Wald Entropy for Ghost-Free, Infinite Derivative Theories of Gravity

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(arXiv:1509.01247)
(arXiv:1503.05568)

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Motivation

- Study the Wald’s entropy for infinite higher derivative gravity.
- Study the area law when gravity is being modified in the UV.
- Entropy of such gravity on (A)dS background.
- Study the result in the context of non-singular bouncing cosmology.

An Old Problem!

- UV is pathological and IR is well behaved.

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R + \ldots)$$

What terms shall we add so that the gravity becomes well behaved in short distances and early times?
(While keeping the general covariance)

Most general higher order action

$$S^{tot} = S^{EH} + S^{UV}$$

$$S^{EH} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}R$$

$$S^{UV} = \int d^4x\sqrt{-g}\left[\left(R^{\mu_1\nu_1\lambda_1\sigma_1}\mathcal{O}_{\mu_1\nu_1\lambda_1\sigma_1}^{\mu_2,\nu_2\lambda_2\sigma_2}R_{\mu_2,\nu_2\lambda_2\sigma_2}\right)+\cdots\right]$$

$\mathcal{O}$ Contains covariant operators such as D'Alembertian operator.

These corrections are expected to arise naturally in string field theory, where they are analogous to have all orders of α' corrections.

E.Witten '86
W. Siegel '01

Simplification

- Using integration by parts, Bianchi identities and the symmetry properties of the Riemann tensor, we simplify the action to

$$S^{tot} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R + \alpha\,(R\mathcal{F}_1(\Box_M)R$$
$$+R_{\mu\nu}\mathcal{F}_2(\Box_M)R^{\mu\nu} + R_{\mu\nu\lambda\sigma}\mathcal{F}_3(\Box_M)R^{\mu\nu\lambda\sigma})]$$

α has inverse of mass squared dimension

$$\Box_M \equiv \Box/M^2$$

$$\mathcal{F}_i(\Box_M) = \sum_{n=0}^{\infty} f_{i_n}(\Box_M)^n$$

Equations of Motion

Biswas, Conroy, Koshelev, Mazumdar (2014)

$$S=\int d^4x\sqrt{-g}\left(\frac{R}{2}+R\mathcal{F}_1(\Box)R+R^{\mu\nu}\mathcal{F}_2(\Box)R_{\mu\nu}+C^{\mu\nu\lambda\sigma}\mathcal{F}_3(\Box)C_{\mu\nu\lambda\sigma}\right)$$

$$\begin{aligned}P^{\alpha\beta}=G^{\alpha\beta}&+4G^{\alpha\beta}\mathcal{F}_1(\Box)R+g^{\alpha\beta}R\mathcal{F}_1(\Box)R-4\left(\nabla^\alpha\nabla^\beta-g^{\alpha\beta}\Box\right)\mathcal{F}_1(\Box)R\\
&-2\Omega_1^{\alpha\beta}+g^{\alpha\beta}(\Omega_{1\sigma}^{\sigma}+\bar{\Omega}_1)+4R_\mu^\alpha\mathcal{F}_2(\Box)R^{\mu\beta}\\
&-g^{\alpha\beta}R_\nu^\mu\mathcal{F}_2(\Box)R_\mu^\nu-4\nabla_\mu\nabla^\beta(\mathcal{F}_2(\Box)R^{\mu\alpha})+2\Box(\mathcal{F}_2(\Box)R^{\alpha\beta})\\
&+2g^{\alpha\beta}\nabla_\mu\nabla_\nu(\mathcal{F}_2(\Box)R^{\mu\nu})-2\Omega_2^{\alpha\beta}+g^{\alpha\beta}(\Omega_{2\sigma}^{\sigma}+\bar{\Omega}_2)-4\Delta_2^{\alpha\beta}\\
&-g^{\alpha\beta}C^{\mu\nu\lambda\sigma}\mathcal{F}_3(\Box)C_{\mu\nu\lambda\sigma}+4C^{\alpha}_{\ \mu\nu\sigma}\mathcal{F}_3(\Box)C^{\beta\mu\nu\sigma}\\
&-4(R_{\mu\nu}+2\nabla_\mu\nabla_\nu)(\mathcal{F}_3(\Box)C^{\beta\mu\nu\alpha})-2\Omega_3^{\alpha\beta}+g^{\alpha\beta}(\Omega_{3\gamma}^{\gamma}+\bar{\Omega}_3)-8\Delta_3^{\alpha\beta}\\
&=T^{\alpha\beta},\end{aligned}$$

