UNIMODULAR GRAVITY REDUX

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JHEP

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Pauli, 1920 (Straumann)

¿From Pauli's discussions with Enz and Thellung we know that Pauli estimated the influence of the zero-point energy of the radiation field – cut off at the classical electron radius – on the radius of the universe, and came to the conclusion that it "could not even reach to the moon".

In units with $\hbar = c = 1$ the vacuum energy density of the radiation field is

$$< \rho >_{vac} = \frac{8\pi}{(2\pi)^3} \int_0^{\omega_{max}} \frac{\omega}{2} \omega^2 d\omega = \frac{1}{8\pi^2} \omega_{max}^4,$$

with

$$\omega_{max} = \frac{2\pi}{\lambda_{max}} = \frac{2\pi m_e}{\alpha}.$$

The corresponding radius of the Einstein universe in Eq.(2) would then be ($M_{pl} \equiv 1/\sqrt{G}$)

$$a = \frac{\alpha^2}{(2\pi)^{\frac{2}{3}}} \frac{M_{pl}}{m_e} \frac{1}{m_e} \sim 31 km.$$

comment of Henry Whitehead: 'It is the snobbishness of the young to suppose that a theorem is trivial because the proof is trivial.'

In Unimodular Gravity the full configuration space is restricted to unimodular metrics

$$\hat{g} \equiv \det\, \hat{g}_{\mu\nu} = -1$$

Action principle

$$S_{\rm in} \equiv \int d^n x \left(-\frac{1}{2\kappa^2} R[\hat{g}] + \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - V(\psi) \right)$$

Potential energy (not only zero mode) naively decoupled from the gravitational field

How to integrate over unimodular metrics?

Transverse Diffeomorphisms $\mathrm{TDiff}(M)$

$$\det \frac{\partial y^\alpha}{\partial x^\mu} = 1$$

(Generating vector is transverse)

$$\partial_\mu \xi^\mu = 0$$

Van der Bij, van Dam and Ng.
TDiff has enough freedom to kill the three extra polarizations: 5-3=2

Convenient new variables $\hat{g}_{\mu\nu} \equiv g^{-\frac{1}{n}}\, g_{\mu\nu}$

Their variation $\delta g_{\alpha\beta}$ is unconstrained

Whereas $\hat{g}^{\alpha\beta}\delta\hat{g}_{\alpha\beta} = 0$

ermann_Weyl_ETH-Bib_Portr_00890.jpg (JPEG Image, 46... http://upload.wikimedia.org/wikipedia/commons/7/78/Herma...

In this variables there is (sort of spurious?) Weyl symmetry in addition to TDiff

Flat Space

$$L \equiv \sum_{i=1}^{4} C_i \mathcal{O}^{(i)}$$

Most general l.c. of dimension 4 operators

$$\mathcal{O}^{(1)} \equiv \frac{1}{4}\partial_m h_{\rho\sigma}\partial^\mu h^{\rho\sigma}$$

$$\mathcal{O}^{(2)} \equiv -\frac{1}{2}\partial_\rho h_{\rho\sigma}\partial^\mu h^{\mu\sigma}$$

$$\mathcal{O}^{(3)} \equiv \frac{1}{2}\partial_\mu h\partial^\lambda h^{\mu\lambda}$$

$$\mathcal{O}^{(4)} \equiv -\frac{1}{4}\partial_\mu h\partial^\mu h$$

E.A., Blas, Garriga & Verdaguer, NPB

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

LTDiff $\quad C_2 = 1$

$$\partial_\mu \xi^\mu = 0$$

Two cases of enhanced symmetry
Both propagate spin 2 ONLY

$C_3 = C_4 = 1$ Fierz-Pauli (LDiff)

$$C_3 = \frac{2}{n}$$

$$C_4 = \frac{n+2}{n^2}$$

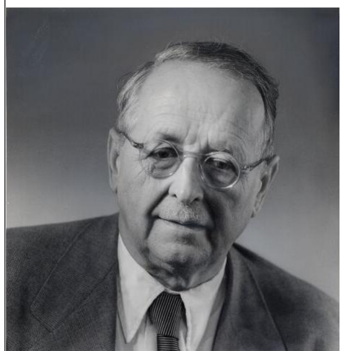
(WTDiff)

WTDiff is a truncation of Fierz-Pauli obtained by

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \frac{1}{n} h \eta_{\mu\nu}$$

But this is NOT A FIELD REDEFINITION, because it is not invertible

Weyl_ETH-Bib_Portr_00890.jpg (JPEG Image, 46... http://upload.wikimedia.org/wikipedia/commons/7/78/Herma...

