Geodesic Completeness and Non-Local Theories of Gravity

The Raychaudhuri Equation and non-singular cosmologies

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arXiv:1509.01247
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Outline

- Review the Raychaudhuri Equation
- Raychaudhuri Equation in General Relativity
- Outline of calculation for a non-local bouncing Cosmologies
  - in FRW
  - around de Sitter space
- Relation to Gravitational Entropy

Central point: Present a method whereby any modified theory of gravity may be tested for singularities under a limited set of assumptions.

The Raychaudhuri Equation

- Timelike:

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 = -\sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}\xi^\mu\xi^\nu$$

- Null:

$$\frac{d\theta}{d\tau} + \frac{1}{2}\theta^2 = -\sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu$$

where ξ^μ and k^μ are congruences of *timelike* and *null geodesics respectively such that* $\xi^\mu\xi_\mu = -1$ and $k^\mu k_\mu = 0$. Expansion $\theta \equiv \nabla_\mu k^\mu$

Wald, General Relativity (1984)

$$\boldsymbol{\frac{d\theta}{d\tau} + \frac{1}{2}\theta^2 = -\sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}k^{\mu}k^{\nu}}$$

May be reduced to the inequality

$$\frac{d\theta}{d\tau} + \frac{1}{2}\theta^2 \leq -R_{\mu\nu}k^{\mu}k^{\nu}$$

From the perfect fluid equation, we find that in GR, we have

$$R_{\mu\nu}k^{\mu}k^{\nu} = \kappa T_{\mu\nu}k^{\mu}k^{\nu} = \kappa(\rho + p) \geq 0$$

by the Null Energy Condition (NEC), so that we write the *null convergence condition* as

$$\frac{d\theta}{d\tau} + \frac{1}{2}\theta^2 \leq 0$$

which implies a spacetime singularity in GR. In other words

$$R_{\mu\nu}k^{\mu}k^{\nu} \geq 0 \Rightarrow \text{Singularity}$$

Wald, General Relativity (1984)

How can this condition be modified so that a spacetime may be rendered singularity-free?

- A spacetime will be geodesically incomplete (admit singularities) unless one of the following is achieved
  - Violation of Appropriate Energy conditions (Weak / Null)
  - **Modification of Classical Einstein Equations**
    - **Requiring $R_{\mu\nu}k^{\mu}k^{\nu} < 0$**

Example 1: FRW

For the non-local action

Biswas, Mazumdar, Siegel [arXiv:hep-th/0508194v2]

$$S = \int d^4x \frac{\sqrt{-g}}{2}\left(M_P^2 R + R\mathcal{F}(\Box)R - 2\Lambda\right)$$

We have the equation of motion

Biswas, Conroy, Koshelev, Mazumdar [arXiv:1308.2319]

$$T_{\mu\nu} = M_P^2 G_{\mu\nu} + g_{\mu\nu}\Lambda + 2\lambda G_{\mu\nu}\mathcal{F}_1(\Box)R + \frac{\lambda}{2}g_{\mu\nu}R\mathcal{F}_1(\Box)R$$
$$- 2\lambda\left(\nabla_\mu\partial_\nu - g_{\mu\nu}\Box\right)\mathcal{F}_1(\Box)R - \lambda\Omega^1_{\mu\nu} + \frac{\lambda}{2}g_{\mu\nu}(\Omega^{\sigma}_{1\sigma} + \bar{\Omega}_1)$$

Contract with $k^\mu k^\nu$ such that $\boldsymbol{k^\mu k_\mu = 0}$ with

$$R_{\mu\nu}k^\mu k^\nu = \frac{(\rho + p) + 2\lambda k^\mu k^\nu \nabla_\mu\partial_\nu\mathcal{F}_1(\Box)R + \lambda k^\mu k^\nu \Omega^1_{\mu\nu}}{(M_P^2 + 2\lambda\mathcal{F}_1(\Box)R)}$$

where

$$\Omega^1_{\mu\nu} = \sum_{n=1}^{\infty} f_{1_n} \sum_{l=0}^{n-1} \nabla_\mu R^{(l)}\nabla_\nu R^{(n-l-1)}, \quad \mathcal{F}(\Box) = \sum_{n=0}^{\infty}\frac{f_n}{M^{2n}}\Box^n$$

