Ward identities and relations between conductivities and viscosities in holography

David Rodríguez Fernández

based on C. Hoyos and D. Rodríguez Fernández arXiv: 1511.01002

IV Postgraduate meeting on theoretical physics.
IFT, Universidad Autónoma de Madrid.

18-th November 2015

Summary

Table of Contents

Gauge/Gravity duality

- The **Gauge/Gravity duality** relates gravity theories on asymptotically Anti-de Sitter manifolds in d dimensions to quantum field theories in $d-1$ dimensions.
- Starts from a conjecture that leads to a map between the free parameters on the gravity theory (length of the string l_s and radius of curvature L) and the QFT (number of colors N and Yang-Mills coupling g_{YM})

$$L^4/l_s^4 \approx \lambda = g_{YM}^2 N$$

	Gauge theory	Gravity theory
Weak form	$N \to \infty$, λ **large**	**Classical supergravity,** $l_s/L \to 0$
	$\Downarrow$	$\Downarrow$
	Strongly coupled	Weakly coupled

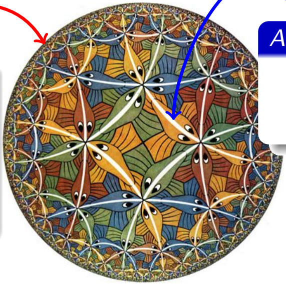
- Certain questions of strongly coupled theories will become clearer and tractable on the gravity side.

boundary

bulk

QFT_d

- strong interacting QFT
- operators: $\langle \mathcal{O} \rangle$, $\langle T_{\mu\nu} \rangle$,...



AdS_{d+1}

- gravity theory
- fields: $g_{\mu\nu}, A_\mu, \phi$

$$e^{-W[\phi,g]} = \left\langle e^{-\int \phi \langle \mathcal{O} \rangle + g^{\mu\nu} \langle T_{\mu\nu} \rangle} \right\rangle_{QFT} = e^{-S_{AdS}[g,\phi]}\Big|_{\mathcal{B}} ,$$

- energy scale in QFT $\Leftrightarrow$ radial coordinate in the gravity dual

Gauge/Gravity duality

However, in many cases, these objects are not directly calculable since we cannot apply perturbation theory if the theory is strongly coupled

$\Downarrow$

With holography, we can deal with it

We are interested in the fluid properties associated to transport of conserved currents for strongly coupled systems

Gauge/Gravity duality

- Near the boundary $\mathcal{B}$,

$$ds^2 \sim \frac{L^2}{z^2}\left[dz^2 + \left(g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + \cdots\right) dx^\mu dx^\nu\right] ,$$

For pure AdS $\rightarrow g^{(0)}_{\mu\nu} = \eta_{\mu\nu}\, , g^{(n)}_{\mu\nu} = 0\, , n > 0$

- In holography,

$$\frac{-2}{\sqrt{g^{(0)}}} \frac{\delta^n \mathcal{S}}{\delta g^{(0)}_{\mu_1\nu_1} \cdots \delta g^{(0)}_{\mu_n\nu_n}}\Bigg|_{\mathcal{B}} = \langle T^{\mu_1\nu_1} \cdots T^{\mu_n\nu_n}\rangle\, ,$$

QFT is at finite temperature $\Leftrightarrow$ Black hole gravity dual

Motivation

The **Ward identities** are relations between the correlation functions that follow from the global symmetries of the theory.

For the two point correlator

$$\widetilde{\Gamma}_{\alpha\beta\mu\nu}(x,x') = \langle T_{\alpha\beta}(x) T_{\mu\nu}(x') \rangle$$

the Ward identities are

First kind

$$\frac{\partial}{\partial x^{\alpha}} \widetilde{\Gamma}^{\alpha\beta\mu\nu}(x,x') \approx 0$$

Well known from near boundary analysis

Second kind

$$\frac{\partial}{\partial x'^{\mu}} \widetilde{\Gamma}^{\alpha\beta\mu\nu}(x,x') \approx 0$$

Necessary further “input”

Motivation

In momentum space,

Kubo formulas

$$\text{Momentum conductivity} \Rightarrow \kappa_{ij} = -\frac{1}{\omega}\,\mathrm{Im}\,\Gamma_{0i0j}(\omega, k),$$

$$\text{Shear viscosity} \Rightarrow \eta = -\frac{1}{\omega}\,\mathrm{Im}\,\Gamma_{xyxy}(\omega, k),$$

$$\text{Bulk viscosity and shear viscosity} \Rightarrow \eta + \frac{\zeta}{2} = -\frac{1}{\omega}\,\mathrm{Im}\,\Gamma_{xxxx}(\omega, k).$$