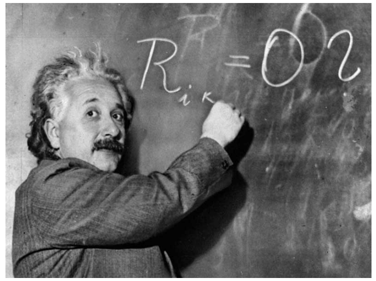


WTDiff is the linear limit of Unimodular Gravity

$$S \equiv -M_P^{n-2} \int d^n x \left(R[\hat{g}] + L_{\text{matt}}[\psi_i, \hat{g}] \right) =$$

$$= -M_P^{n-2} \int d^n x \; |g|^{\frac{1}{n}} \; \left(R + \frac{(n-1)(n-2)}{4n^2} \frac{\nabla_\mu g \nabla^\mu g}{g^2} + L_{\text{matt}}[\psi_i, |g|^{-\frac{1}{n}} \; g_{\mu\nu}] \right)$$

einstein.jpg (JPEG Image, 800 × 600 pixels) http://calnewport.com/blog/wp-content/uploads/2007/1



EM reminiscent of Einstein's 1919 traceless Theory (E.A. JHEP)

$$R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} + \frac{(n-2)(2n-1)}{4n^2} \left(\frac{\nabla_\mu g \nabla_\nu g}{g^2} - \frac{1}{n} \frac{(\nabla g)^2}{g^2} g_{\mu\nu} \right) - \frac{n-2}{2n} \left(\frac{\nabla_\mu \nabla_\nu g}{g} - \frac{1}{n} \frac{\nabla^2 g}{g^2} g_{\mu\nu} \right)$$

$$= M^{2-n} \left(T_{\mu\nu} - \frac{1}{n} T g_{\mu\nu} \right)$$

UG is NOT equivalent to GR in the gauge g=-1

(Although classically it is quite similar)
(work in progress)

The Bianchi identities bring the trace back into the game

$$\frac{n-2}{2n}\nabla_\mu R = -\frac{1}{n}\nabla_\mu T$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - Cg_{\mu\nu} = T_{\mu\nu}$$

The constant piece of the potential, V0, does not source the c.c.

Quantum corrections

Does the cc get generated by quantum corrections?

$$g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \qquad |\bar{g}| = 1$$

Gauge symmetry: Tdiff(M) & Weyl(M)

Need of BRST gauge fixing

Nielsen-Kallosh ghosts (30)

$$\nabla_\mu c^{T\mu} = 0.$$

$$h_{\mu\nu}^{(0,0)},\ c_\mu^{(1,1)},\ b_\mu^{(1,-1)},\ f_\mu^{(0,0)},\ \phi^{(0,2)}$$
$$\pi^{(1,-1)},\ \pi'^{(1,1)},\ \bar{c}^{(0,-2)},\ c'^{(0,0)}$$
$$c^{(1,1)},\ b^{(1,-1)},\ f^{(0,0)},$$

The operator involving $h_{\mu\nu},\ f$ and c'

Is non-minimal.

Usual heat kernel techniques do not work here.

Its determinant has been computed using the (quite unwieldy) Barvinsky-Vilkovisky technique

Background EM

$$\bar{R}_{\mu\nu} = \frac{1}{n}\ \bar{R}\bar{g}_{\mu\nu}$$

$$\bar{R} = \text{constant}$$

$$\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} = \frac{1}{n}\bar{R}^2 = \text{constant}$$

$$W_4 = E_4 + 2\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} - \frac{2}{3}\bar{R}^2 = E_4 + \text{constant}$$

Topological density modulo a dynamically irrelevant term

Divergences proportional to the EM can be eliminated by a one-loop field redefinition

$$\delta S = \int \frac{\delta S}{\delta g_{\mu\nu}} \; (a \; R_{\mu\nu} + b \; R \; g_{\mu\nu})$$

Point transformations leave formally invariant the path integral

Counterterms that vanish on-shell are gauge dependent (Kallosh)

Ergo, it is possible to find a gauge where the theory is finite, even off-shell

$$Z[\bar{A}] \equiv \int \mathcal{D}A \; e^{-S_c[\bar{A}+A]+\int \frac{\delta S_c}{\delta A}\big|_{\bar{A}} A - \xi \int L_{gf}}$$

