# The Geometry of Generations

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[1506.soon] with Y. H. He, V. Jejjala, C. Matti

# Vacuum Moduli Spaces of Gauge Theories



Supersymmetric quantum field theories have scalars $\rightarrow$ a complicated vacuum space of possible field vevs $\langle\phi_i\rangle$, characterized by certain flat directions

- Vacuum configuration is any set of field values $\{\phi_i^0\}$ such that $V(\phi_i^0, \bar{\phi}_i^0) = 0$

$$V(\phi_i, \bar{\phi}_i) = \sum_i \left|\frac{\partial W}{\partial \phi_i}\right|^2 + \frac{1}{2}\sum_A g_A^2 \left(\sum_i \phi_i^\dagger T^A \phi_i\right)^2$$

  where $\phi_i$ is the lowest (scalar) component of superfield $\Phi_i$ with charge $q_i$

- The vacuum moduli space $\mathcal{M}$ is the space of all possible solutions $\phi^0$ to these F and D-flatness conditions

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- The vacuum moduli space $\mathcal{M}$ is the space of all possible solutions $\phi^0$ to these F and D-flatness conditions

$\Rightarrow$ This manifold $\mathcal{M}$ may have special structure that correlates with certain phenomenological properties

- These structures need NOT be directly related to gauge invariance or discrete symmetries – *otherwise unexplained from traditional field theory perspective*
- Identifying such structures may aid in top-down model building from string compactification

# Approach via Algebraic Geometry



$\Rightarrow$ To every solution of the F-flatness conditions there exists a solution to the D-flatness conditions in the orbit of the complexified gauge group $\mathcal{G}^C$:

$$\boxed{\mathcal{M} = \mathcal{F}//\mathcal{G}^C}$$

where $\mathcal{F}$ is the space of all F-flat field configurations

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⇒ Consider a theory defined by a certain gauge group and set of fields $\{\Phi_i\}$

- Consider a minimal generating set of Gauge Invariant Operators (GIOs) $D_j = \{r_j(\{\phi_i\})\}$
- This defines a polynomial ring $S = \mathbb{C}\left[r_1, \ldots, r_k\right]$
- F-flatness corresponds to the Jacobian ideal $\langle \partial W / \partial \phi_i \rangle$ in the polynomial ring $R = \mathbb{C}\left[\phi_1, \ldots, \phi_n\right]$
- The polynomial map $D$ is a map from the quotient ring $\mathcal{F} \simeq R / \langle \partial W / \partial \phi_i \rangle$ to the ring $S$
- The image of this map gives $\mathcal{M}$, defined as an affine variety in $S$: $\mathcal{M} \simeq \mathrm{Im}\left(\mathcal{F} \stackrel{D=GIO}{\longrightarrow} S\right)$

# Computational Algorithm



Strategy: consider the set of gauge invariants, composed of monomials in the $\phi_i$, and back substitute solutions to the F-flatness conditions

$\Rightarrow$ This becomes an elimination problem, suitable for computational methods

## INPUTS

- A theory defined by a superpotential $W = W(\{\phi_i\})$ with $i = 1, \ldots, n$
- Basis of GIOs $\{r_j(\{\phi_i\})\}$, with $j = 1, \ldots, k$

## ALGORITHM

- Define the polynomial ring $R = \mathbb{C}[\phi_{i=1,\ldots,n}, y_{j=1,\ldots,k}]$
- Consider the ideal $I = \langle \frac{\partial W}{\partial \phi_i}; y_j - r_j(\phi_i) \rangle$
- Eliminate all variables $\phi_i$ from $I \subset R$, giving the ideal $\mathcal{M}$ in terms of $y_j$

OUTPUT: $\mathcal{M}$ as an affine variety in $\mathbb{C}\,[y_1, \ldots, y_k]$

$\Rightarrow$ Algorithm equivalent to asking for all relations among the GIOs which satisfy the F-flatness conditions (i.e. all possible syzygies among the coordinates)

# Restricting to MSSM Electroweak Sector



⇒ Would ultimately like to study is the full, renormalizable MSSM superpotential

- Seven species of chiral superfields ⇒ 49 scalar fields ($n = 49$)
- 973 total GIOs ($k = 973$)  **T. Gherghetta, C. Kolda, S. Martin**, *Nucl. Phys.*, **B468** (1996) 37

⇒ Set vevs for $u^i_L$, $u^i_R$, $d^i_L$, $d^i_R$ to zero by hand

⇒ This leaves $n = 13$ scalar fields and $k = 22$ GIOs

**J. Gray, YH. He, V. Jejjala and BDN**, *Phys. Lett.*, **B638** (2006) 253
**J. Gray, YH. He, V. Jejjala and BDN**, *Nucl. Phys.*, **B750** (2006) 1

| Operator | Explicit Sum | Index | Number |
|---|---|---|---|
| $LH_u$ | $L^\alpha_i H^\beta \epsilon_{\alpha\beta}$ | $i = 1, 2, 3$ | 3 |
| $H_u H_d$ | $H_\alpha \overline{H}_\beta \epsilon^{\alpha\beta}$ | – | 1 |
| $LLe$ | $L^i_\alpha L^j_\beta e^k \epsilon^{\alpha\beta}$ | $i, j = 1, 2, 3;\ k = 1, \ldots, j-1$ | 9 |
| $LH_d e$ | $L^i_\alpha \overline{H}_\beta \epsilon^{\alpha\beta} e^j$ | $i, j = 1, 2, 3$ | 9 |

$$W = C^0 H_u H_d + C^3_{ij} L_i H_d e_j = C^0 \sum_{\alpha,\beta} H_\alpha \overline{H}_\beta \epsilon^{\alpha\beta} + \sum_{i,j} C^3_{ij} e^j \sum_{\alpha,\beta} L^i_\alpha \overline{H}_\beta \epsilon^{\alpha\beta}$$

