Isotropy theorem of oscillating cosmological fields

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Cosmology

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- Dark matter: WIMPs, etc
- Accelerated expansion: modification of GR or a fluid such that $p/\rho < -1/3$
- Termalized causally unconnected regions and flatness problem $\Rightarrow$ Inflation?

Coherent scalars

Coherent scalar fields are ubiquitous in cosmology,

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Coherent fast oscillating scalar

The general analysis for a minimally coupled scalar under a power-law potential was made by M. S. Turner[1],

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{n}\phi^n \;\Rightarrow \langle p\rangle = \frac{n-2}{n+2}\,\langle\rho\rangle$$

[1] Phys. Rev. D28 (1983) 1243

Vector fields in cosmology

On the other hand, for vector fields the evolution is clearly anisotropic. However, there have been proposals in order to deal with this problem,

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[2] Cervero, Jacobs; Phys. Lett. B78, 427 (1978)

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1. Particular solutions: Triads of orthogonal vectors
2. Large number, N, of randomly oriented fields[2],

$$\frac{T^i_j}{p_k} \sim \frac{1}{\sqrt{N}}$$

[2] Golovnev, Mukhanov, Vanchurin; JCAP 0806,009 (2008)

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1. Particular solutions: Triads of orthogonal vectors
2. Large number, N, of randomly oriented fields,

$$\frac{T^i_j}{p_k} \sim \frac{1}{\sqrt{N}}$$

3. Average isotropy for abelian massive fields linearly polarized[2].

[2] Dimopoulos; Phys. Rev. D 74, 083502 (2006)

Coherent vector field evolution

Let us assume a plain FLRW metric,

$$ds^2 = dt^2 - a(t)^2 d\vec{x}^2.$$

the action of a minimally coupled vector field reads,

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(A^2)$$

The homogeneous solution evolves following the equation,

$$A_0 = 0$$

$$\ddot{A}_i + H\dot{A}_i - 2V'(A^2)A_i = 0$$

Coherent vector field evolution

And the energy momentum tensor results,

$$T^{\mu}{}_{\nu} = \frac{1}{4} F_{\rho\lambda} F^{\rho\lambda} g^{\mu}{}_{\nu} - F^{\rho\mu} F_{\rho\nu} + V(A^2) g^{\mu}{}_{\nu} - 2V'(A^2) A^{\mu} A_{\nu}$$

By components,

$$\rho \equiv T^{0}{}_{0} = \frac{1}{2} \frac{\dot{A}_i \dot{A}_j \delta^{ij}}{a^2} + V(A^2)$$

$$p_k \equiv -T^{k}{}_{k} = \frac{1}{2} \frac{\dot{A}_i \dot{A}_j}{a^2} \delta^{ij} - \frac{\dot{A}_k \dot{A}_k}{a^2} - V(A^2) - 2V'(A^2) \frac{A_k A_k}{a^2}, \; k = 1, 2, 3$$

$$T^{i}{}_{j} = \frac{\dot{A}_i \dot{A}_j}{a^2} + 2V'(A^2) \frac{A_i A_j}{a^2} \; ; \; T^{i}{}_{0} = 0$$

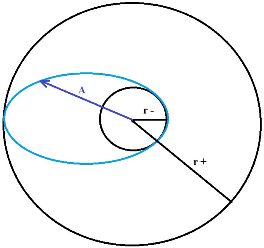
As it can be seen the EMT is anisotropic, in general.

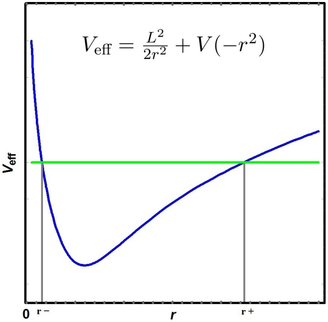
Coherent vector field evolution

If we consider fast oscillations $\dot{A}_i \sim \mathcal{O}\left(\omega_{\rm eff} A_i\right) \gg \mathcal{O}\left((\dot{a}/a)A_i\right)$ and redefine $r_i \equiv A_i/a$

$$\ddot{r}_i + \left(2V'(-r^2) + \mathcal{O}\left(H^2\right)\right) r_i = 0$$

we can neglect the terms order $\mathcal{O}\left(H^2\right)$. Thus, the field behaves analogously to a point particle in a potential and we can exploit classical mechanics results,