The Set-up

- Homogenous and Isotropic FRW metric: $ds^2 = -dt^2 + a^2(t)dr^2$
- With generic symmetric ‘bouncing’ scale factor, i.e.
  - $a(t) = a_0 + a_2 t^2 + a_4 t^4 + \cdots$ = **Even Function** $\Rightarrow \dot{a}(0) = \dddot{a}(0) = \cdots = 0$
  - $H(t) = \frac{\dot{a}(t)}{a(t)}$ = **Odd Function** $\Rightarrow H(0) = \ddot{H}(0) = \cdots = 0$
  - $R = 6\left(\dot{H} + 2H^2\right) = \mathrm{R_0} + \mathrm{R_2 t^2} + \cdots =$ **Even Function**
- Accelerated expansion of the Universe, i.e. $\ddot{a} > 0 \Rightarrow R_0 > 0$
- Require $R_{\mu\nu} k^\mu k^\nu < 0$ to avoid singularities

Outline of Calculation

- At bounce $t \to 0$, to avoid singularities we require

$$R_{\mu\nu}k^{\mu}k^{\nu} = \frac{(\rho + p) + 2(k^0)^2\partial_t^2(\mathcal{F}(\Box)R)}{M_p^2 + 2\mathcal{F}(\Box)R} < 0$$

- We then employ the diffusion equation method at $t \to 0$ to find

$$(\mathcal{F}(\Box)R)(0) = R_0\mathcal{F}(y), \;\; (\partial_t^2\mathcal{F}(\Box)R)(0) = 2R_2\mathcal{F}(y)$$

Calcagni, Montobbio, Nardelli, arXiv:0705.3043

- Substituting, gives the following set of inequalities

$$\frac{\rho + p}{2R_0} \lessgtr yR_0\mathcal{F}(y), \quad \frac{M_P^2}{2R_0} \gtrless -\mathcal{F}(y)$$

- Where we have defined $y \equiv -\frac{2R_2}{R_0}$, $R(t) = R_0 + R_2t^2 + \cdots$ and $R_0 > 0$

Ghost-Free Condition

Biswas, Mazumdar, Siegel [arXiv:hep-th/0508194v2]

$$\mathcal{F}(\Box) = \frac{a(\Box/M^2) - 1}{\Box/M^2} \implies \text{Ghost-free}$$

- Where $a(\Box/M^2)$ is an entire function which we choose to be

$$\mathcal{F}(\Box) = \frac{e^{-\Box/M^2} - 1}{\Box/M^2}$$

- So that in our derived set of inequalities

$$\frac{\rho + p}{2R_0} \lesseqgtr yR_0\mathcal{F}(y), \quad \frac{M_P^2}{2R_0} \gtreqless -\mathcal{F}(y)$$

- Are constrained due to the particular nature of this function

$$\mathcal{F}(y < 0) < -1 \text{ and } -1 \leq \mathcal{F}(y \geq 0) < 0 \qquad y \equiv -\frac{2R_2}{R_0}, R_0 > 0$$

Results

$$\frac{\rho + p}{2R_0} \lesseqgtr yR_0\mathcal{F}(y), \quad \frac{M_P^2}{2R_0} \gtreqless -\mathcal{F}(y)$$

- Assume $R_2 \geq 0$: Without violating the NEC ($\rho + p \geq 0$), the upper signs then give

$$\frac{(\rho + p)}{2R_0} > 0, \ \frac{M_P^2}{2R_0} < 1$$

- Assume $R_2 \leq 0$, the lower signs give

$$\frac{M_P^2}{2R_0} < -\mathcal{F}(y) \leq 1$$

- Which is analogous to the previous inequality
- Thus $R_{\mu\nu}k^{\mu}k^{\nu} < 0$ without violating the NEC
- That is, we have shown that a non-local cosmology may be rendered **geodesically past-complete** for any value of R_2

  - Recall: $y \equiv -\frac{2R_2}{R_0}$ and $R_0 > 0$

Example 2: Around de Sitter space

For the non-local action

Biswas, Mazumdar, Siegel [arXiv:hep-th/0508194v2]

$$S = \int d^4x \frac{\sqrt{-g}}{2} \left(M_P^2 R + R\mathcal{F}(\Box)R - 2\Lambda \right)$$