⇒ Flavor mixing matrices $C_{ij}$ generated randomly

- Dimensionality of some coefficients suppressed (irrelevant for topology)

# SU(2) Doublets and the Grassmannian Manifold



⇒ Consider a simple $SU(2)$ theory with $N_D$ doublets $\varphi_I$ ($I = 1, \ldots, N_D$)

- GIOs are monomials of the form

$$(\varphi_I \varphi_J) \equiv \varphi_I^\alpha \varphi_J^\beta \epsilon_{\alpha\beta}$$

- Antisymmetric under species interchange $I \leftrightarrow J$
- Full set of relations among these GIOs are themselves redundant $\longrightarrow$ syzygies exist
- In particular, they are subject to the Plücker relations

$$(\varphi_I \varphi_J)(\varphi_K \varphi_L) = (\varphi_I \varphi_K)(\varphi_J \varphi_L) + (\varphi_I \varphi_L)(\varphi_K \varphi_J)$$

⇒ Therefore, moduli space not trivial ($\mathbb{C}^p$), but instead the Grassmannian manifold $\mathrm{Gr}(N_D, 2)$, with dimension $2(N_D - 2)$

- $D = \{LL\} \longrightarrow \mathrm{Gr}(3,2) \simeq \mathbb{P}^2$
- $D = \{LL,\ L\overline{H}\} \longrightarrow \mathrm{Gr}(4,2)$
- $D = \{LL,\ L\overline{H},\ LH,\ H\overline{H}\} \longrightarrow \mathrm{Gr}(5,2)$

# Hypercharge and the Segrè Map



⇒ Some of the $SU(2)$-invariants $\varphi_I$ have non-vanishing hypercharge

- Let $\chi_I^{+1}$ and $\chi_J^{-1}$ be $SU(2)$-invariants with hypercharge $\pm 1$
  - ⋆ EXAMPLE: $\chi_I^{+1} = \{LL, \, L\overline{H}\}$
  - ⋆ EXAMPLE: $\chi_J^{-1} = \{e\}$
- $SU(2) \times U(1)_Y$-invariants built from $(\chi_I^{+1}\chi_J^{-1})$ are subject to additional constraint

$$(\chi_I^{+1}\chi_J^{-1})(\chi_K^{+1}\chi_L^{-1}) = (\chi_I^{+1}\chi_L^{-1})(\chi_K^{+1}\chi_J^{-1})$$

⇒ These equations define the Segrè embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$

$$\begin{array}{ccccc} LL \, \text{or} \, L\overline{H} & & e & & LLe \, \text{or} \, L\overline{H}e \\ \mathrm{Gr}(3,2) = \mathbb{P}^2 & \times & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^8 \\ [x_0 : x_1 : x_2] & & [z_0 : z_1 : z_2] & \to & x_i z_j \end{array} ,$$

- An affine cone over a projective variety of dimension 4, realized as the intersection of nine quadratic polynomials in $\mathbb{P}^8$
- Corresponding Hilbert series (of the first kind) is $H(t) = \frac{1+4t+t^2}{(1-t)^5}$
- *Palindromic* numerator for $H(t)$ indicates this manifold is Calabi-Yau

# Importance of the $LLe$ Operator



⇒ The richness of the vacuum structure of the MSSM EW sector comes from an unlikely place: the $LLe$ operator

⇒ These operators are subject to the relations

$$(L^i_\alpha L^j_\beta e^k \epsilon^{\alpha\beta})(L^m_\gamma L^n_\delta e^p \epsilon^{\gamma\delta}) = (L^m_\alpha L^n_\beta e^k \epsilon^{\alpha\beta})(L^i_\gamma L^j_\delta e^p \epsilon^{\gamma\delta})$$

- An operator with a common $e^i$ field is linearly proportional to another set of operators with a common $e^j$ field ($i \neq j$).
- With the labelling $y_{i+j-2+3(k-1)} = L^i_\alpha L^j_\beta e^k \epsilon^{\alpha\beta}$, the above relations become the ideal

$$\begin{aligned}\langle & y_1y_5 - y_2y_4,\ y_1y_6 - y_3y_4,\ y_2y_6 - y_3y_5, \\ & y_1y_8 - y_2y_7,\ y_1y_9 - y_3y_7,\ y_2y_9 - y_3y_8, \\ & y_4y_8 - y_5y_7,\ y_4y_9 - y_6y_7,\ y_5y_9 - y_6y_8\rangle\end{aligned}$$

