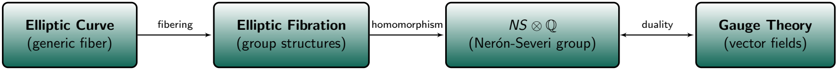

expansion of M-theory 3-form:
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- group structure on elliptic curves
- group structure in the circle-reduced gauge theory
- mediation by a homomorphism to $NS \otimes \mathbb{Q}$

Mordell-Weil group

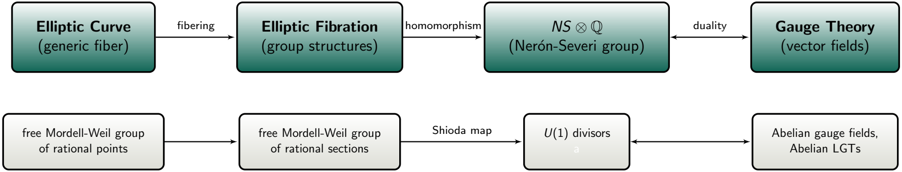
[Aspinwall, Fukae, Guralnik, Katz, Morrison, Oguiso, Shimada, Shioda, Yamada, Yang]

The K-rational points on an elliptic curve E form a finitely generated Abelian group:

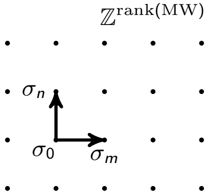
$$E(K) \cong \mathbb{Z}^{\mathrm{rank(MW)}} \oplus \mathbb{Z}_r$$

$\rightarrow$ split into free part and torsion subgroup

Free part:



- zero-section σ_0, generating rational sections σ_m, σ_n



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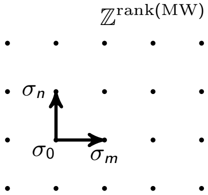
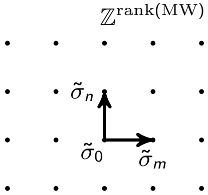


$\longmapsto$

MW-shift with $\sigma_m - \sigma_0$



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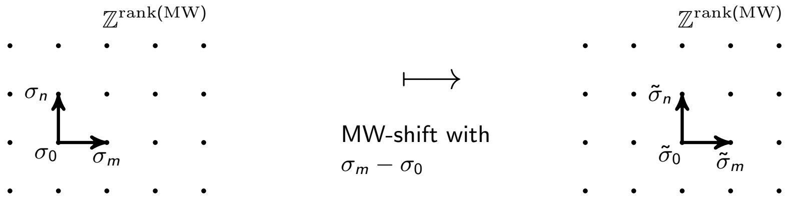

Effect in the circle-reduced theory:

- Abelian large gauge transformation of the vector A^m

$$A^m \mapsto A^m - A^0$$

(mixing with KK-vector A^0)

- In terms of 6D fields $\hat{A}^m$:

$$\hat{A}^m \mapsto \hat{A}^m + \mathrm{d}y \qquad \text{with } y \sim y + 2\pi$$

$$\hat{\Phi} \mapsto e^{iq_m y}\hat{\Phi} \qquad \text{for a field } \hat{\Phi} \text{ with charge } q_m$$

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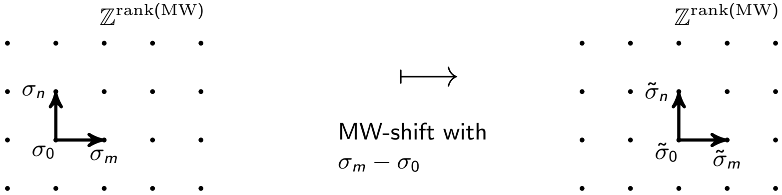

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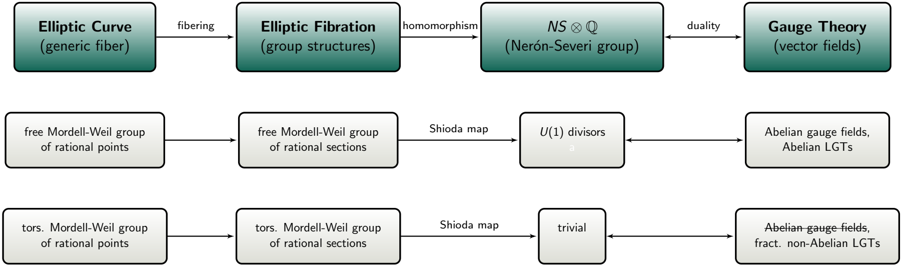
One-to-one correspondence

free Mordell-Weil group
$\Updownarrow$
Abelian large gauge transformations

Torsion part of the Mordell-Weil group:

[Aspinwall, Morrison; Mayrhofer, Morrison, Till, Weigand]



- zero-section σ_0
  generators of the free part σ_m
  generators of the torsion part σ_r
- shift all sections by $\sigma_r - \sigma_0$

Effect in the circle-reduced theory:

Fractional non-Abelian large gauge transformation

Torsion part of the Mordell-Weil group:

[Aspinwall, Morrison; Mayrhofer, Morrison, Till, Weigand]

$\Rightarrow$ non-simply connected non-Abelian gauge group
$\Rightarrow$ refinement of the coweight lattice
$\Rightarrow$ certain large gauge transformations with fractional coefficients c^I are compatible with the circle boundary conditions

For a state $\hat{\Phi}$ with weight w:

$$\hat{\Phi} \mapsto e^{i c^I w_I y} \hat{\Phi} \qquad c^I \in \mathbb{Q} \text{ but } c^I w_I \in \mathbb{Z}$$

Indeed: torsional MW-shift along $\sigma_r - \sigma_0$

$\Rightarrow$ fractional non-Abelian large gauge transformation

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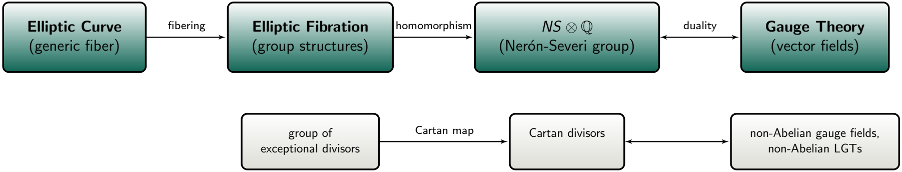
One-to-one correspondence

torsional Mordell-Weil group
⇕
fract. non-Abelian large gauge transformations

Group of exceptional divisors in the fibration

[Grimm, AK]



Group of exceptional divisors:

- elements: $\mathrm{span}_{\mathbb{Z}}(F_I)$ with F_I blow-up divisors (∼ Mordell-Weil lattice)
- group operation: addition of divisors (∼ Mordell-Weil group law)

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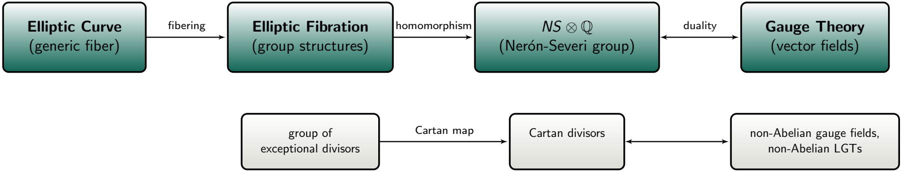

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"Cartan map":

- choose a zero-node $\Sigma_0 \in \mathrm{span}_{\mathbb{Z}}(F_I)$ (∼ choose a zero-section)
- the generators are $\Sigma_I := \Sigma_0 + F_I$ (∼ Mordell-Weil generators)
- Cartan map: $D_I := \Sigma_I - \Sigma_0 - [(\Sigma_I - \Sigma_0) \cdot \Sigma_0 \cdot D^\alpha] D_\alpha$

  (∼ Shioda map: $D_m := \sigma_m - \sigma_0 - [(\sigma_m - \sigma_m) \cdot \sigma_0 \cdot D^\alpha] D_\alpha$)

→ Homomorphism from the group of exceptional divisors to Nerón-Severi

One-to-one correspondence

group of exceptional divisors $\Leftrightarrow$ non-Abelian large gauge transformations

Interesting aspect:

[Braun, Grimm, Keitel; Klevers, Peña, Oehlmann, Piragua, Reuter]

$\exists$ Generic fibers with fibration-independent non-Abelian gauge symmetry!

