Yukawa Couplings in Heterotic Calabi-Yau Models

Andre Lukas

University of Oxford

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based on:

- 1506.nnnn with Evgeny Buchbinder, Philip Candelas, Stefan Blasneag
- relates to 0904.2186 with Lara Anderson, James Gray, Dan Grayson, Yang-Hui He
- longer-term project, involving Lara Anderson, James Gray, Burt Ovrut

Outline

- Introduction: Heterotic Yukawa couplings
- Yukawa unification?
- Current status of calculation
- Our laboratory: the tetra-quadric CY
- Example 1: up-Yukawa couplings
- Example 2: singlet Yukawa couplings
- Conclusion

Introduction: Heterotic Yukawa couplings

- Consider heterotic string on CY 3-fold X
- observable bundle $V \to X$ with structure group $H \subset E_8$
- low-energy gauge group $G = \mathcal{C}_{E_8}(H)$
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Matter multiplets described by harmonic (0,1) forms:

$$\nu_i \in H^1(X, E_i) \qquad \bar{\partial}_{E_i}\nu_i = \bar{\partial}^\dagger_{E_i}\nu_i = 0 \qquad i = 1, 2, 3$$

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$$\lambda(\nu_1, \nu_2, \nu_3) = \int_X \Omega \wedge \nu_1 \wedge \nu_2 \wedge \nu_3$$

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Holomorphic Yukawa couplings can be computed algebraically!

Normalisation (proportional to):

$$(\nu_i, \mu_i) := \int_X \nu_i \wedge \bar{\star}_{E_i} \mu_i = \frac{1}{2} \int_X J \wedge J \wedge \nu_i \wedge (H \bar{\mu}_i)$$

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Algebraic computation (probably) not possible. Requires methods of differential geometry.

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SM Yukawa couplings are obtained after taking quotient by discrete symmetry Γ, adding a Wilson line and keeping the Γ-invariant parts.

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Wilson line described by Γ-representations χ_2, χ_3 satisfying $\chi_2^2 \otimes \chi_3^3 = 1$. For $\Gamma = \mathbb{Z}_2$ we have $\chi_2 = (1)$ and $\chi_3 = (0)$.

$$\chi_H = \chi_2^* = (1)$$

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This holds for any symmetry Γ and all types of Yukawa couplings.

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Yukawa unification may arise from additional symmetries which constraints the upstairs Yukawa couplings λ_{IJ} .

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limitations:

- Sometimes not obvious how to carry out in practice when objects isomorphic to Cech representatives are used.

- Normalisation unkown and cannot be computed in this language.

We would like to....

- understand how to compute hol. Yukawa couplings using differential geometry language.
- clarify how such a differential geometry calculation relates to the algebraic one
- set the scene for a computation of the normalisation which requires differential geometry.

Our laboratory: the tetra quadric CY

Tetra-quadric: defined as zero locus of multi-degree (2,2,2,2) polynomial in ambient space $\mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

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Leads to structure groups $S(U(1)^n) \subset SU(n)$ and gauge groups $E_6,\ SO(10),\ SU(5)$

Tetra-quadric is simplest CICY which leads to line bundle standard models.

Line bundle cohomology on the tetra-quadric

Kozsul sequence: $0 \to N^* \otimes \mathcal{L} \to \mathcal{L} \to L \to 0$ where
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Kozsul sequence: $0 \to N^* \otimes \mathcal{L} \to \mathcal{L} \to L \to 0$ where
$N = \mathcal{O}_{\mathcal{A}}(2,2,2,2)$ and $L = \mathcal{L}|_X$

$$\cdots \to H^1(\mathcal{A}, N^* \otimes \mathcal{L}) \xrightarrow{p} H^1(\mathcal{A}, \mathcal{L}) \xrightarrow{i^*} H^1(X, L)$$
$$\xrightarrow{\delta} H^2(\mathcal{A}, N^* \otimes \mathcal{L}) \xrightarrow{p} H^2(\mathcal{A}, \mathcal{L}) \to \cdots$$

easier

$$H^1(X, L) \cong \mathrm{Coker}\left(H^1(\mathcal{A}, N^* \otimes \mathcal{L}) \xrightarrow{p} H^1(\mathcal{A}, \mathcal{L}) \right) \oplus$$
$$\mathrm{Ker}\left(H^2(\mathcal{A}, N^* \otimes \mathcal{L}) \xrightarrow{p} H^2(\mathcal{A}, \mathcal{L}) \right)$$

harder (co-boundary map)

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  $\kappa = 1 + |z|^2$

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"For negative degree maps, replace variables by derivatives."

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Yukawa couplings vanish due to structure of cohomology.

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Can always be explicitly integrated, or calculated algebraically:

$$\lambda(\nu_1, \nu_2, \nu_3) = 16\pi^3 c\mu(P, Q, R) \qquad \mu(P, Q, R) = \tilde{P}\tilde{Q}\tilde{R}$$

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- Case 3: More than one ν_i originates from ambient 2-form

Slightly more complicated but can always be integrated.