We have the equation of motion

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$$T_{\mu\nu} = M_P^2 G_{\mu\nu} + g_{\mu\nu}\Lambda + 2\lambda G_{\mu\nu}\mathcal{F}_1(\Box)R + \frac{\lambda}{2} g_{\mu\nu} R\mathcal{F}_1(\Box)R$$

$$- 2\lambda \left(\nabla_\mu \partial_\nu - g_{\mu\nu}\Box\right) \mathcal{F}_1(\Box)R - \lambda\Omega^1_{\mu\nu} + \frac{\lambda}{2} g_{\mu\nu}(\Omega_{1\sigma}^{\ \sigma} + \bar{\Omega}_1)$$

Perturb around de Sitter space using $g_{\mu\nu} \to \bar{g}_{\mu\nu} + h_{\mu\nu}$, giving

$$\left(M_P^2 + 24H^2\lambda\tilde{f}_{1_0}\right)\left(r^\mu_\nu - \frac{1}{2}\delta^\mu_\nu r\right) = T^\mu_\nu + 2\lambda\left(\nabla^\mu\partial_\nu - \delta^\mu_\nu\Box\right)\tilde{\mathcal{F}}_1(\Box)r - 6\lambda H^2\delta^\mu_\nu\tilde{\mathcal{F}}_1(\Box)r$$

where $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} \frac{f_n}{M^{2n}}\Box^n$ and $r = \delta R$

Background metric: $ds^2 = -dt^2 + e^{2Ht}\boldsymbol{dr}^2$

$$\left(M_P^2 + 24H^2\lambda\tilde{f}_{1_0}\right)\left(r^\mu_\nu - \frac{1}{2}\delta^\mu_\nu r\right) = T^\mu_\nu + 2\lambda\left(\nabla^\mu\partial_\nu - \delta^\mu_\nu\Box\right)\tilde{\mathcal{F}}_1(\Box)r - 6\lambda H^2\delta^\mu_\nu\tilde{\mathcal{F}}_1(\Box)r$$

- From the linearised EOM given above, we may deduce the following
- (1) The **Null Energy Condition**

$$\rho + p = \frac{1}{3}\left(M_P^2 + 24H^2\lambda f_{1_0}\right)\left(r - 4r^0_0\right) - 2\lambda\left(\partial_t^2 - H\partial_t\right)\mathcal{F}_1(\Box)r,$$

  - Obtained by taking the 00 and ij-components. Null energy Condition requires $\rho + p \geq 0$
- (2) **Contracting** with null vectors k^μ , such that $k^\mu k_\mu = 0$, gives the contribution to the Raychaudhuri equation.

$$r^\mu_\nu k^\nu k_\mu = \left(M_P^2 + 24\lambda H^2 f_{1_0}\right)^{-1}(k^0)^2\left[(\rho + p) + 2\lambda\left(\partial_t^2 - H\partial_t\right)\mathcal{F}_1(\Box)r\right] < 0$$

- Combining (1) and (2) gives the general condition for such a non-local theory, around de Sitter, to be free of singularities, which is simply

$$r < 4r^0_0$$

Gravitational Entropy

- To compute the gravitational entropy, we rewrite the action (with $\lambda = M_P^2 \alpha$)

$$I = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left(R - 2M_P^{-2}\Lambda + \alpha R \mathcal{F}(\Box) R \right)$$

- Which was found to be

Conroy, Mazumdar, Talaganis, Teimouri [arXiv:1509.01247]

$$S_I = \frac{A_H^{dS}}{4G_4} \left(1 - 8\alpha M_P^{-2}\Lambda \right),$$

- Where we chose the ghost-free form

Biswas, Mazumdar, Siegel [arXiv:hep-th/0508194v2]

$$\mathcal{F}(\Box) = \frac{e^{-\Box/M^2} - 1}{\Box/M^2}$$

- The primary thing to note here is that a non-physical *negative entropy* state is realised for

$$8\Lambda\alpha > M_P^2 \quad \text{i.e.} \quad \mathrm{M_P^4} - 8\lambda\Lambda < 0$$

Entropy and the bounce

- Recall that the condition for avoiding singularities is as follows

$$r^\mu_\nu k^\nu k_\mu = \left(M_P^2 + 24\lambda H^2 f_{1_0}\right)^{-1} (k^0)^2 \left[(\rho + p) + 2\lambda\left(\partial_t^2 - H\partial_t\right)\mathcal{F}_1(\Box) r\right] < 0$$

i.e.

