Black Hole Horizons

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Black holes in higher dimensions

- General Relativity in 4 dimensions admits a unique class of black hole solutions, which are parametrized by mass, charge and angular momentum (*No-hair Theorem)*.
- However in dimensions $D > 4$, uniqueness theorems no longer exist.
- In $D = 5$, black rings, with horizon topology $S^2 \times S^1$, were first discovered in pure Einstein gravity [Emparan, Reall], and later also in minimal supergravity [Elvang, Emparan, Mateos, Reall].
- In String/M-theory we shall consider gravitational systems in ten and eleven dimensions and more unusual black hole solutions are expected.
- Finding the full black hole solution is a notoriously hard. However the problem of studying *near-horizon solutions* is manageable.

Near-horizon solutions

- Examining near-horizon geometries is a useful method for determining which types of black hole solutions can, or cannot, exist.
- Also for asymptotically AdS black holes, one can look at the dual CFT description of the near-horizon geometry. The isometries on the gravity side play the role of conformal group on the gauge side.
- The isometries can be dynamically enhanced. Often supersymmetric near horizon solutions experience *supersymmetry enhancement*, which implies *symmetry enhancement*.
- Lichnerowicz Theorems, together with Index Theory arguments, represent a powerful tool to establish whether supersymmetry enhancement occurs or not.

Lichnerowicz Theorems in Supergravity

Consider supersymmetric solutions, i.e. solutions which admit Killing spinors ϵ. Let us define a certain Dirac operator $\mathcal{D}$.

Lichnerowicz Theorems establish the following 1:1 correspondence

$$\epsilon \text{ is Killing spinor} \qquad \Longleftrightarrow \qquad \mathcal{D}\epsilon = 0 \ .$$

Those types of theorems have been proven for near-horizon geometries in

- $D = 11$ supergravity [Gutowski, Papadopoulos]
- type IIB [Gran, Gutowski, Papadopoulos]
- and type IIA supergravity (both massive and massless) [Gran, Gutowski, Kayani, Papadopoulos].

Q: **How general are those Lichnerowicz Theorems in supergravity?**

A: We shall consider α' corrections....

- α' corrections of $D = 11$, type IIA and type IIB supergravity are not easy to deal with.

- but they are manageable in *Heterotic Supergravity* !

Outline

- Near-horizon geometries in Heterotic Supergravity at tree level
- Supersymmetry enhancement
- Near-horizon geometries in anomaly corrected Heterotic Supergravity
- Lichnerowicz Theorem
- Supersymmetry enhancement?
- Conclusions

Heterotic Supergravity at tree level

The bosonic fileds of heterotic supergravity are the metric g, a real scalar field (the dilaton) Φ, a real 3-form H, and a non-abelian 2-form field F.

The bosonic field equations are:

$$\begin{aligned}
R_{MN} - \frac{1}{4} H_{ML_1L_2} H_N{}^{L_1L_2} + 2\nabla_M \nabla_N \Phi &= 0 \\
\nabla^M \left(e^{-2\Phi} H_{MN_1N_2} \right) &= 0 \\
\nabla^M \left(e^{-2\Phi} F_{MN} \right) + \frac{1}{2} e^{-2\Phi} H_{NL_1L_2} F^{L_1L_2} &= 0 \\
\nabla^2 \Phi - 2\nabla^M \Phi \nabla_M \Phi + \frac{1}{12} H_{L_1L_2L_3} H^{L_1L_2L_3} &= 0
\end{aligned}$$

The Bianchi identity associated with the 3-form is

$$dH = 0$$

We further assume that the solution is *supersymmetric*, i.e. there exists a Majorana-Weyl Killing spinor ϵ satisfying the Killing spinor equations (KSE)

$$
\begin{aligned}
\nabla^{(+)}_M \epsilon \equiv \big(\nabla_M - \frac{1}{8} H_{MN_1N_2}\Gamma^{N_1N_2}\big)\epsilon &= 0 \\
\big(\Gamma^M \nabla_M \Phi - \frac{1}{12} H_{N_1N_2N_3}\Gamma^{N_1N_2N_3}\big)\epsilon &= 0 \\
F_{MN}\Gamma^{MN}\epsilon &= 0
\end{aligned}
$$

$\nabla^{(+)}$ is the connection with torsion H.