From a generalization of the virial theorem,

$$\partial_t\left(r_i\dot{r}_j\right) = \dot{r}_i\dot{r}_j + 2V'(-r^2)r_ir_j$$

If we make a temporal average, $H^{-1} \gg T \gg \omega_{\text{eff}}^{-1}$,

$$\langle\partial_t\left(r_i\dot{r}_j\right)\rangle = \frac{1}{T}\int_{t+T}^{t} dt'\partial_t\left(r_i\dot{r}_j\right) = \frac{r_i(t+T)\dot{r}_j(t+T) - r_i(t)\dot{r}_j(t)}{T}$$

If the system is bounded and fast oscillating,

Generalized virial theorem

$$\langle\partial_t\left(r_i\dot{r}_j\right)\rangle = 0 + \mathcal{O}\left(H\omega_{\text{eff}}\, r_ir_j\right) \Rightarrow \left\langle\dot{r}_i\dot{r}_j + 2V'(-r^2)r_ir_j\right\rangle \approx 0$$

Using the generalized virial theorem, we reach two important results,

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- The energy momentum tensor is diagonal and isotropic,

$$\langle T^{\mu}_{\nu} \rangle = \text{Diag}\left(\langle \rho \rangle , \langle p \rangle , \langle p \rangle , \langle p \rangle\right)$$

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- For power-law potentials, $V = \lambda(A_\mu A^\mu)^n$, the behaviour of the equation of state results,

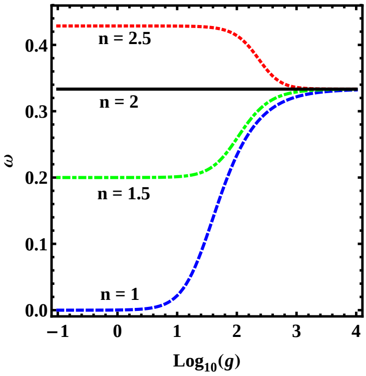
$$\omega = \frac{\langle p \rangle}{\langle \rho \rangle} = \frac{n-1}{n+1}$$

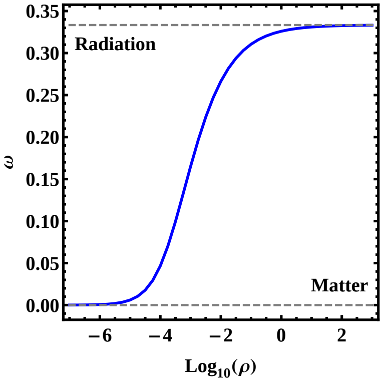
equivalent to the scalar case!

Non-abelian vector fields

The same results are found for non-abelian vector fields,

$$\mathcal{L} = \frac{1}{4} F^{a}_{\mu\nu} F^{a\,\mu\nu} - V(A^2), \quad D_\mu \equiv \partial_\mu - i g A^a_\mu T^a$$





Phys. Rev. D 87, 043523 (2013)

Non-abelian vector fields + gauge fixing term

And also if we give momentum to the zero component,

$$\mathcal{L} = \frac{1}{4} F^a_{\mu\nu} F^{a\ \mu\nu} + \frac{\xi}{2} \left(\nabla_\mu A^{a\ \mu} \right)^2 - V(A^2),$$

Nothing makes those cases special
…maybe the average isotropy is a
general feature ?!

Belinfante-Rosenfeld EMT

Considering a Minkowski space-time and a lagrangian

$$\mathcal{L} = \mathcal{L}\left[\phi^A, \partial_\mu \phi^A\right]$$

Under an infinitesimal translation: $x^\mu \longrightarrow x^\mu + \delta a^\mu$

$$0 = \delta \int d^4x \mathcal{L} = -\int d^4x \delta a_\nu \partial_\mu \Theta^{\mu\nu} \longrightarrow \partial_\mu \Theta^{\mu\nu} = 0$$

Which is the Noether current associated to the symmetry under space-time translations, the canonical energy-momentum tensor

$$\Theta^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^A\right)} \partial^\nu \phi^A$$