$\Rightarrow$ $\exists$ Corresponding group structure on the generic fiber???

One-to-one correspondence

group of exceptional divisors ⇔ non-Abelian large gauge transformations

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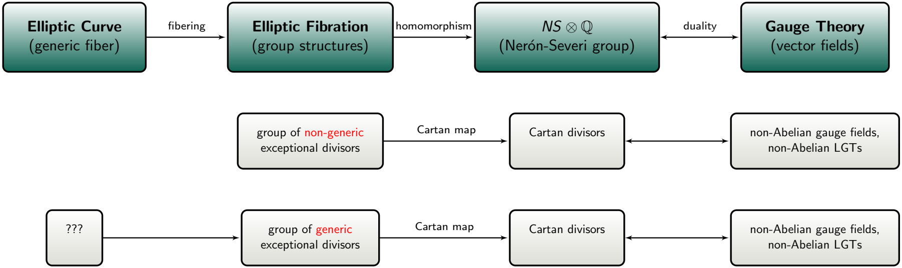


Anomaly cancellation

[Grimm, AK]

Group actions (large gauge transformations) on one-loop Chern-Simons terms in 5D:

$$S_{\mathrm{CS}} = \int k_{mnp} \, A^m \wedge F^n \wedge F^p \, .$$

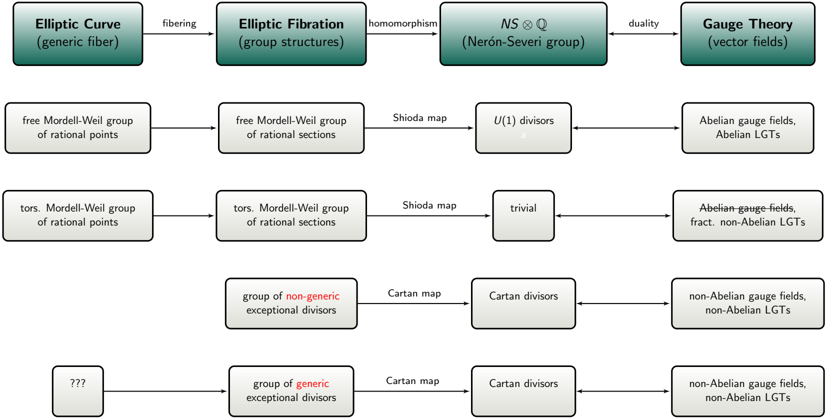
Two ways to evaluate a large gauge transformation on k_{mnp}:

1. dual transformation to gauge fields A^m
2. loop-calculation with gauge-transformed fields

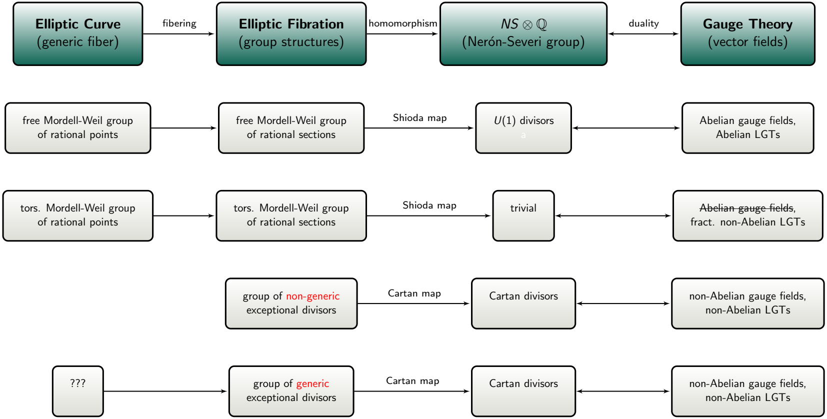
→ **Consistency of both approaches iff anomalies are canceled in 6D!**

- Mordell-Weil group
  ⇒ cancellation of Abelian and mixed gauge anomalies
- group of exceptional divisors
  ⇒ cancellation of non-Abelian and mixed gauge anomalies

Conclusions



Conclusions



Work in progress/to do:

- Group on the elliptic curve for generic exceptional divisors?
- Investigate Tate-Shafarevich group in this spirit