$$\Omega_1^{\alpha\beta}=\sum_{n=1}^{\infty}f_{1_n}\sum_{l=0}^{n-1}\nabla^\alpha R^{(l)}\nabla^\beta R^{(n-l-1)},\quad \bar{\Omega}_1=\sum_{n=1}^{\infty}f_{1_n}\sum_{l=0}^{n-1}R^{(l)}R^{(n-l)},$$

$$\Omega_2^{\alpha\beta}=\sum_{n=1}^{\infty}f_{2_n}\sum_{l=0}^{n-1}R_\nu^{\mu;\alpha(l)}R_\mu^{\nu;\beta(n-l-1)},\quad \bar{\Omega}_2=\sum_{n=1}^{\infty}f_{2_n}\sum_{l=0}^{n-1}R_\nu^{\mu(l)}R_\mu^{\nu(n-l)},$$

$$\Delta_2^{\alpha\beta}=\frac{1}{2}\sum_{n=1}^{\infty}f_{2_n}\sum_{l=0}^{n-1}[R_\sigma^{\nu(l)}R^{(\beta|\sigma|;\alpha)(n-l-1)}-R_\sigma^{\nu;(\alpha(l)}R^{\beta)\sigma(n-l-1)}]_{;\nu},$$

$$\Omega_3^{\alpha\beta}=\sum_{n=1}^{\infty}f_{3_n}\sum_{l=0}^{n-1}C^{\mu;\alpha(l)}_{\nu\lambda\sigma}C_\mu^{\ \nu\lambda\sigma;\beta(n-l-1)},\quad \bar{\Omega}_3=\sum_{n=1}^{\infty}f_{3_n}\sum_{l=0}^{n-1}C^{\mu(l)}_{\nu\lambda\sigma}C_\mu^{\ \nu\lambda\sigma(n-l)},$$

$$\Delta_3^{\alpha\beta}=\frac{1}{2}\sum_{n=1}^{\infty}f_{3_n}\sum_{l=0}^{n-1}[C^{\lambda\nu(l)}_{\ \ \sigma\mu}C_\lambda^{\ (\beta|\sigma\mu|;\alpha)(n-l-1)}-C^{\lambda\nu\ \ ;(\alpha(l)}_{\ \ \sigma\mu}C_\lambda^{\ \beta)\sigma\mu(n-l-1)}]_{;\nu}.$$

Linearised EoM

$$S = \int d^4x \sqrt{-g} \left[R + R\mathcal{F}_1(\Box)R + R_{\mu\nu}\mathcal{F}_2(\Box)R^{\mu\nu} + R_{\mu\nu\alpha\beta}\mathcal{F}_3(\Box)R^{\mu\nu\alpha\beta} \right]$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}\,.$$

$$R_{\rho\mu\sigma\nu} = \frac{1}{2}\left(\partial_\sigma\partial_\mu h_{\rho\nu} + \partial_\nu\partial_\rho h_{\mu\sigma} - \partial_\nu\partial_\mu h_{\rho\sigma} - \partial_\sigma\partial_\rho h_{\mu\nu}\right)\,,$$

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma\partial_\mu h^\sigma_\nu + \partial_\nu\partial_\sigma h^\sigma_\mu - \partial_\nu\partial_\mu h - \Box h_{\mu\nu}\right)\,,$$

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h\,,$$

$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and $h = \eta^{\mu\nu}h_{\mu\nu}$.

