An introduction to Gauge Invariants and Hilbert Series

Alessandro Pini

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Universidad de Oviedo

Contents of the talk

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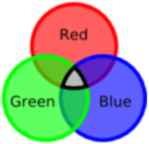
Gauge invariants in QCD

- Gauge group $G = SU(3)$
- Global symmetries $F = SU(3) \times U(1)_B$



Problem: How to construct gauge invariants?

Mesons

Composition of a color and anti-color: $q_i \overline{q_j} \Rightarrow$ neutral under the $U(1)_B$

Baryons

Composition of the three colors: $q_i q_j q_k \Rightarrow$ charged under the $U(1)_B$

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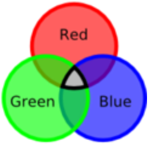

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- **Problem:** How to classify them in a generic QFT ? $\Rightarrow$ Hilbert Series
  We will focus on the mesons.

$\mathcal{N} = 1$ QFT and structure of the moduli space

$$S = \int d^4x \left[\int d^4\theta \Phi_i e^V \Phi_i^\dagger + \left(\frac{1}{4g^2} \int d^2\theta \mathrm{Tr}[W_\alpha W^\alpha] + \int d^2\theta W(\Phi) + \mathrm{h.c.} \right) \right]$$

The space of vacua is a manifold specified by

- F-flatness condition

$$F_i = \frac{\partial W}{\partial \phi_i} |_{\phi_i = \phi_{i0}} = 0$$

- D-flatness condition

$$D_A = \sum_i \phi_{i0}^\dagger T_A \phi_{i0} = 0 \quad \text{with} \quad A = 1, ..., \mathrm{Dim}[\mathrm{Lie(G)}]$$

- It's parametrized by gauge invariant operators

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Problem: How to get information regarding the structure of the moduli space as a manifold? $\Rightarrow$ Hilbert series

Contents of the talk

The superpotential and the F-terms

$$W = \mathrm{Tr}[\Phi_1[\Phi_2, \Phi_3]] \quad \Rightarrow \quad \mathrm{F}_i : \partial_{\Phi_i} W = [\Phi_j, \Phi_k] = 0$$

Let's consider only the field Φ_1. The gauge invariants are

$$\mathbb{I}, \quad \mathrm{Tr}[\Phi_1], \quad \mathrm{Tr}[\Phi_1^2], \quad \mathrm{Tr}[\Phi_1^3], \quad \ldots \quad \mathrm{Tr}[\Phi_1^n]$$

Let's introduce the **fugacity**

$$\Phi_1 \mapsto t_1, \quad \text{such that} \quad t_1 \in \mathbb{C} \quad \text{and} \quad |t_1| < 1$$

So that summing

$$1 + t_1 + t_1^2 + t_1^3 + \ldots + t_1^n + \ldots = \sum_{n=0}^{\infty} t_1^n = \frac{1}{1 - t_1}$$

Gauge invariants of the form $\mathrm{Tr}[\Phi_1^i \Phi_2^j \Phi_3^k]$ with $i, j, k \geq 0$

If $\forall \quad \Phi_i \mapsto t_i$ we get

$$1 + t_1 + t_2 + t_3 + t_1^2 + t_1 t_2 + \ldots = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} t_1^i t_2^j t_3^k = \frac{1}{(1 - t_1)(1 - t_2)(1 - t_3)}$$

Coefficient of the expansion $\overset{1:1}{\longleftrightarrow}$ gauge invariant operator

E.g. we have

- $1 \leftrightarrow \mathbb{I}$.
- $t_i \leftrightarrow \Phi_i$
- $t_i t_j \leftrightarrow \Phi_i \Phi_j$
- ...