∇ is the Levi-Civita connection.

Gaussian Null Co-ordinates

Assumptions

- spacetime contains an (extremal) Killing horizon, null hypersurface associated with V, symmetry of the full solution.
- there is a Killing spinor ϵ well-defined on the horizon

Following [Friedrich, Racz, Wald], one can introduce a Gaussian null co-ordinate system, with co-ordinates u, r, y^I, such that $V = \frac{\partial}{\partial u}$, the horizon is at $r = 0$, and the metric is

$$ds^2 = 2drdu + 2rh_I du dy^I - r^2 \Delta du du + \gamma_{IJ} dy^I dy^J$$

where Δ, h_I and γ_{IJ} are u-independent scalar, 1-form and metric of the spatial cross section of the horizon $\mathcal{S}$, which we shall assume **compact** and **without boundary**.

Then we perform the **near-horizon limit** [Reall et al.]

$$r \to \sigma r \qquad u \to \frac{u}{\sigma} \qquad y^I \to y^I \qquad \sigma \to 0$$

the metric remains invariant in form, but Δ, h_I and γ_{IJ} no longer depend by r, only by y.

$$ds^2 = 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j$$

$$\mathbf{e}^+ = du \qquad \mathbf{e}^- = dr + rh - \frac{1}{2}r^2\Delta du \qquad \mathbf{e}^i = e^i{}_J dy^J$$

The near-horizon limit only exists for *extremal* black holes.

Moreover, we require that all bosonic fields are well-defined and regular in the near-horizon limit.

- The dilaton

$$\Phi = \Phi(y)$$

- The 3-form field $H = dB$

$$H = \mathbf{e}^{+} \wedge \mathbf{e}^{-} \wedge N + r\mathbf{e}^{+} \wedge Y + W$$

N, Y, W are *only* y-dependent 1, 2, 3-forms, constrained by $dH = 0$.

Now that we know explicitly the u and r dependence of all bosonic fields, it is straightforward to integrate up the "$+$" and "-" components of the gravitino KSE for any Killing spinor.

Using some global analysis on the spinors

$$\Delta = 0 \qquad N = h \qquad Y = dh$$

We split the Killing spinors in positive and negative light-cone chiralities

$$\epsilon = \epsilon_+ + \epsilon_- \qquad\qquad \Gamma_+\epsilon_+ = 0 \qquad\qquad \Gamma_-\epsilon_- = 0$$

and we have

$$\epsilon_+ = \eta_+ + \frac{1}{2}uh_i\Gamma^i\Gamma_+\eta_- \ , \qquad\qquad \epsilon_- = \eta_-$$

where $\eta_\pm = \eta_\pm(y)$ do *not* depend on r and u.

The spinors $\eta_\pm$ must satisfy a number of differential and algebraic conditions as a consequence of the spatial components of the KSE.

Supersymmetry enhancement

The reduced KSE are:

$$
\begin{aligned}
(\tilde{\nabla}_i - \frac{1}{8} W_{i\ell_1\ell_2}\Gamma^{\ell_1\ell_2})\eta_\pm &= 0 \\
dh_{ij}\Gamma^{ij}\eta_\pm &= 0 \\
\Big((2d\Phi \pm h)_i\,\Gamma^i - \frac{1}{6} W_{ijk}\Gamma^{ijk} \Big)\eta_\pm &= 0
\end{aligned}
$$

η_+ satisfies " $+$ " $\quad\Longrightarrow\quad$ $\eta_- = \Gamma_-\Gamma^i h_i \eta_+$ satisfies " $-$ "

$\Longrightarrow$ doubling of supersymmetry!

key point: $\tilde{\nabla}^{(+)} h_i = 0$ from global analysis

$$
\tilde{\nabla}^i\tilde{\nabla}_i h^2 + (h - 2d\Phi)^j \tilde{\nabla}_j h^2 = 2\big(\tilde{\nabla}^{(i} h^{j)}\big)^2 + \frac{1}{2}\big((dh - i_h W)_{ij}\big)^2
$$