- These are precisely the nine quadratics that define the Segrè variety

# Importance of the $LLe$ Operator



⇒ The nature of these relations, and the resulting manifold, depends crucially on the number of generations $N_f$

- $N_f = 1$, no $LLe$ operator possible. Moduli space is empty
- $N_f = 2$, only two $LLe$ operators and no relation possible: $\mathcal{M} = \mathbb{C}^2$
- $N_f = 3$, $\mathcal{M}$ is the Segrè variety

⇒ For $N_f \geq 4$, there are now new relations, such as

$$(L^i_\alpha L^j_\beta \epsilon^{\alpha\beta})(L^k_\gamma L^\ell_\delta \epsilon^{\gamma\delta}) + (L^i_\alpha L^k_\beta \epsilon^{\alpha\beta})(L^\ell_\gamma L^j_\delta \epsilon^{\gamma\delta}) + (L^i_\alpha L^\ell_\beta \epsilon^{\alpha\beta})(L^j_\gamma L^k_\delta \epsilon^{\gamma\delta}) = 0$$

- These relations are themselves redundant – a rich set of syzygies
- Now the Grassmanian no longer degenerates to projective space, yielding the Segrè variety
- Varieties are Calabi-Yau spaces that are not explicit toric varieties

⇒ In general, the $LLe$ moduli space is (the affine cone over) the Grassmanian $\mathrm{Gr}(N_f, 2) \times \mathbb{P}^{N_f - 1}$ with dimension $3N_f - 4$

# MSSM Electroweak Sector: No Superpotential



⇒ If we start with $W = 0$, we are studying the moduli space determined solely by the relations among various GIOs

- Bilinears only $\{LH,\, H\overline{H}\}$: no relations, $\mathcal{M} = \mathbb{C}^4$
- $LLe$ OR $L\overline{H}e$: $\mathcal{M}$ is the Segrè variety
- $LLe$ AND $L\overline{H}e$: $\mathcal{M}$ is a seven-dimensional variety, defined by

$$\begin{array}{ccccc} \{LL,\, L\overline{H}\} & & e & & \{LLe\, L\overline{H}e\} \\ \mathrm{Gr}(4,2) & \times & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^{17} \\ [x_0 : x_1 : x_2 : x_3 : x_4 : x_5] & & [z_0 : z_1 : z_2] & \rightarrow & x_i z_j \end{array}$$

subject to a single Plücker relation, with $H(t) = (1 + 11t + 15t^2 + 3t^3)/(1-t)^7$

- All GIOs $\{LH,\, H\overline{H}\, LLe,\, L\overline{H}e\}$: $\mathcal{M}$ a nine-dimension Calabi-Yau, defined by

$$\begin{array}{ccccc} \{LL,\, L\overline{H},\, LH,\, H\overline{H}\} & & e & & \{LLe\, L\overline{H}e, LH, H\overline{H}\} \\ \mathrm{Gr}(5,2) & \times & \mathbb{P}^2 & \longrightarrow & \mathbb{C}^{22} \\ [x_0 - x_5 : x_6 : x_7 : x_8 : x_9] & & [z_0 : z_1 : z_2] & \rightarrow & (x_i z_j, x_6, x_7, x_8, x_9) \end{array}$$

subject to five Plücker relations, with $H(t) = (1 + 13t + 28t^2 + 13t^3 + t^4)/(1-t)^9$

# MSSM with Charged Leptons



$$W = C^0 \sum_{\alpha,\beta} H_\alpha \overline{H}_\beta \epsilon^{\alpha\beta} + \sum_{i,j} C^3_{ij} \sum_{\alpha,\beta} L^i_\alpha \overline{H}_\beta e^j \epsilon^{\alpha\beta}$$

⇒ F-term equations force the following combinations to vanish

- $F_H$: $\quad 0 = C^0 \overline{H}_\beta \epsilon^{\alpha\beta}$
- $F_{\overline{H}}$: $\quad 0 = C^0 H_\alpha \epsilon^{\alpha\beta} + \sum_{i,j} C^3_{ij} L^i_\alpha e^j \epsilon^{\alpha\beta}$
- $F_{L^j}$: $\quad 0 = C^3_{ij} e^i \overline{H}_\beta \epsilon^{\alpha\beta}$
- $F_{e^i}$: $\quad 0 = \sum_{i,j} C^3_{ij} L^j_\alpha \overline{H}_\beta \epsilon^{\alpha\beta}$

⇒ Summary of these constraints:

- $\overline{H}$ vanishes, $L\overline{H}e$, $H\overline{H}$ not involved in vacuum manifold
- $F_{\overline{H}}$ equation relates $LH$ and $LLe$ operators in the vacuum

⇒ Therefore, $\mathcal{M}$ determined solely by the $LLe$ relations ⇒ the Segrè embedding

⇒ This only happens for $N_f = 3$ and $N_h = 1$!