Example 1: up-Yukawa couplings

Standard model based on SU(5) GUT with line bundles

$$L_1 = \mathcal{O}_X(-1,0,0,1)\ ,\ L_2 = \mathcal{O}_X(-1,-3,2,2)\ ,\ L_3 = \mathcal{O}_X(0,1,-1,0)$$
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Relevant line bundles for up-Yukawa coupling:

$$\begin{array}{lll} K_1 = L_2^* \otimes L_5^* & 3\ \mathbf{5}^H_{2,5} & \hat{\nu}_1 = \kappa_3^{-2} Q_{(0,2,-2,0)} d\bar{z}_3 \\ K_2 = L_5 & 4\ \mathbf{10}_2 & \hat{\nu}_2 = \kappa_4^{-2} R_{(1,1,0,-2)} d\bar{z}_4 \\ K_3 = L_2 & 8\ \mathbf{10}_5 & \hat{\omega} = \kappa_1^{-3}\kappa_2^{-5} S_{(-3,-5,0,0)} d\bar{z}_1 \wedge d\bar{z}_2 \end{array}$$

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where

$$\begin{array}{rcl} Q &=& q_0 + q_1 z_2 + q_2 z_2^2 \\ R &=& r_0 + r_1 z_1 + r_2 z_2 + r_3 z_1 z_2 \\ S &=& s_0 + s_1 \bar{z}_2 + s_2 \bar{z}_2^2 + s_3 \bar{z}_2^3 + s_4 \bar{z}_1 + s_5 \bar{z}_1 \bar{z}_2 + s_6 \bar{z}_1 \bar{z}_2^2 + s_7 \bar{z}_1 \bar{z}_2^3 \end{array}$$

Yukawa couplings, explicit calculation:

$$\lambda(Q,R,S) = \frac{1}{\pi}\int_{\mathbb{C}^4} \frac{QRS}{\kappa_1^3\kappa_2^5\kappa_3^2\kappa_4^2} d^4z\, d^4\bar{z}$$

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$$\begin{aligned}
\mu(Q,R,S) &= \left(q_0\partial_{y_0}^2 + q_1\partial_{y_0}\partial_{y_1} + q_2\partial_{y_1}^2\right)\left(r_0\partial_{x_0}\partial_{y_0} + r_1\partial_{x_1}\partial_{y_0} + r_2\partial_{x_0}\partial_{y_1} + r_3\partial_{x_1}\partial_{y_1}\right) \\
&\quad \left(s_0x_0y_0^3 + s_1x_0y_0^2y_1 + s_2x_0y_0y_1^2 + s_3x_0y_1^3 + s_4x_1y_0^3 + s_5x_1y_0^2y_1 + s_6x_1y_0y_1^2 + s_7x_1y_1^3\right) \\
&= 2\,[3q_0r_0s_0 + 3q_0r_1s_4 + q_0r_2s_1 + q_0r_3s_5 + q_1r_0s_1 + q_1r_1s_5 + \\
&\qquad q_1r_2s_2 + q_1r_3s_6 + q_2r_0s_2 + q_2r_1s_6 + 3q_2r_2s_3 + 3q_2r_3s_7]\ .
\end{aligned}$$

After taking quotient by $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ and adding Wilson line:

$$\lambda^{(u)} = \frac{\pi^3}{3} \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right)$$

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with associated line bundles

$$\begin{aligned}
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The same model has a coupling

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Yukawa couplings for a 5-parameter family of tetra-quadrics:

$$\lambda(\nu_1, \nu_2, \nu_3) = -\frac{1}{\pi} \int_{\mathbb{C}^4} \frac{Q \mathcal{R} S}{\kappa_1^4 \kappa_2^6 \kappa_3^4 \kappa_4^5} d^4 z \, d^4 \bar{z}$$

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Still need to find kernel $M\mathbf{a} = 0$ where

$$M = \begin{pmatrix}
24c_6 & 0 & 0 & 0 & 4c_3 & 4c_6 & 0 & 0 & 0 & 24c_5 & 0 & 0 & 3c_4 & 0 & 0 \\
24c_5 & 0 & 6c_2 & 0 & 4c_6 & 4c_3 & 0 & 6c_2 & 0 & 24c_6 & 0 & 0 & -3c_4 & 0 & 0 \\
24c_4 & 24c_6 & 0 & 6c_2 & 4c_6-4c_4 & 4c_3+4c_4 & 6c_2 & 0 & 24c_5 & -24c_4 & 12c_2 & 0 & 3c_1 & 3c_4 & 2c_2 \\
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0 & 24c_6 & 0 & 0 & 4c_6 & 4c_5 & 0 & 0 & 24c_3 & 0 & 0 & 0 & 0 & 3c_4 & 0 \\
0 & 0 & 12c_6 & 12c_6 & 8c_2 & 8c_2 & 12c_3 & 12c_5 & 0 & 0 & 0 & 0 & 0 & 0 & 4c_4 \\
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0 & 0 & 12c_6 & 12c_6 & 0 & 0 & 12c_5 & 12c_3 & 0 & 0 & 0 & 0 & 6c_2 & 6c_2 & 4c_4 \\
0 & 0 & 12c_3+12c_4 & 12c_4+12c_5 & 8c_2 & 8c_2 & 12c_6-12c_4 & 12c_6-12c_4 & 0 & 0 & 0 & 0 & 6c_2 & 6c_2 & 4c_1-4c_4
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This is generically rank 1, but will be generally rank 2 away from the 5-parameter family. For $c_3 = c_5$ the Higgs remains massless even if $\langle S^i \rangle \neq 0$.

Conclusion

- Calculating Yukawa couplings in string theory is crucial in order to make contact with physics.

- Much remains to be done for Yukawa couplings in heterotic Calabi-Yau models with arbitrary vector bundles.

- We can now compute the holomorphic (perturbative) Yukawa couplings for heterotic line bundle models, both algebraically and in terms of differential geometry.

- First explicit calculation of complex structure dependence: rank of hol. Yukawa couplings can change in complex structure moduli space

Much remains to be done:

- Compute hol. Yukawa couplings for other manifolds.
- Compute hol. Yukawa couplings for non-Abelian bundles.
- Find a way to work out the normalisation.
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