If we **unrefine**, i.e. $t_i \mapsto t$, we get

$$\frac{1}{(1-t_1)(1-t_2)(1-t_3)} \mapsto \frac{1}{(1-t)^3} = 1 + \boxed{3t} + \boxed{6t^2} + \boxed{10t^3} + \ldots$$

$\Delta = 1$, $\quad \mathrm{Tr}[\Phi_1],\ \mathrm{Tr}[\Phi_2],\ \mathrm{Tr}[\Phi_3].$

$\Delta = 2$, $\quad \mathrm{Tr}[\Phi_1\Phi_2],\ \mathrm{Tr}[\Phi_1\Phi_3],\ \mathrm{Tr}[\Phi_2\Phi_3],\ \mathrm{Tr}[\Phi_1^2],\ \mathrm{Tr}[\Phi_2^2],\ \mathrm{Tr}[\Phi_3^2].$

$\Delta = 3$, $\quad \mathrm{Tr}[\Phi_1^2\Phi_2],\ \mathrm{Tr}[\Phi_1^2\Phi_3],\ \mathrm{Tr}[\Phi_2^2\Phi_1],\ \mathrm{Tr}[\Phi_2^2\Phi_3],\ \mathrm{Tr}[\Phi_3^2\Phi_1],$
$\mathrm{Tr}[\Phi_3^2\Phi_2],\ \mathrm{Tr}[\Phi_1\Phi_2\Phi_3],\ \mathrm{Tr}[\Phi_1^3],\ \mathrm{Tr}[\Phi_2^3],\ \mathrm{Tr}[\Phi_3^3].$

Therefore if we **unrefine**

Coefficient of the expansion $\overset{1:1}{\longleftrightarrow}$ total number of gauge invariant operators with given Δ

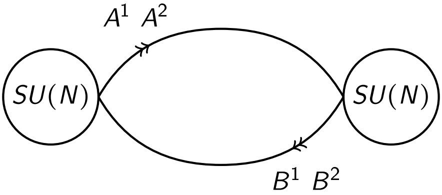
Moduli space of vacua

3 *independent* mesonic operator $M_i = \mathrm{Tr}[\Phi_i] \quad \Rightarrow \quad \mathbb{C}^3$

In general, given a set of charges $(k_1, ..., k_n)$, we can define **Hilbert Series** $H(t_1, ...t_n)$

$$H(t_1, ...t_n) = \sum_{i_1, i_2, ... i_k} c_{k_1, k_2, ..., k_n} t_1^{k_1} t_2^{k_2} ... t_n^{k_n}$$

- The coefficient $c_{k_1, k_2, ..., k_n}$ is the number of operators with charge $(k_1, ..., k_n)$
- Mesonic gauge invariant operator $M_i \leftrightarrow$ fugacity t_i



- $G = SU(N) \times SU(N)$
- $\mathcal{N} = 1 \quad U(1)_R$, $\Delta_{A_i} = \Delta_{B_i} = \frac{1}{2}$

- $W = \mathrm{Tr}[A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1] = \mathrm{Tr}[\det A_i \; B_j] \quad SU(2)_A \times SU(2)_B$
- Baryonic-symmetry $U(1)_B : A_i \to e^{i\theta} A_i, B_j \to e^{-i\theta} B_j$

The F-terms are

$$\partial_{A_1} W = B_1 A_2 B_2 - B_2 A_2 B_1, \quad \partial_{A_2} W = -B_1 A_1 B_2 + B_2 A_1 B_1$$
$$\partial_{B_1} W = A_2 B_2 A_1 - A_1 B_2 A_2, \quad \partial_{B_2} W = -A_2 B_1 A_1 + A_1 B_1 A_2$$

The mesonic fields

- Only *one* field e.g. ~~$\mathrm{Det}[A_i]$~~ , ~~$\mathrm{Det}[B_j]$~~ $\Rightarrow$ charged under $U(1)_B$
- Using *two* fields

$$x = \mathrm{Tr}[A_1 B_1], \quad y = \mathrm{Tr}[A_2 B_2], \quad z = \mathrm{Tr}[A_1 B_2], \quad w = \mathrm{Tr}[A_2 B_1]$$

Using F-terms we get:

- The **mesons commute**, i.e.

$$xz = zx, \quad xw = wx, \quad yz = zy, \quad yw = wy, \quad xy = yx, \quad zw = wz$$

- The mesons satisfy the **relation**

$$xy = zw$$

that is the conifold equation in $\mathbb{C}^4$.