# MSSM with Seesaw Neutrinos



⇒ Add in three right-handed neutrinos: now 16 total scalar fields, 25 GIOs

$$W = C^0 \sum_{\alpha,\beta} H_\alpha \overline{H}_\beta \epsilon \alpha\beta + \sum_{i,j} C^3_{ij} \sum_{\alpha,\beta} L^i_\alpha \overline{H}_\beta e^j \epsilon^{\alpha\beta} + \sum_{i,j} C^4_{ij} \nu^i \nu^j + \sum_{i,j} C^5_{ij} \sum_{\alpha,\beta} L^j_\alpha H_\beta \nu^i \epsilon^{\alpha\beta}$$

⇒ F-term equations force the following combinations to vanish

- $F_H$: $$0 = \sum_{i,j} C^5_{ij} \nu^i L^j_\alpha \epsilon^{\alpha\beta} - C^0 \overline{H}_\alpha \epsilon^{\alpha\beta}$$
- $F_{\overline{H}}$: $$0 = C^0 H_\alpha \epsilon^{\alpha\beta} + \sum_{i,j} C^3_{ij} L^i_\alpha e^j \epsilon^{\alpha\beta}$$
- $F_{L^j}$: $$0 = \sum_{i,j} C^5_{ij} \nu^i H_\beta \epsilon^{\alpha\beta} + C^3_{ij} e^i \overline{H}_\beta \epsilon^{\alpha\beta}$$
- $F_{\nu^i}$: $$0 = \sum_j C^4_{ij} \nu^i + \sum_{i,j} C^5_{ij} L^j_\alpha H_\beta \epsilon^{\alpha\beta}$$
- $F_{e^i}$: $$0 = \sum_{i,j} C^3_{ij} L^j_\alpha \overline{H}_\beta \epsilon^{\alpha\beta}$$

# MSSM with Seesaw Neutrinos



⇒ Summary of these equations

- GIOs $\nu$, $LH$, $H\overline{H}$ and $L\overline{H}e$ all vanish in the vacuum
- Vacuum manifold again determined by $LLe$ operators, but we do NOT obtain the Segrè variety

⇒ New constraint emerges from $F_{\overline{H}}$ equation: $\sum_{ij} C^3_{ij} e^i L^j_\alpha L^k_\beta \epsilon^{\alpha\beta} = 0$

- These are three (free $k$ index) new linear relations on the $LLe$ operators
- The defining ideal is modified to

$$\langle y_1y_5 - y_2y_4,\ y_1y_6 - y_3y_4,\ y_2y_6 - y_3y_5,\ y_1y_8 - y_2y_7$$
$$y_1y_9 - y_3y_7,\ y_2y_9 - y_3y_8,\ y_4y_8 - y_5y_7,\ y_4y_9 - y_6y_7$$
$$y_5y_9 - y_6y_8, y_1 - y_9,\ y_2 - y_6,\ y_4 - y_8\rangle$$

⇒ Defines a variety of dimension three; choose d.o.f. to be $[y_3 : y_5 : y_7]$

$$\begin{array}{lll} y1 \longrightarrow x_0x_2; & y_2 \longrightarrow x_0x_1; & y_3 \longrightarrow x_0^2; \\ y_4 \longrightarrow x_1x_2; & y_5 \longrightarrow x_1^2; & y_1 \longrightarrow x_1x_0 \\ y_7 \longrightarrow x_2^2; & y_8 \longrightarrow x_2x_1; & y_9 \longrightarrow x_2x_0 \end{array}$$

⇒ This is the Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$

# MSSM with Seesaw Neutrinos



$\Rightarrow$ How did this happen?

- New relation $\sum_{ij} C^3_{ij} e^i L^j_\alpha L^k_\beta \epsilon^{\alpha\beta} = 0$ effectively identifies $LL$ coordinates to $e$ coordinates
- Thus, the two $\mathbb{P}^2$ factors in the Segrè embedding are now identified

$$\begin{array}{ccccc} LL & & e & & LLe \\ \mathbb{P}^2 & \times & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^8 \\ [x_0 : x_1 : x_2] & & [z_0 : z_1 : z_2] & \rightarrow & x_i z_j \end{array},$$

becomes simply

$$\begin{array}{ccc} \mathbb{P}^2 & \rightarrow & \mathbb{P}^5 \\ [x_0 : x_1 : x_2] & \rightarrow & [x_0^2 : x_0 x_1 : x_1^2 : x_0 x_2 : x_1 x_2 : x_2^2] \end{array},$$

$\Rightarrow$ A similar identification between the variables of $\mathrm{Gr}(4,2)$ and $\mathbb{P}^3$ occurs for the case of $N_f = 4$ with seesaw neutrinos

- But here the identification is not complete, and a Severi variety does not emerge

# MSSM with Dirac Neutrinos



⇒ Something different occurs in cases where the Majorana mass term is absent

$$W = C^0 \sum_{\alpha,\beta} H_\alpha \overline{H}_\beta \epsilon \alpha\beta + \sum_{i,j} C^3_{ij} \sum_{\alpha,\beta} L^i_\alpha \overline{H}_\beta e^j \epsilon^{\alpha\beta} + \sum_{i,j} C^5_{ij} \sum_{\alpha,\beta} L^j_\alpha H_\beta \nu^i \epsilon^{\alpha\beta}$$