$$P^{\alpha\beta} = -\frac{1}{2}\Big[a(\Box)\Box h^{\alpha\beta} + b(\Box)\partial_\sigma(\partial^\alpha h^{\sigma\beta} + \partial^\beta h^{\sigma\alpha}) + c(\Box)\left(\partial^\alpha\partial^\beta h + \eta^{\alpha\beta}\partial_\mu\partial_\nu h^{\mu\nu}\right)$$
$$+ d(\Box)\eta^{\alpha\beta}\Box h + f(\Box)\Box^{-1}\partial^\alpha\partial^\beta\partial_\mu\partial_\nu h^{\mu\nu}\Big]\,,$$

where we have defined the functions a, b, c, d, f

$$a(\Box) = 1 + 2\mathcal{F}_2(\Box)\Box + 8\mathcal{F}_3(\Box)\Box\,,$$
$$b(\Box) = -1 - 2\mathcal{F}_2(\Box)\Box - 8\mathcal{F}_3(\Box)\Box\,,$$
$$c(\Box) = 1 - 8\mathcal{F}_1(\Box)\Box - 2\mathcal{F}_2(\Box)\Box\,,$$
$$d(\Box) = -1 + 8\mathcal{F}_1(\Box)\Box + 2\mathcal{F}_2(\Box)\Box$$
$$f(\Box) = 8\mathcal{F}_1(\Box)\Box + 4\mathcal{F}_2(\Box)\Box + 8\mathcal{F}_3(\Box)\Box.$$

Bianchi Identities

$$a(\Box)\Box h_{\mu\nu} + b(\Box)\partial_\sigma(\partial_\nu h^\sigma_\mu + \partial_\mu h^\sigma_\nu) + c(\Box)(\eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \partial_\mu\partial_\nu h) + \eta_{\mu\nu} d(\Box)\Box h + f(\Box)\Box^{-1}\partial_\sigma\partial_\lambda\partial_\mu\partial_\nu h^{\lambda\sigma} = -2\kappa\tau_{\mu\nu} \text{ with } \kappa = M_p^{-2}$$

$$-\kappa\tau\nabla_\mu\tau^\mu_\nu = 0 = \underbrace{(c+d)}_{=0}\Box\partial_\nu h + \underbrace{(a+b)}_{=0}\Box h^\mu_{\nu,\mu} + \underbrace{(b+c+f)}_{=0} h^{\alpha\beta}_{,\alpha\beta\nu}$$

Which implies:

$$a + b = 0; \quad c + d = 0; \quad b + c + f = 0 .$$

As long as F's are analytical functions around $\Box = 0$
We have

$$a(0) = c(0) = -b(0) = -d(0) = 1 \text{ and } f(0) = 0$$

This insures we recover GR in IR limit.

Graviton Propagator

P. Van Nieuwenhuizen (1973)
Talaganis,Biswas,Mazumdar (2015)

- Using the field equations

$$\Pi^{-1\,\lambda\sigma}_{\mu\nu} h_{\lambda\sigma} = \kappa \tau_{\mu\nu}$$

One obtains the graviton propagator using the spin projection operators corresponding to the spin-2, the two scalars, and the vector degree of freedom. (See Spyros Talk)

For D-dim propagator see (arXiv:1509.01247)

$$\Pi = \frac{P^2}{ak^2} + \frac{P^0_s}{(a-3c)k^2}$$

Assuming a=c, such that we can take the continuous limit from UV to IR

$$\lim_{k^2\to 0} \Pi \sim \frac{1}{a(k^2)}\left[\frac{P^2}{k^2} - \frac{P^0_s}{2k^2}\right] \to \left[\frac{P^2}{k^2} - \frac{P^0_s}{2k^2}\right]$$

Ghost free action as no new propagating degree of freedom other than the massless graviton.