Gauge fixing & ghost terms are together BRST exact

$$L_{gf} = sF_{gf}$$

Let us examine the dependence of the partition function with the gauge fixing

$$\frac{\partial Z}{\partial \xi} = \int \mathcal{D}A \left(\int L_{gf}\right) e^{-S(\bar{A}+A)+\frac{\partial S}{\partial A}|_{\bar{A}}+\xi\int L_{gf}} =$$

$$= \int \mathcal{D}A \; s\left(\int F_{gf}\right) e^{-S(\bar{A}+A)+\frac{\partial S}{\partial A}|_{\bar{A}}+\xi\int L_{gf}} =$$

$$= \int \mathcal{D}A \left(\int F_{gf}\right) s\left(S(\bar{A}+A) - \left.\frac{\partial S}{\partial A}\right|_{\bar{A}}\right) e^{-S(\bar{A}+A)+\frac{\partial S}{\partial A}|_{\bar{A}}+\xi\int L_{gf}}$$

The classical action is gauge invariant, then BRST invariant

$$sS_c = 0$$

On shell $\left.\frac{\delta S_c}{\delta A}\right|_{\bar{A}} = 0$ $\qquad \frac{\partial Z}{\partial \xi} = 0$

The one loop quantum correction reads

$$S_{\infty} = \frac{1}{16\pi^2(n-4)} \int d^4x \left(\frac{119}{90} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{83}{120} R^2 \right) =$$
$$= \frac{1}{16\pi^2(n-4)} \int d^4x \left(\frac{119}{90} E_4 + \text{constant} \right) \qquad (25)$$

This term is non-dynamical; in GR this is not the case (Christensen-Duff)

$$S_{GR} = \frac{1}{16\pi^2 (n-4)} \int \sqrt{|\bar{g}|}\, d^4x \left(-\frac{1142}{135} \Lambda + \frac{53}{45} W_4 \right) \qquad (26)$$

This result is not a consequence of $\bar{g} = -1.$

Rather, it is Weyl symmetry that forbids dimension zero operators

$$\mathcal{O}^{(0)} \equiv \int (-\bar{g})^{\beta} \; d^n x$$

It is possible to work out GR in the gauge g=-1.
CC is still generated there; there is no symmetry in GR to prevent it from appearing.

How certain are we of the belief that the effective string gravitational equations are Diff invariant and not only TDiff invariant?

Impossible to tell the difference from any first quantized approach...

Session Two slides

Background Field Splitting

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \bar{g}^{\frac{1}{n}}\, h_{\mu\nu},$$

In terms of a unimodular background $\bar{g}_{\mu\nu} = \bar{g}^{\frac{1}{n}}\, \tilde{g}_{\mu\nu}.$

$$S_{UG}[g_{\mu\nu} = \bar{g}_{\mu\nu} + \bar{g}^{\frac{1}{n}}\, h_{\mu\nu}] = S_{UG}[g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}],$$

All covariant derivatives etc are defined wrt the background

In order not to clutter formulas

$$\tilde{g}_{\mu\nu} = g_{\mu\nu}$$

BRST

$$s_D g_{\mu\nu} = s_W g_{\mu\nu} = 0$$
$$s_D h_{\mu\nu} = \nabla_\mu c^T_\nu + \nabla_\nu c^T_\mu + c^{T\rho}\nabla_\rho h_{\mu\nu} + \nabla_\mu c^{T\rho} h_{\rho\nu} + \nabla_\nu c^{T\rho} h_{\rho\mu}$$
$$s_W h_{\mu\nu} = 2c\,(g_{\mu\nu} + h_{\mu\nu})\,.$$

TDiff ghosts are transverse $\qquad \nabla_\mu c^{T\mu} = 0$

Quadratic piece of the classical action

$$\mathcal{L}_2 = \frac{1}{4} h^{\mu\nu}\bar{\nabla}^2 h_{\mu\nu} - \frac{n+2}{4n^2} h\bar{\nabla}^2 h + \frac{1}{2}\bar{\nabla}_\mu h^{\mu\alpha}\bar{\nabla}_\nu h^\nu_\alpha - \frac{1}{n}\bar{\nabla}_\mu h\bar{\nabla}_\nu h^{\mu\nu} + $$
$$+ \frac{1}{2} h^{\alpha\beta} h^\mu_\beta \bar{R}_{\mu\alpha} - \frac{1}{n} h h^{\mu\nu}\bar{R}_{\mu\nu} - \frac{1}{2} h^{\mu\nu} h^{\alpha\beta}\bar{R}_{\mu\alpha\nu\beta} - \frac{1}{2n} h^{\mu\nu} h_{\mu\nu}\bar{R} + \frac{1}{2n^2} h^2 \bar{R}.$$