- $F_{\overline{H}} = 0$ still gives $\sum_{ij} C^3_{ij} e^i L^j_\alpha L^k_\beta \epsilon^{\alpha\beta} = 0$
  - ⋆ Effectively identifies $LL$ coordinates to $e$ coordinates
  - ⋆ Would seem to imply a Veronese variety
- $F_{\nu^i} = 0$ now implies $\sum_{ij} C^5_{ij} \nu^i L^j_\alpha L^k_\beta e^\ell \epsilon_{\alpha,\beta} = 0$
  - ⋆ Would-be $\nu^i$ d.o.f. now related back to the remaining $LLe$ d.o.f.
  - ⋆ So this is a Veronese-like embedding into a higher-dimensional space

⇒ We refer to it as the "deformed Veronese" variety, which is defined by

$$\begin{array}{ccccc} LL \simeq e & & \nu & & \{\nu,\, LLe\} \\ \mathbb{P}^2 & \times & \mathbb{C} & \longrightarrow & \mathbb{C}^9 \\ [x_0 : x_1 : x_2] & & [\lambda] & \to & [x_0^2 : x_0 x_1 : x_1^2 : x_0 x_2 : x_1 x_2 : x_2^2 : \lambda x_0 : \lambda x_1 : \lambda x_2] \end{array} ,$$

which is a non-compact, 4D toric CY with $H(t) = (1 + 5t + t^2)/(1-t)^4$

# Example of Outcomes with Bilinear Deformations



⇒ Inclusion of both gauge-invariant trilinears gives a trivial background

⇒ Computation of vacuum manifold $\mathcal{M}$ for various bilinear deformations

| Deformation | $L\overline{H}e$ | $LLe$ | $L\overline{H}e + LH\nu$ | $LLe + LH\nu$ |
|---|---|---|---|---|
| none | Segrè $\times \mathbb{C}^4$ | def. Ver. $\times \mathbb{C}$ | def. Ver. $\times \mathbb{C}$ | $(10\|6,\ 14\|3^6 2^3)$ |
| $+H\overline{H}$ | Segrè | point | def. Ver. | $\mathbb{C}^3$ |
| $+LH$ | $\mathbb{C}$ | conifold | def. Ver. $\times \mathbb{C}$ | $(10\|6,\ 14\|3^6 2^3)$ |
| $+LH + H\overline{H}$ | $\mathbb{C}$ | point | def. Ver. | $\mathbb{C}^3$ |
| $+\nu^2$ | Segrè $\times \mathbb{C}^4$ | def. Ver. $\times \mathbb{C}$ | Veronese $\times \mathbb{C}$ | Veronese $\times \mathbb{C}$ |
| $+\nu^2 + H\overline{H}$ | Segrè | point | Veronese | $\emptyset$ |
| $+\nu^2 + LH$ | $\mathbb{C}$ | conifold | conifold $\times \mathbb{C}^2$ | conifold $\times \mathbb{C}^2$ |
| $+\nu^2 + LH + H\overline{H}$ | $\mathbb{C}$ | point | conifold $\times \mathbb{C}^2$ | $\emptyset$ |

⇒ A systematic treatment of the nature of R-parity from the point of view of geometry is now underway

- “Interesting” geometry (non-trivial structure and small dimensionality) seems to prefer the “natural” R-parity assignment
- Seesaw mechanism plus $\mu$-term seems to demand the “natural” R-parity assignment

# Future Prospects and Potential



⇒ Ultimately we expect string theory to motivate/identify the connection between compactification geometry and geometry of the gauge theory vacuum space

- For example, the MSSM likes the Veronese embedding
- The Veronese variety is a Severi variety with underlying $SU(3)$ isometry in which two $\mathbb{P}^2$ factors are identified
- This is getting closer to the sorts of data that can be sought after in large classes of string compactifications

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⇒ Vacuum geometry cares about *both* the superpotential *and* the GIO set which defines the polynomial ring

- Two theories with the same $W$ but different GIO sets *give different moduli spaces*!
- Example: imposing $R = (-1)^{-L}$ and $U(1)_L$ symmetries do not give the same answers for the MSSM superpotential
- There may be a geometrical origin for R-parity conservation that is not obviously due to field theory symmetries

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⇒ Finally, non-trivial geometries *require* $N_f \geq 3$. Small dim($\mathcal{M}$) requires $N_f < 4$. **Three generations are special**.

# Backup Slides

# Interlude on Complexified Gauge Groups



⇒ $N = 1$ SUSY action on previous page has very large symmetry group

$$\Phi_i \rightarrow g \cdot \Phi_i \ ; \qquad e^V \rightarrow (g^\dagger)^{-1} e^V g^{-1}$$

where $g = e^{i\Lambda}$ and $\Lambda$ is a chiral superfield

⇒ Normally we don't see all of this invariance because we work in Wess-Zumino gauge

$$V_a = -\theta\sigma_\mu\bar{\theta}v_a^\mu + i\theta^2\overline{\theta\lambda}_a - i\bar{\theta}^2\theta\lambda_a + \frac{1}{2}\theta^2\bar{\theta}^2 D_a$$

- Residual gauge symmetry is just the usual one with real parameters
- In this gauge the D-flatness condition is $D_a = \sum_i q_i \phi_i^\dagger t_a \phi_i = 0$
- NOTE: This constraint is not holomorphic in the fields $\phi_i$