Belinfante-Rosenfeld EMT

This tensor is not unique, a new piece $\partial_\rho \tilde{\Theta}^{\rho\mu\nu}$ antisymmetric in the first two indexes can always be added. However, as the energy momentum tensor must be symmetric this extra term is fixed

$$T^{\mu\nu} = \Theta^{\mu\nu} - \frac{1}{2}\partial_\rho \left(S^{\rho\mu\nu} + S^{\mu\nu\rho} - S^{\nu\rho\mu} \right)$$

where

$$S^{\mu\nu\rho} \equiv \Pi^\mu_A \Sigma^{\nu\rho} \phi^A$$

$$\Pi^\mu_A \equiv \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^A \right)} \ , \quad \Sigma^{\nu\rho} \equiv \text{Lorentz group generators}$$

The Belinfante-Rosenfeld EMT can be written in a curved space-time in a straightforward way by using minimal coupling

$$T^{\mu\nu} = \Theta^{\mu\nu} - \frac{1}{2}\nabla_\rho \left(S^{\rho\mu\nu} + S^{\mu\nu\rho} - S^{\nu\rho\mu} \right)$$

Averaging anisotropy

Let us consider a Friedmann-Lemaître-Robertson-Walker metric and a homogeneous field. The responsible of the anisotropies is the non-canonical piece of the EMT.

$$\nabla_0 \tilde{\Theta}^{0\mu\nu} = \partial_0 \tilde{\Theta}^{0\mu\nu} + \left(\Gamma^{\rho}_{\rho\gamma} \tilde{\Theta}^{\gamma\mu\nu} + \Gamma^{\mu}_{\rho\gamma} \tilde{\Theta}^{\rho\gamma\nu} + \Gamma^{\nu}_{\rho\gamma} \tilde{\Theta}^{\rho\mu\gamma} \right)$$

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If the field oscillates with a frequency much higher than the expansion rate, we can neglect the scale factor derivatives.
As the leading term is a total derivative vanishes in average,

$$\left\langle \nabla_\gamma \tilde{\Theta}^{\gamma\mu\nu} \right\rangle \approx \left\langle \partial_0 \tilde{\Theta}^{0\mu\nu} \right\rangle = \frac{\tilde{\Theta}^{0\mu\nu}(t+T) - \tilde{\Theta}^{0\mu\nu}(t)}{T}$$

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$$\frac{\left\langle \nabla_\gamma \tilde{\Theta}^{\gamma\mu\nu} \right\rangle}{\langle T^{00} \rangle} \sim \mathcal{O}\left(\frac{H}{\omega_{\text{eff}}} \right)$$

Results

- The average energy momentum tensor becomes diagonal and isotropic

$$\left\langle T^{00}\right\rangle = \left\langle \Pi^0_A \partial_0 \phi^A - \mathcal{L}\right\rangle ;\ \left\langle T^{ii}\right\rangle = \left\langle -g^{ii}\mathcal{L}\right\rangle ;\ \left\langle T^{0i}\right\rangle = \left\langle T^{jk}\right\rangle = 0$$

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- Using these results we can also express the average equation of state in this suggestive form

$$\omega = \frac{\langle p\rangle}{\langle \rho\rangle} = \frac{\langle \mathcal{L}\rangle}{\left\langle \Pi^0_A \partial_0 \phi^A - \mathcal{L}\right\rangle} = \frac{\langle \mathcal{L}\rangle}{\langle \mathcal{H}\rangle}$$

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- For theories with $\mathcal{H} = \left(\lambda^{AB} g_{00} \Pi^0_A \Pi^0_B\right)^{n_T} + \left(M^{AB}\phi^A\phi^B\right)^{n_v}$

$$\omega = \frac{2n_V}{1+\frac{n_V}{n_T}} - 1;\ \ \text{if } n_T = 1 \Rightarrow \omega = \frac{n_V - 1}{n_V + 1}$$

Isotropy theorem of cosmological fields

Theorem

For an inertial observer, the metric in Riemann normal coordinates

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{3}R_{\mu\alpha\nu\beta}\, x^\alpha x^\beta + \dots$$

If the following conditions holds, the energy momentum is diagonal and isotropic in average.

- *The lagrangian depends only on the fields and their gradients which are minimally coupled with gravity.*
- *The field evolves rapidly,*

$$|R^\gamma_{\lambda\mu\nu}| \ll \omega^2_{\mathit{eff},\ A} \ \text{ and } \ |\partial_i S^{\mu\nu\rho}| \ll |\partial_0 S^{\mu\nu\rho}|$$

- ϕ^A *and* Π^0_A *are bounded during its evolution.*

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Conclusions

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Thanks for your attention!