Ghost Free Condition

- The theory will be ghost free if "a" is an entire function and "a-3c" has at most a single zero.
- Since, we do not wish to introduce any new extra degrees of freedom other than the massless graviton throughout the IR to the UV:

$$a(\Box_M) = c(\Box_M)$$
$$\Rightarrow 2\mathcal{F}_1(\Box_M) + \mathcal{F}_2(\Box_M) + 2\mathcal{F}_3(\Box_M) = 0\,.$$

Wald's Entropy

R. M. Wald
V. Iyer (1993-94)

- consider a simple static, homogeneous and isotropic metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2$$

For a spherically symmetric Black hole we define the entropy as

$$S_W = -8\pi \oint \left(\frac{\delta \mathcal{L}}{\delta R_{rtrt}} \right)^{(0)} q(r) d\Omega^2$$

where we construct two normal directions along r and t with

$$\oint \equiv \oint_{r=r_H,\ t=\mathrm{const}}$$

The area of the horizon:

$$\mathrm{Area} = \oint q(r) d\Omega^2$$

Where,

$$q(r)d\Omega^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Wald's Entropy

$$S^{tot} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R + \alpha\,(R\mathcal{F}_1(\Box_M)R$$
$$+R_{\mu\nu}\mathcal{F}_2(\Box_M)R^{\mu\nu} + R_{\mu\nu\lambda\sigma}\mathcal{F}_3(\Box_M)R^{\mu\nu\lambda\sigma})]$$

$$S_W = \frac{\text{Area}}{4G}\,[1 + \alpha\,\{2\mathcal{F}_1(\Box_M) + \mathcal{F}_2(\Box_M) \;\; + 2\mathcal{F}_3(\Box_M)\}\,R]$$

$$a(\Box_M) = c(\Box_M)$$
$$\Rightarrow 2\mathcal{F}_1(\Box_M) + \mathcal{F}_2(\Box_M) + 2\mathcal{F}_3(\Box_M) = 0\,.$$

- Holographic nature of gravity remains preserved.
- Higher order corrections yield zero entropy .
- Holography is an IR effect.
- Ground state of gravity?

$$S_W = S_W^{EH} = \frac{Area}{4G}$$

What is the entropy in the linearised limit?

- Assume that we modify the metric with two Newtonian potential such that the r and t directions take the following form

$$ds^2 = -(1 + 2\Phi(r))dt^2 + (1 - 2\Psi(r))dr^2$$

$$S_W = \frac{Area}{4G}\{1 + 2\Psi - 2\Phi + \alpha\left[2\mathcal{F}_1(\Box_M) + \mathcal{F}_2(\Box_M) + 2\mathcal{F}_3(\Box_M)\right](-2\Phi'')\}$$

Note that when $\Psi = \Phi$ and $2F_1 + F_2 + 2F_3 = 0$, for any source term within the linearised limit, the gravitational entropy duly reduces to that of EH entropy.

Other scenarios?

- What if we don’t consider spherically mertic? Namely Φ and Ψ are different?
- What if we introduce extra propagating degrees of freedom apart from the massless graviton? (Such as in f(R)-theories) (Implies “a” and “c” are not equal). This is interesting in the context of cosmological singularity.

Black hole entropy in D-dim

- Our result can be generalised to D-dim.

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2_{D-2}$$

$$A_H = \oint_{r=r_H, t=const} r^{D-2} d\Omega^2_{D-2}$$

$$S_W = -8\pi \oint_{r=r_H, t=const} \frac{\delta \mathcal{L}}{\delta R_{rtrt}} r^{D-2} d\Omega^2_{D-2}$$

$$S_W = \frac{A_H}{4G_D}\big[1 + \alpha(2\mathcal{F}_1(\Box)R - \mathcal{F}_2(\Box) \times (g^{rr}R^{tt} + g^{rr}R^{rr}) - 4\mathcal{F}_3(\Box)R^{rtrt})\big]$$

Black hole entropy in D-dim

- We can decompose the entropy equation into (r,t) and its spherical components.

$$R = g^{\mu\nu} R_{\mu\nu} = g^{ab} R_{ab} + g^{\bar{m}\bar{n}} R_{\bar{m}\bar{n}},$$

Where the dimensions can be obtain via:

$$g^{\mu\nu} g_{\mu\nu} \equiv g^{ab} g_{ab} + g^{\bar{m}\bar{n}} g_{\bar{m}\bar{n}} = 2 + (D-2) = D.$$