# Interlude on Complexified Gauge Groups



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- NOTE: This constraint is not holomorphic in the fields $\phi_i$

$\Rightarrow$ Now imagine choosing a less restrictive gauge such that

$$V_a = C_a - \theta\sigma_\mu\bar{\theta}v_a^\mu + i\theta^2\overline{\theta\lambda}_a - i\bar{\theta}^2\theta\lambda_a + \frac{1}{2}\theta^2\bar{\theta}^2 D_a$$

**Buccella, Derendinger, Ferrara and Savoy**, *Phys. Lett.*, **B115** (1982)
**C. Procesi and G. Schwarz**, *Phys. Lett.*, **B161** (1985)
**M. Luty and W. Taylor**, *Phys. Rev.*, **D53** (1996)

# Interlude on Complexified Gauge Groups



- Residual gauge symmetry is now the complexification $\mathcal{G}^C$ of group $\mathcal{G}$
- The D-flatness condition in this gauge becomes

$$\frac{\partial}{\partial C_a} \sum_i \phi_i^\dagger e^C \phi_i = 0$$

$\Rightarrow$ Now imagine some $\phi_i^0$ which satisfies the above and $\partial W/\partial \phi_i|_{\phi_i=\phi_i^0}$.

- Use the extra gauge invariance to define $\hat{\phi}_i^0 \equiv e^{C/2}\phi_i^0$

$$\frac{\partial}{\partial \hat{C}_a} \sum_i (\hat{\phi}_i^0)^\dagger e^{\hat{C}} \hat{\phi}_i^0 = \frac{\partial}{\partial \hat{C}_a} \sum_i X(e^{\hat{C}/2}\hat{\phi}_i^0) = 0; \qquad X(\phi) \equiv \phi^\dagger \phi$$

- F-flatness conditions holomorphic and invariant under $\mathcal{G}^C$
- Can always perform such a transformation to take $C_a \rightarrow 0$, giving an F- and D-flat solution in WZ gauge
- The D-flatness conditions are now trivial: just a gauge-fixing condition!

$\Rightarrow$ **Gauge invariant holomorphic operators form a basis for these $D$-orbits**

# Attacking the MSSM



⇒ Seven species of chiral superfields ⇒ 49 scalar fields ($n = 49$)

⇒ All 973 possible GIOs tabulated below ($k = 973$)

**T. Gherghetta, C. Kolda, S. Martin**, *Nucl. Phys.*, **B468** (1996)

| Operator | Explicit Sum | Index | Number |
|---|---|---|---|
| $LH_u$ | $L_i^\alpha H^\beta \epsilon_{\alpha\beta}$ | $i = 1, 2, 3$ | 3 |
| $H_u H_d$ | $H_\alpha (H_d)_\beta \epsilon^{\alpha\beta}$ | NA | 1 |
| $LLe$ | $L_\alpha^i L_\beta^j e^k \epsilon^{\alpha\beta}$ | $i, j = 1, 2, 3;\ k = 1, \ldots, j-1$ | 9 |
| $LH_d e$ | $L_\alpha^i (H_d)_\beta e^j \epsilon^{\alpha\beta}$ | $i, j = 1, 2, 3$ | 9 |
| $udd$ | $u_a^i d_b^j d_c^k \epsilon^{abc}$ | $i, j = 1, 2, 3;\ k = 1, \ldots, j-1$ | 9 |
| $QdL$ | $Q_{a,\alpha}^i d_a^j L_\beta^k \epsilon^{\alpha\beta}$ | $i, j, k = 1, 2, 3$ | 27 |
| $QuH_u$ | $Q_{a,\alpha}^i u_a^j (H_u)_\beta \epsilon^{\alpha\beta}$ | $i, j = 1, 2, 3$ | 9 |
| $QdH_d$ | $Q_{a,\alpha}^i d_a^j (H_d)_\beta \epsilon^{\alpha\beta}$ | $i, j = 1, 2, 3$ | 9 |
| $QQQL$ | $Q_{a,\beta}^i Q_{b,\gamma}^j Q_{c,\alpha}^k L_\delta^l \epsilon^{abc} \epsilon^{\beta\gamma} \epsilon^{\alpha\delta}$ | $i, j, k, l = 1, 2, 3;\ i \neq k, j \neq k,$ $j < i, (i, j, k) \neq (3, 2, 1)$ | 24 |
| $QuQd$ | $Q_{a,\alpha}^i u_a^j Q_{b,\beta}^k d_b^l \epsilon^{\alpha\beta}$ | $i, j, k, l = 1, 2, 3$ | 81 |
| $QuLe$ | $Q_{a,\alpha}^i u_a^j L_\beta^k e^l \epsilon^{\alpha\beta}$ | $i, j, k, l = 1, 2, 3$ | 81 |
| $uude$ | $u_a^i u_b^j d_c^k e^l \epsilon^{abc}$ | $i, j, k, l = 1, 2, 3;\ j < i$ | 27 |
| $QQQH_d$ | $Q_{a,\beta}^i Q_{b,\gamma}^j Q_{c,\alpha}^k (H_d)_\delta \epsilon^{abc} \epsilon^{\beta\gamma} \epsilon^{\alpha\delta}$ | $i, j, k, l = 1, 2, 3; i \neq k, j \neq k,$ $j < i, (i, j, k) \neq (3, 2, 1)$ | 8 |
| $QuH_d e$ | $Q_{a,\alpha}^i u_a^j (H_d)_\beta e^k \epsilon^{\alpha\beta}$ | $i, j, k = 1, 2, 3$ | 27 |
| $dddLL$ | $d_a^i d_b^j d_c^k L_\alpha^m L_\beta^n \epsilon^{abc} \epsilon_{ijk} \epsilon^{\alpha\beta}$ | $m, n = 1, 2, 3;\ n < m$ | 3 |