Let's consider the abelian case $N = 1$, then

- $\Rightarrow A_i$ and B_j are c-numbers $\Rightarrow$ commute.
- generic mesonic operator $\mathrm{Tr}[A_i^n B_j^n]$.
- $\forall\ n$ we have $(n+1)^2$ different operators.

$t \leftrightarrow$ mesonic operator $\Rightarrow$

$$\mathrm{HS}(t) = \sum_{n=0}^{\infty} (n+1)^2 t^n = \frac{1+t}{(1-t)^3} = \boxed{1} + \boxed{4t} + \boxed{9t^2} + \dots$$

$\Delta = 0 \quad \Rightarrow \quad \mathbb{I}$

$\Delta = 1,\ x = \mathrm{Tr}[A_1 B_1],\ \ y = \mathrm{Tr}[A_2 B_2],\ \ z = \mathrm{Tr}[A_1 B_2],\ \ w = \mathrm{Tr}[A_2 B_1]$

The conifold $\mathcal{C}$ - fourth part: summary

$\Delta = 2, \qquad x^2, \quad y^2, \quad z^2, \quad w^2, \quad xz, \quad xw, \quad xy, \quad yz, \quad yw$

We could also have zx, wx, yx, zy, wy and zw.

the mesons commute

the relation $xy = zw$

Coefficient of the expansion $\overset{1:1}{\longleftrightarrow}$ Total number of gauge invariant operators with given Δ up to relations

Contents of the talk

Unrefined Hilbert Series $H(t)$: the general structure

$$H(t) = \frac{Q(t)}{(1-t)^{p}}$$

- Polynomial of integer coefficients
- Dimension of the embedding space

The polynomial $Q(t)$

- $Q(t) = 1 \Rightarrow$ there are not relations.
- $Q(t) = \prod_{j=1}^{M}(1-t^{d_j}) \Rightarrow$ **complete intersection.** with $d_j \in \mathbb{N}$

number of relations + dimension of the moduli space = number of generators

- $Q(t) \neq \prod_{j=1}^{M}(1-t^{d_j}) \Rightarrow$ **not complete intersection**

Unrefined Hilbert Series $H(t)$: the PE

It's useful to use the **Plethystic Exponetial**

Given a function $f(t) \mid f(0) = 0$ $\qquad \mathrm{PE}[f(t)] = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f(t^n)\right)$

- $$\mathcal{N} = 4 \; SYM \; 4d \qquad H(t) = \frac{1}{(1-t)^3} = \mathrm{PE}[3t]$$
- $$\text{Conifold} \;\; \mathcal{C} \qquad H(t) = \frac{1-t^2}{(1-t)^4} = \mathrm{PE}[4t - t^2]$$

$\Rightarrow$ Compact way to summarize the information

Unrefined Hilbert Series $H(t)$: the PLog

The inverse function of the PE is the **Plethystic Logarithm**

$$\mathrm{PLog}[f(t)] = \sum_{n=0}^{\infty} \frac{\mu(n)}{n} \mathrm{Log}[f(t^n)]$$

E.g.

- $\mathcal{N} = 4$ SYM $4d$, $\mathrm{PLog}[H(t)] = 3t$
- Conifold $\mathcal{C}$, $\mathrm{PLog}[H(t)] = 4t - t^2$

  generators ($4t$) relations (t^2)

- Not complete intersection $\Rightarrow$ infinite series.