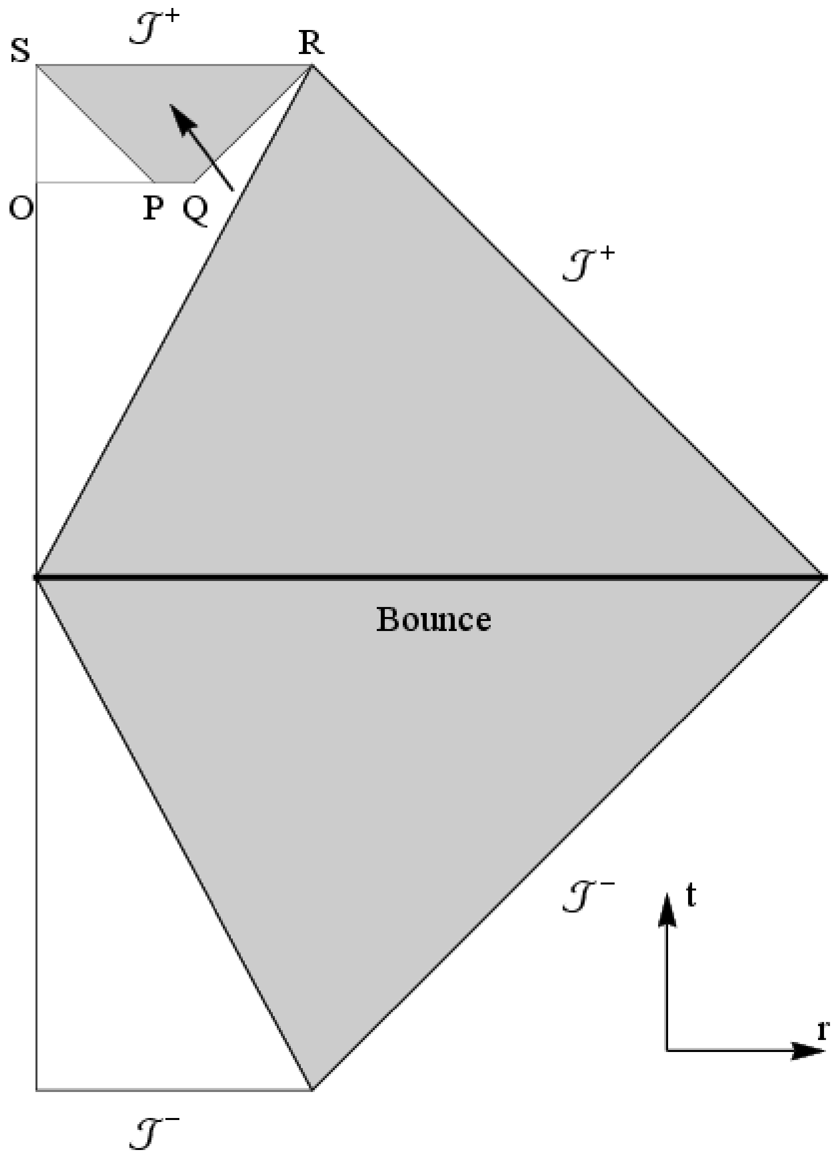
$$\frac{M_P^2}{M_P^4 - 8\lambda\Lambda} \gtrless 0, \qquad \left(\partial_t^2 - H\partial_t\right)\mathcal{F}_1(\Box) r \lessgtr 0$$

- As we have seen, to ensure we avoid negative entropy, we may discount the lower signs as $M_P^2 - 8\lambda\Lambda > 0$. Thus, we simply require