Expanding the scalar curvature into Ricci and Riemann tensors, along with the properties of the static, spherically symmetric metric

$$g^{rr} R^{tt} + g^{rr} R^{rr} = -g_{tt} R^{tt} - g_{rr} R^{rr} = -g^{ab} R_{ab}$$

$$-4R_{rtrt} = 2g^{ab} R_{ab} - 2g^{ab} g^{\bar{m}\bar{n}} R_{\bar{m}a\bar{n}b}$$

Black hole entropy in D-dim

- In a static, spherically symmetric background

$$S_W = \frac{A_H}{4G_D}[1 + \alpha(2\mathcal{F}_1(\Box) + \mathcal{F}_2(\Box) + 2\mathcal{F}_3(\Box))\times$$
$$g^{ab}R_{ab} + 2\alpha(\mathcal{F}_1(\Box)g^{\bar{m}\bar{n}}R_{\bar{m}\bar{n}} - \mathcal{F}_3(\Box)g^{ab}g^{\bar{m}\bar{n}}R_{\bar{m}a\bar{n}b})].$$

The angular components of the Ricci tensor are given by

$$R_{\theta_n\theta_n} = \sin^{-2}(\theta_n)R_{\theta_{n+1}\theta_{n+1}}$$

n runs from 1 to (d-2) satisfying each angular direction.

For the given metric:

$$R_{\theta_1\theta_1} = (D-3) - (D-3)f(r) - rf'(r) = 0$$

With solution:

$$f(r) = 1 - \frac{\mu}{r^{D-3}}$$

Considering the metric and the solution one realises Schwarzschild solution.

Black hole entropy in D-dim

Thus, when considering a Schwarzschild solution, all $R_{\theta_i\theta_i}$ components, will vanish. This is a consequence of the axisymmetric properties of the solution.

Therefore:

$$S_W = \frac{A_H}{4G_D}[1 + \alpha(2\mathcal{F}_1(\Box) + \mathcal{F}_2(\Box) + 2\mathcal{F}_3(\Box)) \times g^{ab}R_{ab}]$$

Using

$$f(\Box) = 0, \text{ i.e. } \quad a(\Box) = c(\Box),$$

$$2\mathcal{F}_1(\Box) + \mathcal{F}_2(\Box) + 2\mathcal{F}_3(\Box) = 0$$

We recover the Area law in to all dimensions.

$$S_W = \frac{A_H}{4G_D}$$

Black hole entropy in D-dim

- We insured that,

1. in the context of a static, spherically symmetric metric, which asymptotes to Minkowski, the holographic nature of gravity is preserved in the IR.
2. The higher-order corrections to the UV do not affect the gravitational entropy as long as the only propagating degrees of freedom are the massless graviton.

D-DIMENSIONAL (A)dS ENTROPY

- The original action must be modified by the cosmological constant.

$$I^{tot} = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g} \big[R - 2\Lambda + \alpha \big(R \mathcal{F}_1(\Box) R + R_{\mu\nu} \mathcal{F}_2(\Box) R^{\mu\nu} + R_{\mu\nu\lambda\sigma} \mathcal{F}_3(\Box) R^{\mu\nu\lambda\sigma} \big) \big] .$$

$$\Lambda = \pm \frac{(D-1)(D-2)}{2l^2}$$

positive sign corresponds to dS, negative to AdS.

Note: hereafter, the topmost sign will refer to dS and the bottom to AdS.

D-DIMENSIONAL (A)dS ENTROPY

- We can obtain the (A)dS metric by taking

$$f(r) = \left(1 \mp \frac{r^2}{l^2}\right)$$

Considering the entropy equation and using the definition of curvatures in (A)dS backgrouds

$$A_H^{(A)dS} \equiv l^{D-2} A_{D-2}$$

$$S_W^{(A)dS} = \frac{A_H^{(A)dS}}{4G_D}[1 + \alpha(2\mathcal{F}_1(\Box) + \mathcal{F}_2(\Box) + 2\mathcal{F}_3(\Box))\times g^{ab}R_{ab} + 2\alpha(\mathcal{F}_1(\Box)g^{\bar{m}\bar{n}}R_{\bar{m}\bar{n}} - \mathcal{F}_3(\Box)g^{ab}g^{\bar{m}\bar{n}}R_{\bar{m}a\bar{n}b})]$$