Contents of the talk

Conclusions

- The Hilbert Series is a powerful tool for the characterization of the moduli space of vacua.
- The Hilbert Series has been applied also in the characterization of the moduli space of instantons and computation of the superconformal index.
- It's also possible to count baryonic charges $\Rightarrow$ baryonic Hilbert series

THANK YOU FOR THE ATTENTION

AND...


...SEE YOU IN OVIEDO NEXT YEAR
¡ HASTA PRONTO !

Appendix 1: $\mathcal{N} = 4$ SYM $G = U(N)$ in $4d$

$$S = \frac{1}{g_{YM}^2} \int d^4x \mathrm{Tr}\left[-\frac{1}{2} F_{\mu\nu}^2 - i\overline{\psi}^a \not{D} \psi_a - (D_\mu \phi^i)^2 + C_i^{ab} \psi_a [\phi^i, \psi_b] + [\phi^i, \phi^j]^2 \right]$$

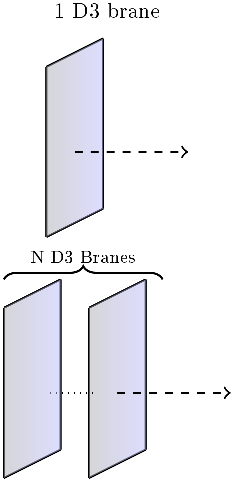
$i, j = 1, \ldots, 6, \quad a, b = 1, \ldots, 4.$

Using the $\mathcal{N} = 1$ language

$$\mathcal{N} = 4 \ \text{ multiplet} \Rightarrow \begin{cases} 1 \ \mathcal{N} = 1 \text{ vector multiplet } W_\alpha : (A_\mu, \psi^4) \\ 3 \ \mathcal{N} = 1 \text{ chiral multiplets } \Phi^i : (\phi^i + i\phi^{i+3}, \psi^i) \end{cases}$$

$$S = \frac{1}{g_{YM}^2} \int d^4x \int d^4\theta \left(W_\alpha^2 + \sum_{i=1}^3 \overline{\Phi}_1 \Phi_i + \int d^2\theta \epsilon_{ijk} \Phi^i \Phi^j \Phi^k \right)$$

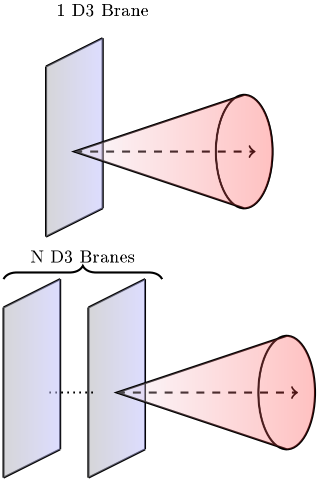
Appendix 2: D3 branes on flat space-time



- 3 complex scalars Φ_i
- the transverse space $\mathbb{C}^3$

- $Sym^N[\mathbb{C}^3] = (\mathbb{C}^3)^N/N!$

Appendix 3: D3 branes at the tip of a cone χ



- Constraints among mesons $\Rightarrow$ manifold χ
- $\mathrm{Sym}[\chi] = \chi^N / N!$

Global symmetries

$$ds^2_{T^{1,1}} = \frac{1}{9}(d\psi + cos\theta_1 d\phi_1 + cos\theta_2 d\phi_2)^2 + \sum_{i=1}^{2} \frac{1}{6}(d\theta_i^2 + sin^2\theta_i d\phi_i^2)$$

$$SU(2)_i : (\psi, \theta_i, \phi_i), \qquad U(1)_R : \psi$$

Baryonic symmetry

- $T^{1,1} \sim S^3 \times S^2$
- reduction of the RR 4-form on $S^3 \mapsto$ vector field on $AdS_5 \mapsto$ baryonic symmetry in the CFT

Appendix 4: The Möbius function $\mu(n)$

$$\mu(n) = \begin{cases} 0 & n \text{ has one or more repetead prime factors} \\ 1 & n = 1 \\ (-1)^n & \text{n is a product of disticnt primes} \end{cases}$$