$$\left(\partial_t^2 - H\partial_t\right)\mathcal{F}_1(\Box) r < 0$$

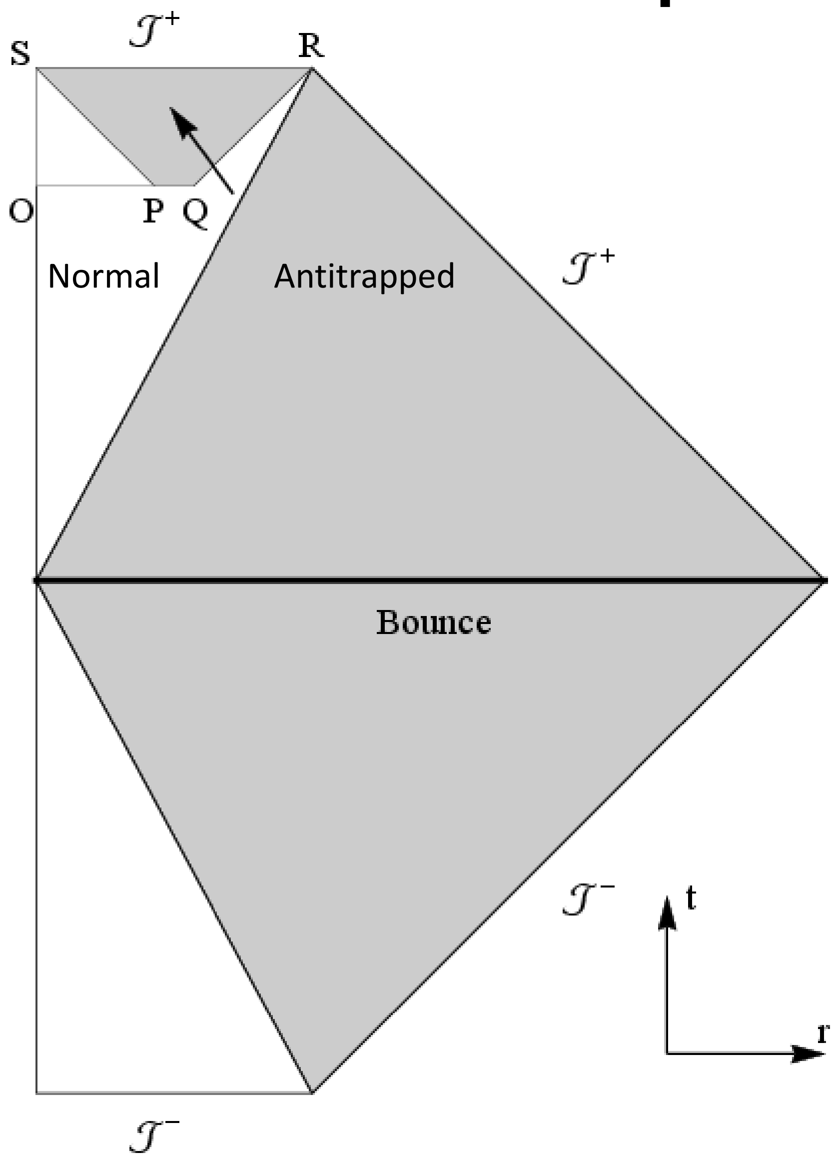
- The solution is *saturated* if we require a **zero entropy** state at the bounce. The nature of entropy at the bounce point $t = 0$ requires further study.
- We may also solve for the linearised EOMs to find $r(t) \rightarrow c_1 e^{\sigma_1 t} + c_2 e^{\sigma_2 t}$ and analytically show that singularities are avoid for all values of $\sigma_{1,2}$ except for $\sigma_1 = \sigma_2 = 0$

Summary



- Showed via the Raychaudhuri Equation that GR admits a spacetime singularity when a suitable energy condition is held.
- Presented a **general method**, applicable to any modified theory of gravity for testing a spacetime for singularities
- Examples of a **non-singular and ghost-free** theories in FRW and around de Sitter were given.
- Relation to gravitational **entropy around the bounce point** was explored.

Notes on Expansion and Surfaces



- Expansion may also be defined as

$$\theta \equiv \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} k^\mu)$$

- In FRW, we then compute the ingoing and outgoing expansions

$$\theta_{IN,OUT} \equiv \frac{2N}{a(t)} \left(H \mp \frac{1}{a(t) r} \right)$$

- **Normal surface ⇒ $\boldsymbol{\theta_{IN} < 0, \theta_{OUT} > 0}$**
- **Antitrapped surface ⇒ $\boldsymbol{\theta_{IN,OUT} > 0}$**
- **Trapped surface ⇒ $\boldsymbol{\theta_{IN,OUT} < 0}$ ⇒ Singularity**
- *Minimally Antitrapped Surface* has vanishing expansion
- Any surface of greater physical size than this is *Antitrapped*
- Antitrapped surfaces have an *apparent horizon* as an inner boundary
  - Cosmological apparent horizon
  - Inflationary apparent horizon