We will examine the relations that arise between ζ, η, κ according to holography boundary

bulk

QFT_d

- strong interacting QFT
- $2+1$ dimensions
- Finite temperature
- Rotational invariance
- Parity symmetry
- **With** conformal invariance
- **Without** conformal invariance
- Correlators

AdS_{d+1}

- Einstein gravity
- $3+1$ dimensions
- Black hole
- Rotational invariance
- Parity symmetry
- **Without** scalar field in $\mathcal{S}$
- **With** scalar field in $\mathcal{S}$
- Linearized fluctuations

Table of Contents

We will consider the

Bulk line element

$$ds^2 = \frac{dz^2}{f(z)} + \left(-f(z)dt^2 + dx^2 + dy^2\right) \,, \quad f(z) = 1 - \left(\frac{z}{z_h}\right)^3 \,,$$

together with the

Bulk Lagrangian density $\mathcal{L}$

$$\mathcal{L} = \mathcal{R} - 2\Lambda \,, \qquad \Lambda = -3 \,.$$

We make linear fluctuations $g \to g + h$. For the shear viscosity, we turn on the h_{0y}, h_{xy} components.

Conserved current

Through the change of variables

$$h_{0y} = z\psi_0, \quad h_{xy} = \frac{z}{\sqrt{f}}\psi_1.$$

we get the coupled Schroedinger equations

$$0 = \psi_1'' - \mathcal{V}_1\psi_1 + \boxed{\frac{\omega k}{f^{3/2}}\psi_0}, \quad 0 = \psi_0'' - \mathcal{V}_0\psi_0 - \boxed{\frac{\omega k}{f^{3/2}}\psi_1}.$$

Conserved current

If we define the "currents"

$$j_{0,1} = \overline{\psi}_{0,1}\psi_{0,1}{}' - \overline{\psi}_{0,1}{}'\psi_{0,1}.$$

we can read off a conserved "Schroedinger" current $\mathcal{J}$,

$$\mathcal{J} = j_0 - j_1\,, \quad \mathcal{J}'_{\text{on-shell}} = 0\,,$$

k parity odd and even split

Given a field $\mathcal{A}$, we define

- Parity odd component: $\mathcal{A}_{\text{odd}}(k) = -\mathcal{A}_{\text{odd}}(-k)$
- Parity even component: $\mathcal{A}_{\text{even}}(k) = \mathcal{A}_{\text{even}}(-k)$
- $$\mathcal{A}(k) = \mathcal{A}_{\text{odd}}(k) + \mathcal{A}_{\text{even}}\,(k)$$

Shear viscosity in a CFT

From computing the current at the horizon $\mathcal{J}|_{Horizon} = \mathcal{J}(z_h)$, we see that is parity invariant. Since it is conserved,

$$[\mathcal{J}|_{\mathcal{B}}]_{\text{odd}} = 0\,,$$

With this "ingredient" we will be able to retrieve some information about the second kind W.i.

Combining

Ward identities from $[\mathcal{J}_B]_{\text{odd}} = 0$

$$\left(\Gamma_{xyxy} - \overline{\Gamma}_{xyxy}\right)_{\text{odd}} = 0\,, \quad \omega\Gamma_{xy0y\,\text{odd}} - k\overline{\Gamma}_{xyxy\,\text{even}} = 0\,,$$

$$+$$

Ward identities from $\partial_\mu \langle T^{\mu\nu}\rangle$

$$\omega\Gamma_{0yxy} + k\Gamma_{xyxy} = 0\,, \quad \omega\Gamma_{0y0y} + k\Gamma_{xy0y} = 0\,,$$

we obtain the

1^{st} generalized Ward identity

$$\left[\omega^2\Gamma_{0y0y} + k^2\overline{\Gamma}_{xyxy}\right]_{\text{even}} = 0\,,$$

Table of Contents

Bulk viscosity in a non-CFT

Now, since $\zeta \neq 0 \implies$

$$\mathcal{L} = \mathcal{R} - 2\Lambda + (\partial\phi)^2 - V(\phi)\,, \quad \phi = \phi(z)\,,$$

and

$$ds^2 = dz^2 + e^{2A(z)}\left(-e^{2B(z)}dt^2 + dx^2 + dy^2\right)\,,$$

This time, we will turn on 5 fluctuations $y_{1,\cdots,5}$. Each of them have well-defined momentum parity.