$i, j, k = 1, 2, 3 \leftrightarrow$ flavor indices, $a, b, c = 1, 2, 3 \leftrightarrow$ color indices, $\alpha, \beta, \gamma = 1, 2 \leftrightarrow SU(2)_L$ indices

# Attacking the MSSM



| Operator | Explicit Sum | Index | Number |
|---|---|---|---|
| $uuuee$ | $u^i_a u^j_b u^k_c e^m e^n \epsilon^{abc}\epsilon_{ijk}$ | $m,n=1,2,3;\ n\leq m$ | 6 |
| $QuQue$ | $Q^i_{a,\alpha}u^j_a Q^k_{b,\beta}u^m_b e^n \epsilon_{\alpha\beta}$ | $i,j,k,m,n=1,2,3;$ as$\{(i,j),(k,m)\}$ | 108 |
| $QQQQu$ | $Q^i_{a,\beta}Q^j_{b,\gamma}Q^k_{c,\alpha}Q^m_{f,\delta}u^n_f \epsilon^{abc}\epsilon^{\beta\gamma}\epsilon^{\alpha\delta}$ | $i,j,k,m=1,2,3;\ i\neq m,$ $j\neq m, j<i,$ $(i,j,k)\neq(3,2,1)$ | 72 |
| $dddLH_d$ | $d^i_a d^j_b d^k_c L^m_\alpha (H_d)_\beta \epsilon^{abc}\epsilon_{ijk}\epsilon_{\alpha\beta}$ | $m=1,2,3$ | 3 |
| $uudQdH_u$ | $u^i_a u^j_b d^k_c Q^m_{f,\alpha} d^n_f (H_u)_\beta \epsilon^{abc}\epsilon_{\alpha\beta}$ | $i,j,k,m=1,2,3;\ j<i$ | 81 |
| $(QQQ)_4LLH_u$ | $(QQQ)_4^{\alpha\beta\gamma}L^m_\alpha L^n_\beta (H_u)_\gamma$ | $m,n=1,2,3;\ n\leq m$ | 6 |
| $(QQQ)_4LH_uH_d$ | $(QQQ)_4^{\alpha\beta\gamma}L^m_\alpha (H_u)_\beta (H_d)_\gamma$ | $m=1,2,3$ | 3 |
| $(QQQ)_4H_uH_dH_d$ | $(QQQ)_4^{\alpha\beta\gamma}(H_u)_\alpha (H_d)_\beta (H_d)_\gamma$ | NA | 1 |
| $(QQQ)_4LLLe$ | $(QQQ)_4^{\alpha\beta\gamma}L^m_\alpha L^n_\beta L^p_\gamma e^q$ | $m,n,p,q=1,2,3;$ $n\leq m;\ p\leq n$ | 27 |
| $uudQdQd$ | $u^i_a u^j_b d^k_c Q^m_{f,\alpha} d^n_f Q^p_{g,\beta} d^q_g \epsilon^{abc}\epsilon_{\alpha\beta}$ | $i,j,k,m,n,p,q=1,2,3;$ $j<i,$ as$\{(m,n),(p,q)\}$ | 324 |
| $(QQQ)_4LLH_de$ | $(QQQ)_4^{\alpha\beta\gamma}L^m_\alpha L^n_\beta (H_d)_\gamma e^p$ | $m,n,p=1,2,3;\ n\leq m$ | 9 |
| $(QQQ)_4LH_dH_de$ | $(QQQ)_4^{\alpha\beta\gamma}L^m_\alpha (H_d)_\beta (H_d)_\gamma e^n$ | $m,n=1,2,3$ | 9 |
| $(QQQ)_4H_dH_dH_de$ | $(QQQ)_4^{\alpha\beta\gamma}(H_d)_\alpha (H_d)_\beta (H_d)_\gamma e^m$ | $m=1,2,3$ | 3 |

In the above we defined $[(QQQ)_4]_{\alpha\beta\gamma} = Q^i_{a,\alpha}Q^j_{b,\beta}Q^k_{c,\gamma}\epsilon^{abc}\epsilon^{ijk}$

$\Rightarrow$ The reason the problem is unsolved after two decades...