Anomaly Corrected Heterotic Supergravity

The Green-Schwarz anomaly cancellation mechanism requires that

$$dH = -\frac{\alpha'}{4}\left(tr(R^{(-)} \wedge R^{(-)}) - tr(F \wedge F)\right) + \mathcal{O}(\alpha'^2)$$

The KSE become

$$\begin{aligned}
\tilde{\nabla}^{(+)}\epsilon \equiv \big(\nabla_M - \frac{1}{8}H_{MN_1N_2}\Gamma^{N_1N_2}\big)\epsilon &= \mathcal{O}(\alpha'^2) \\
\big(\Gamma^M\nabla_M\Phi - \frac{1}{12}H_{N_1N_2N_3}\Gamma^{N_1N_2N_3}\big)\epsilon &= \mathcal{O}(\alpha'^2) \\
F_{MN}\Gamma^{MN}\epsilon &= \mathcal{O}(\alpha')
\end{aligned}$$

Near-horizon geometries

Gaussian null co-ordinates and the near horizon limit follow the same construction done for the tree-level case.

All fields, both bosonic and fermionic, receive corrections $\mathcal{O}(\alpha')$

$$g = g^{[0]} + \alpha' g^{[1]} + \mathcal{O}(\alpha'^2) \qquad \epsilon = \epsilon^{[0]} + \alpha' \epsilon^{[1]} + \mathcal{O}(\alpha'^2)$$

Since H is no longer closed, in the near-horizon limit it takes the form

$$H = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge N + r\mathbf{e}^+ \wedge Y + W$$

where N, Y, W are *only* y dependent 1, 2, 3-forms.

The KSE can be integrated along the "+" and "-" directions.
Using some global analysis on the spinor

$$\Delta = \mathcal{O}(\alpha'^2) \qquad N = h \qquad Y = dh$$

$$\epsilon_+ = \eta_+ + \frac{1}{2} u h_i \Gamma^i \Gamma_+ \eta_- + \mathcal{O}(\alpha'^2)\ , \qquad \epsilon_- = \eta_- + \mathcal{O}(\alpha'^2)$$

The reduced KSE are:

$$
\begin{aligned}
\tilde{\nabla}_i^{(+)}\eta_\pm \equiv \Big(\tilde{\nabla}_i - \frac{1}{8}W_{ijk}\Gamma^{jk}\Big)\eta_\pm &= \mathcal{O}(\alpha'^2) \\
\mathcal{A}\eta_\pm \equiv \Big(\Gamma^i\tilde{\nabla}_i\Phi \pm \frac{1}{2}h_i\Gamma^i - \frac{1}{12}W_{ijk}\Gamma^{ijk}\Big)\eta_\pm &= \mathcal{O}(\alpha'^2)
\end{aligned}
$$

Due to presence of α' corrections,

η_+ satisfies " $+$ " $\quad\not\Longrightarrow\quad$ $\eta_- = \Gamma_-\Gamma^i h_i\eta_+$ satisfies " $-$ "

$\Longrightarrow$ no SUSY enhancement??

key point: $\tilde{\nabla}^{(+)}h_i = \mathcal{O}(\alpha')$ by global analysis

$$
\begin{aligned}
\tilde{\nabla}^i\tilde{\nabla}_i h^2 + (h-2d\Phi)^j\tilde{\nabla}_j h^2 &= 2\big(\tilde{\nabla}^{(i}h^{j)}\big)^2 + \frac{1}{2}\big((dh - i_h W)_{ij}\big)^2 \\
&+ \frac{\alpha'}{4}\Big(2(i_h dh)^2 - (\tilde{R}^{(-)}_{i\ell_1\ell_2\ell_3})^2 + (\tilde{F}_{i\ell})^2\Big)
\end{aligned}
$$

Lichnerowicz Theorem

Can we identify Killing spinors $\eta_\pm$ with the zero modes of a certain Dirac operator?

