Instantons and Donaldson invariants

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$$Q_{\mathcal{M}} : H^2(\mathcal{M}, \mathbb{Z}) \times H^2(\mathcal{M}, \mathbb{Z}) \to \mathbb{Z}$$

$$Q_{\mathcal{M}}(a, b) = \int_{\mathcal{M}} a \wedge b$$

where $a, b \in H^2(\mathcal{M}, \mathbb{Z})$.

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- **Donaldson invariants** are polynomials on (specific) cycles of $H^n(\mathcal{M})$. Physically: *correlation functions of operators of the corresponding gauge theory on $\mathcal{M}$.*

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- Here we will show how to compute the partition function of the topologically twisted $\mathcal{N} = 2$ SYM living on a toric variety.
- A toric variety can be thought of, geometrically, as a generalization of the complex projective space $\mathbb{P}^n$.

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- Actually one gets the cohomology ring

$$\mathcal{H} = \frac{\text{Kernel}\mathcal{Q}}{\text{Image}\mathcal{Q}}$$

  and the various observables are elements of the cohomology groups.

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- For bundles with non-vanishing first Chern class, c_1, we have

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  where $\omega \in H^2(\mathcal{M}, \mathbb{R})$ and $\mathcal{V}$ is a standard $\mathcal{Q}$-exact term that makes localization possible.

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- How it works? The $\mathcal{Q}$-exactness of the $\mathcal{V}$ term requires we set the fermions to zero.
- Then the instanton partition function only receives contributions from point-like instantons of the fixed points of $\mathbb{C}^2$ [Nekrasov] and $\mathcal{Z}_{\text{inst}}^{\mathbb{C}^2}$ can be calculated exactly.

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- Susy is equivariant with respect to the maximal torus $U(1)^{N+2}$ acting on $\mathbb{P}^2$. The $U(1)^2$ factor is due to the vector field V. Actually this $U(1)^2$ factor corresponds to the same Ω-background of Nekrasov with parameters $\epsilon_1^{(l)}$ and $\epsilon_2^{(l)}$. Nekrasov only had one set of $\epsilon' s$ since he worked in $\mathbb{C}^2$ [Nekrasov 2002] .

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l	$\epsilon_1^{(l)}$	$\epsilon_2^{(l)}$
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- Also note that for each patch the Coulomb branch parameters change slightly, we have [Nakajima et. al.]

$$
\begin{aligned}
a_i^{(0)} &= a_i + p_i\epsilon_1 + q_i\epsilon_2 \\
a_i^{(1)} &= a_i + p_i(\epsilon_1 - \epsilon_2) - r_i\epsilon_2 \\
a_i^{(2)} &= a_i + q_i(\epsilon_2 - \epsilon_1) - r_i\epsilon_1
\end{aligned}
$$

where $\vec{p}, \vec{q}, \vec{r}$ are some parameters whose sum is N, while $i = 1, \ldots, N$.

Exact partition function

- The partition function for $\mathbb{P}^2$ is then [Goottsche et. al., Bershtein et. al.]

$$\mathcal{Z}_{\text{full}}^{\mathbb{P}^2}(\mathbf{q}, x, z, y; \epsilon_{1,2}) = \sum_{\{p_i, q_i, r_i\}} \oint da \prod_{l=0}^{2} \mathcal{Z}_{\text{full}}^{\mathbb{C}^2}(\mathbf{q}^{(l)}; a^{(l)}, \epsilon_{1,2}^{(l)}) y^{c_1^{(l)}}$$

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  that is, for $\mathbb{P}^2$ we need to compute Nekrasov's partition function on each $\mathbb{C}^2$ patch separately and then glue them together.

Exact partition function

- The partition function for $\mathbb{P}^2$ is then [Goottsche et. al., Bershtein et. al.]

$$\mathcal{Z}_{\text{full}}^{\mathbb{P}^2}(\mathbf{q}, x, z, y; \epsilon_{1,2}) = \sum_{\{p_i, q_i, r_i\}} \oint da \prod_{l=0}^{2} \mathcal{Z}_{\text{full}}^{\mathbb{C}^2}(\mathbf{q}^{(l)}; a^{(l)}, \epsilon_{1,2}^{(l)}) y^{c_1^{(l)}}$$

  that is, for $\mathbb{P}^2$ we need to compute Nekrasov's partition function on each $\mathbb{C}^2$ patch separately and then glue them together.