The EOMS will not enable to find a conserved quantity that easily

$\Downarrow$

In order to get it, it will be necessary to add further fields (η) that do not form part of our initial problem.

Conserved current

From the general expression for the linearized EOMS,

$$y_i'' + a_{ij}y_j' + b_{ij}y^j = 0\,, \text{ or } K^{-1}(Ky')' + by = 0\,, K' = Ka\,,$$

one can consider an

effective Lagrangian

$$L = (\eta^\dagger)'Ky' - \eta^\dagger Kby + (y^\dagger)'K^\dagger\eta' - y^\dagger b^\dagger K^\dagger \eta\,,$$

which is invariant under $U(1)$ gauge and parity transformations

Auxiliary fields

With this, now we have so the wanted conserved current

Noether current

$$J = (\eta^\dagger)' K y - \eta^\dagger K y' + (y^\dagger)' K^\dagger \eta - y^\dagger K^\dagger \eta' \,, \quad J'_{\text{on-shell}} = 0 \,,$$

The presence of the auxiliary fields η ensure that we get the equations of motion for y_i.

$\Downarrow$

Boundary conditions for η? $\rightarrow$ Multiple choices

Ward identities

Unlike it happened when studying the CFT, the current at the horizon has a term odd under momentum reflection,

$$[J|_{\text{Horizon}}]_{\text{odd}} \neq 0 \implies [J|_{\text{Boundary}}]_{\text{odd}} \neq 0$$

However, we can still draw useful information. For example: if only parity odd source $y_5^{(0)} \neq 0$

$$\left[\Gamma_{0x0x} - \overline{\Gamma}_{0x0x}\right]_{\text{odd}} = 0\,,$$

Ward identity

On the other hand, if we turn on parity even-odd terms, we get the desired

2^{nd} generalized Ward identity

$$\left[\omega^2\Gamma_{0x0x} + k^2\overline{\Gamma}_{xxxx}\right]_{\text{even}} = \left(\omega^2 + k^2\right) P + k\omega W_{\text{odd}}$$

- which relates the momentum conductivity and the bulk viscosity, but it depends on an "ambiguous" term W_{odd}.
- Because of $W_{\text{odd}} \neq 0$, we cannot completely determine the relation without first solving the equations for the fluctuations.

Table of Contents

Conclusions

For $k <<$

$$\kappa_{ij} \simeq \kappa_{ij}^{(0)} + (k^2\delta^{ij} - k^i k^j)\kappa_T^{(2)} + k^i k^j \kappa_L^{(2)} + \cdots,$$

$$\eta \simeq \eta^{(0)} + O(k^2), \quad \zeta = \zeta^{(0)} + O(k^2).$$

$+$

Kubo formulas

$+$

generalized W.ids.

$$\left[\omega^2\Gamma_{0y0y} + k^2\overline{\Gamma}_{xyxy}\right]_{\text{even}} = 0,$$

$$\left[\omega^2\Gamma_{0x0x} + k^2\overline{\Gamma}_{xxxx}\right]_{\text{even}} = \left(\omega^2 + k^2\right) P + k\omega W_{\text{odd}}$$

$\Downarrow$

relations between $\zeta \Leftrightarrow \eta \Leftrightarrow \kappa$

Conclusions

Expanding $W_{\text{odd}} \simeq kW_{\text{odd}}^{(1)} + \cdots$. we get

$$\kappa_{yy}^{(2)} = \frac{1}{\omega^2}\eta^{(0)},$$

$$\kappa_{xx}^{(2)} = \frac{1}{\omega^2}\left(\eta^{(0)} + \frac{\zeta^{(0)}}{2} - \mathrm{Im}\, W_{\text{odd}}^{(1)}\right).$$

$\Downarrow$

Agrees with field theory

$\Downarrow$

Has the right structure, but we do not know from general arguments what is the contribution from W_{odd}.

Conclusions

W_{odd} contains two kind of contributions,

- **One kind** comes from $J|_{\text{Boundary}}$ because it depends on the boundary conditions of the auxiliary fields, which can be fixed in various ways.
- The **second kind** depends on $J|_{\text{Horizon}}$ and it cannot be determined without explicitly solving the EOMS.

Still, we have gained useful information about the W.ids in holography.

THANK YOU!