# Hilbert Series



$\Rightarrow$ Hilbert series provides technology for enumerating GIOs in a supersymmetric quantum field theory

$\Rightarrow$ For a variety $\mathcal{M} \subset \mathbb{C}\left[y_1, \ldots, y_k\right]$, the Hilbert series is the formal series

$$H(t) = \sum_{n=-\infty}^{\infty} \dim \mathcal{M}_n \, t^n = \frac{P(t)}{(1-t)^d}$$

- Hilbert series encodes information about the chiral ring and geometric features of the variety
- dim($\mathcal{M}_n$) is the number of independent polynomials of degree $n$ on $\mathcal{M}$

$\Rightarrow$ A palindromic Hilbert Series obeys $H(t) = \sum_{k=0}^{N} a_k t^k$ with $a_k = a_{N-k}$

- By a theorem of Stanley, the corresponding algebraic variety is Calabi-Yau (in the sense of a trivial canonical sheaf)

# Segrè Embedding Explicitly



To see the Segrè embedding explicitly let $x_i$ be the coordinates of one $\mathbb{P}^2$ (representing the three independent $LL$ operators), and let $z_i$ be coordinates of the second $\mathbb{P}^2$ (representing the three independent $e_i$ operators)

⇒ The Segrè embedding is given by

$$\begin{array}{ccccc} \mathbb{P}^2 & \times & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^8 \\ [x_0 : x_1 : x_2] & & [z_0 : z_1 : z_2] & & x_i z_j \end{array},$$

⇒ The $LLe$ syzygies are then recovered when we identify the coordinates $y_i$ of $\mathbb{P}^8$ via

$$\begin{array}{ccc} y_1 \to z_0 x_2, & y_2 \to z_0 x_1, & y_3 \to z_0 x_0, \\ y_4 \to z_1 x_2, & y_5 \to z_1 x_1, & y_6 \to z_1 x_0, \\ y_7 \to z_2 x_2, & y_8 \to z_2 x_1, & y_9 \to z_2 x_0, \end{array}$$

# Severi Varieties



⇒ Any smooth non-degenerate algebraic variety $X$ of (complex) dimension $n$ embedded into $\mathbb{P}^m$ with $m < \frac{3}{2}n + 2$ has the property that its secant variety Sec(X) is equal to $\mathbb{P}^m$ [Hartshorne-Zak]

- Limiting case where $m = \frac{3}{2}n + 2$ and Sec(X) $\neq \mathbb{P}^n$ is a Severi variety

⇒ All Severi varieties have been classified [Zak]; there are only four:

1. $n = 2$: The Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$
2. $n = 4$: The Segrè variety $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$;
3. $n = 8$: The Grassmannian $\mathrm{Gr}(6, 2)$ of two-planes in $\mathbb{C}^6$, embedded into $\mathbb{P}^{14}$
4. $n = 16$: The Cartan variety of the orbit of the highest weight vector of a certain non-trivial representation of $E_6$

⇒ Only the first two cases involve products of projective spaces

# Severi Varieties



⇒ There exist precisely four division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$

⇒ If we imagine the projective planes formed from each of these division algebras, the complexification of these spaces are precisely homeomorphic to the four Severi varieties

| Projective Plane | Severi Variety | Homogenous Space |
|---|---|---|
| $\mathbb{RP}^2$ | $\mathbb{CP}^2$ | $SU(3)/S(U(1) \times U(2))$ |
| $\mathbb{CP}^2$ | $\mathbb{CP}^2 \times \mathbb{CP}^2$ | $SU(3)^2/S(U(1) \times U(2))^2$ |
| $\mathbb{HP}^2$ | $\mathrm{Gr}(6,2)$ | $SU(6)/S(U(2) \times U(4))$ |
| $\mathbb{OP}^2$ | $S$ | $E_6/\mathrm{Spin}(10) \times U(1)$ |

# One Generation MSSM



⇒ Drop all flavor indices ($i = j = k = 1$) so now $n = 9$

⇒ There are now only 9 GIOs (one of each variety)

$$LH_u,\ H_uH_d,\ QdL,\ QuH_u,\ QdH_d,\ LH_de,\ QuQd,\ QuLe,\ QuH_de$$

⇒ Simplified superpotential

$$\begin{aligned} W_0 \quad &= \quad \lambda^0 \sum_{\alpha,\beta} H_u^\alpha H_d^\beta \epsilon_{\alpha\beta} + \lambda^1 \sum_{\alpha,\beta,a} Q_{a,\alpha}(H_u)_\beta u_a \epsilon^{\alpha\beta} \\ &\quad + \lambda^2 \sum_{\alpha,\beta,a} Q_{a,\alpha}(H_d)_\beta d_a \epsilon^{\alpha\beta} + \lambda^3 \sum_{\alpha,\beta} L_\alpha (H_d)_\beta e \epsilon^{\alpha\beta} \end{aligned}$$

⇒ Computation of vacuum manifold $\mathcal{M}$ for various deformations

| $W_0+?$ | $\dim(\mathcal{M})$ | $\mathcal{M}$ | $W_0+?$ | $\dim(\mathcal{M})$ | $\mathcal{M}$ |
|---|---|---|---|---|---|
| $0$ | $1$ | $\mathbb{C}$ | $QuQd$ | $1$ | $\mathbb{C}$ |
| $LH_u$ | $0$ | point | $QuLe$ | $1$ | $\mathbb{C}$ |
| $QdL$ | $0$ | point | $QuH_de$ | $1$ | $\mathbb{C}$ |