If it would work, than from the Index of the Dirac operator we could count the number of zero modes and hence the number of supersymmetries.

Define the modified connection with torsion

$$\hat{\nabla}_i \equiv \tilde{\nabla}^{(+)} + \kappa \Gamma_i \mathcal{A}$$

and the modified near-horizon Dirac operator

$$\mathcal{D} = \Gamma^i \tilde{\nabla}_i^{(+)} + q\mathcal{A}$$

where κ and q are real numbers.

Consider the functional

$$\mathcal{I} \equiv \int_{\mathcal{S}} e^{c\Phi}\Big(\langle \hat{\nabla}_i \eta_\pm, \hat{\nabla}^i \eta_\pm\rangle - \langle \mathcal{D}\eta_\pm, \mathcal{D}\eta_\pm\rangle\Big) \quad , \qquad c \in \mathbb{R}$$

$\langle\ ,\ \rangle$ is a $Spin(8)$ invariant inner product positive definite.

$$\begin{aligned}
\mathcal{I} \;&=\; \left(8\kappa^2 - \frac{1}{6}\kappa\right)\int_{\mathcal{S}} e^{-2\Phi} \parallel \mathcal{A}\,\eta_\pm \parallel^2 + \int_{\mathcal{S}} e^{-2\Phi}\langle \eta_\pm, \Psi\mathcal{D}\eta_\pm\rangle \\
&-\; \frac{\alpha'}{64}\int_{\mathcal{S}} e^{-2\Phi}\left(2 \parallel d\!\!\!/h\,\eta_\pm \parallel^2 + \parallel \tilde{F\!\!\!/}\eta_\pm \parallel^2 - \parallel \tilde{R}^{(-)}_{\ell_1\ell_2,\,ij}\Gamma^{\ell_1\ell_2}\eta_\pm \parallel^2\right) + \mathcal{O}(\alpha'^2)
\end{aligned}$$

(set $q = \frac{1}{12}$ and $c = -2$). If $0 < \kappa < \frac{1}{48}$, then

$$\mathcal{D}\eta_\pm = \mathcal{O}(\alpha'^2) \qquad \Longrightarrow \qquad \tilde{\nabla}^{(+)}\eta_\pm = \mathcal{O}(\alpha')\ , \quad \mathcal{A}\eta_\pm = \mathcal{O}(\alpha')\ .$$

Lichnerowicz Theorem does not ensure SUSY enhancement.

However it gives extra conditions

$$\tilde{F}_{ij}\Gamma^{ij}\eta_\pm = \mathcal{O}(\alpha')\ , \qquad\qquad dh_{ij}\Gamma^{ij}\eta_\pm = \mathcal{O}(\alpha')$$

Supersymmetry enhancement?

Through the analysis, one has to consider separately the two cases whether

$$\eta_-^{[0]} \neq 0 \qquad \text{or} \qquad \eta_+^{[0]} \neq 0$$

- In the $\eta_-^{[0]} \neq 0$ case, using local analysis

$$\tilde{\nabla}^{(+)} h_i = \mathcal{O}(\alpha'^2)$$

$\Longrightarrow \qquad \exists$ SUSY enhancement!

- However it does not work for the $\eta_+^{[0]} \neq 0$ case ...

Conclusions

Summary

- At tree level, Lichnerowicz Theorems are not needed.
  Global analysis of $\tilde{\nabla}^2 h^2 \quad \Longrightarrow \quad$ SUSY enhancement
- α' corrected, $\tilde{\nabla}^2 h^2$ analysis + Lichnerowicz are insufficient to imply SUSY enhancement, *but* do imply conditions on other fields (e.g. $\tilde{\not{F}} \eta_\pm = \mathcal{O}(\alpha')$)
- If $\eta_-^{[0]} \neq 0 \quad \Longrightarrow \quad$ SUSY enhancement because $\tilde{\nabla}^{(+)} h_i = \mathcal{O}(\alpha'^2)$
  by local analysis

Open question

Any argument to prove (or disprove) supersymmetry enhancement?

Thanks for your attention