$$A_{D-2} = (2\pi^{\frac{D-1}{2}})/\Gamma[\tfrac{D-1}{2}]$$

$$R_{\mu\nu\lambda\sigma} = \pm\frac{1}{l^2}g_{[\mu\lambda}g_{\nu]\sigma}, \quad R_{\mu\nu} = \pm\frac{D-1}{l^2}g_{\mu\nu}, R = \pm\frac{D(D-1)}{l^2},$$

$$S_W^{(A)dS} = \frac{A_H^{(A)dS}}{4G_D}(1 \pm \frac{2\alpha}{l^2}\{f_{1_0}D(D-1) + f_{2_0}(D-1) + 2f_{3_0}\})$$

Note that f's are now only coefficients as they are not acting on any curvature.

4-dim (A)dS entropy

- We note that upon using 4-dim the entropy's contribution takes the form of

$$12f_{1_0} + 3f_{2_0} + 2f_{3_0}$$

The difference from Minkowski case is due to the nature of propagator in (A)dS.

- Gauss Bonnet Gravity

$$\mathcal{L}_{GB} = \frac{\alpha}{16\pi G_D}\{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\}$$

Shu (2008)

$$S = \frac{l^{D-2}A_{D-2}}{4G_D}\left(1 + \frac{\alpha D(D-2)(D-3)}{2l^2}\right)$$

Our result can be matched upon reducing the f's to appropriate coefficients.

NON-SINGULAR BOUNCING COSMOLOGY AND HOLOGRAPHIC ENTROPY

- The applications of seeking (A)dS gravitational entropy for an infinite derivative theory of gravity is to understand the initial conditions for the Universe.
- Non-locality in gravity solves the cosmological singularity problem, at least in the context of homogeneous and isotropic metric, such as Friedmann-Robertson-Walker (FRW) background. (T. Biswas, A. Mazumdar and W. Siegel [hep-th/0508194])

NON-SINGULAR BOUNCING COSMOLOGY AND HOLOGRAPHIC ENTROPY

$$I_R = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left(R - 2\Lambda + \alpha R \mathcal{F}_1(\Box) R\right)$$

α>0 to ensure that gravity remains ghost-free. Conroy, Koshelev, Mazumdar (2014)

A reduced action of this type has been studied in where it was shown in a FRW spacetime (consequently dS), that null rays can be made past-complete without violating any relevant energy conditions, thus replacing the cosmological singularity with a bounce at t = 0.

Well developed example:

$$a(\Box) = e^{-\Box/M^2}$$

$$\mathcal{F}_1(\Box) = \frac{e^{-\Box/M^2} - 1}{\Box/M^2}$$

NON-SINGULAR BOUNCING COSMOLOGY AND HOLOGRAPHIC ENTROPY

- What's the gravitational entropy at the time of bounce for a cosmological constant dominated universe?

$$S_R^{dS} = \frac{A_H^{dS}}{4G_4}\left(1 - \frac{24\alpha}{l^2}\right)$$

Observations:

- The entropy of a cosmological constant dominated universe is less that what we have in EH gravity.
- Upon specific choice of dimensionful α the entropy vanishes entirely.

Conclusion and future directions

- We developed our understanding for infinite derivative theory of gravity. A theory which is ghost- and singularity free.
- We obtained the entropy over (A)dS background.
- We gained some insight regarding the bouncing cosmology.
- Could our Universe have begun its journey with a zero gravitational entropy? As a zero entropy state for any system would be equivalent to realising a ground state of the system.
- In our case, it is the graviton which realises its ground state in the presence of Λ and non-local gravity.
- Could this lead to a new state of gravity such that our Universe would yield a condensation of gravitons, at the moment of bounce, similar to the Bose-Einstein condensate with a zero entropy state?
- Towards (A)dS / CFT?...

Thanks!