- Nekrasov's instanton partition function has a **combinatorial** nature. Information is encoded in partitions of the instanton number represented by **Young diagrams**.

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and where

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Here A is the Young diagram's *arm function* and L its *leg function* while $\mathbf{q} = e^{2\pi i\tau}$.

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  Here A is the Young diagram's *arm function* and L its *leg function* while $\mathbf{q} = e^{2\pi i\tau}$.
- Once we have calculated the partition function on each patch we have to *glue* the patches together. But this is **not trivial**.

Example: $U(2)$ and $k = 1$

- This is the simplest example. The Young configurations that contribute are

$$\square \oplus \emptyset \text{ and } \emptyset \oplus \square$$

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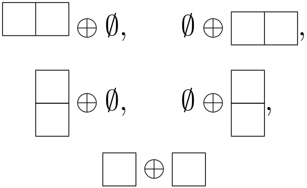
- Using the equivariant variables $\epsilon_1^{(l)}, \epsilon_2^{(l)}$ for $l = 0, 1, 2$ we can easily write down the full partition function as the product

$$\mathcal{Z}_{\text{inst}}^{\mathbb{C}^2} = \mathcal{Z}_{\text{inst}}^{\mathbb{C}^2_{(0)}} \mathcal{Z}_{\text{inst}}^{\mathbb{C}^2_{(1)}} \mathcal{Z}_{\text{inst}}^{\mathbb{C}^2_{(2)}}$$

$U(2)$ and $k = 2$

- This is slightly more complicated since we have 5 Young configurations in total



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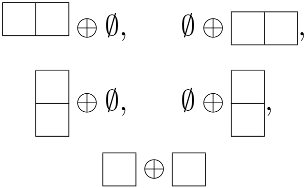

- For example, for the first configuration we have

$$Z_{\text{hyp}}^{\{2\oplus\emptyset\}} = \frac{(2\epsilon_1^2\epsilon_2 a_{12})^{-1}}{(-\epsilon_1+\epsilon_2)(a_{21}+2\epsilon_1+\epsilon_2)(a_{21}+\epsilon_1+\epsilon_2)(a_{12}-\epsilon_1)}$$

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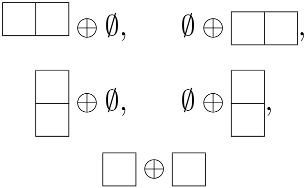

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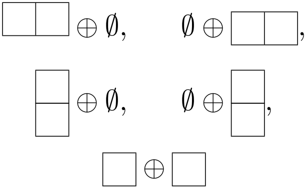

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- Symmetries: $Z_{\text{hyp}}^{\{\emptyset\oplus 2\}}(a_{12}) = Z_{\text{hyp}}^{\{2\oplus\emptyset\}}(a_{21})$
- Find the rest of the configurations, then **glue** the three $\mathbb{C}^2$ contributions together.

Full partition function on $\mathbb{P}^2$ and Donaldson invariants

- I have not told you that we also need to consider the $\mathcal{Z}_{\text{perturbative}}$ and $\mathcal{Z}_{1-\text{loop}}$ contributions as well. Each of those gets a contribution from each $\mathbb{C}^2$ patch of $\mathbb{P}^2$ as well.

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- Nekrasov: instanton partition function is the same as the equivariant Donaldson invariants.
- At the non-equivariant limit $\epsilon_{1,2} \to 0$ then we get the corresponding Donaldson invariants.

Conclusion

- It is possible to calculate the partition function of certain susy gauge theories (we can add matter too) on 4-manifolds admitting certain isometries.
- We have shown how to do this for compact toric varieties with the representative being $\mathbb{P}^2$. Other known example is $\mathbb{P}^1 \times \mathbb{P}^1$. Other examples?

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- One hopes to learn something about the nature of susy gauge field theories since the above constructions give access **exact** non-perturbative calculations.

Thank you!

Thank you! And a big thanks to the **organizers**!