Non-unimodular background can be easily gotten by going to the Jordan frame

In order to get a nilpotent BRST operator

$$s = s_D + s_W$$

It is necessary to define $[\mathcal{D} c^{T\mu}]$,

Better use unconstrained fields

$$c^T_\mu = \Theta_{\mu\nu} c^\nu = (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu - R_{\mu\nu})\, c^\nu = (Q_{\mu\nu} - \nabla_\mu \nabla_\nu)\, c^\nu$$

Swapping transversality with gauge symmetry

$$c_\nu \to \nabla_\nu f$$

New ghosts needed in ordet to get

$$s_D^2 = s_W^2$$

$$\{s_D, s_W\} = 0$$

Batalin-Vilkovisky

(Grassmann number,   Ghost number)

$h_{\mu\nu}$ c_μ

$h_{\mu\nu}^{(0,0)},\ c_\mu^{(1,1)},\ b_\mu^{(1,-1)},\ f_\mu^{(0,0)},\ \phi^{(0,2)}$ Diff

$\pi^{(1,-1)},\ \pi'^{(1,1)},\ \bar{c}^{(0,-2)},\ c'^{(0,0)}$ U(1)

$c^{(1,1)},\ b^{(1,-1)},\ f^{(0,0)},$ Weyl

$$\nabla_\mu (c^{\rho T} \nabla_\rho c^{T\mu}) = 0, \quad \nabla_\mu \left[(Q^{-1})^\mu_\nu \left(c^{\rho T} \nabla_\rho c^{T\nu} \right) \right] = 0.$$

$$s_D c^{T\mu} = c^{T\rho} \nabla_\rho c^{T\nu},$$

Results as above follow

$$S_{gauge-fixing} = \int d^n x \sqrt{|g|}\, s \left(X_{TD} + X_W \right).$$

X=(1,-1)

- Free invertible kinetic terms
- Quadratic in the quantum fields

field	s_D	s_W
$g_{\mu\nu}$	0	0
$h_{\mu\nu}$	$\nabla_\mu c^T_\nu + \nabla_\nu c^T_\mu + c^{\rho T}\nabla_\rho h_{\mu\nu} + \nabla_\mu c^{\rho T} h_{\rho\nu} + \nabla_\nu c^{\rho T} h_{\rho\mu}$	$2c^{(1,1)}\left(g_{\mu\nu} + h_{\mu\nu}\right)$
$c^{(1,1)\mu}$	$\left(Q^{-1}\right)^\mu_\nu \left(c^{\rho T}\nabla_\rho c^{T\nu}\right) + \nabla^\mu \phi^{(0,2)}$	0
$\phi^{(0,2)}$	0	0
$b^{(1,-1)}_\mu$	$f^{(0,0)}_\mu$	0
$\phi^{(0,0)}_\mu$	0	0
$\bar{c}^{(0,-2)}$	$\pi^{(1,-1)}$	0
$\pi^{(1,-1)}$	0	0
$c'^{\,(0,0)}$	$\pi'^{\,(1,1)}$	0
$\pi'^{\,(1,1)}$	0	0
$c^{(1,1)}$	$c^{T\rho}\nabla_\rho c^{(1,1)}$	0
$b^{(1,-1)}$	$c^{T\rho}\nabla_\rho b^{(1,-1)}$	$f^{(0,0)}$
$f^{(0,0)}$	$c^{T\rho}\nabla_\rho f^{(0,0)}$	0

Table 1. BRST transformations of the fields involved in the path integral.

The TDiff Sector

$$X_{TD} = b_\mu^{(1,-1)} \left[F^\mu + \rho_1 f^{\mu(0,0)}\right] + \bar{c}^{(0,-2)} \left[F_2^\mu c_\mu + \rho_2 \pi^{\prime\ (1,1)}\right] + \\ + c^{\prime\ (0,0)} \left[F_1^\mu b_\mu^{(1,-1)} + \rho_3 \pi^{(1,-1)}\right]$$

F_μ Contains the graviton and is akin to the usual Faddeev-Popov gauge fixing

$$\int d^n x \sqrt{|g|}\, s X_{TD} = \int d^n x\, f_\mu^{(0,0)} \left(F^\mu + \rho_1 f^{\mu(0,0)}\right) - b_\mu^{(1,-1)} s F^\mu + \\ + \pi^{(1,-1)} \left(F_2^\mu c_\mu^{(1,1)} + \rho_2 \pi^{\prime\ (1,1)}\right) + \bar{c}^{(0,-2)} F_2^\mu \nabla_\mu \phi^{(0,2)} + \\ + \pi^{\prime\ (1,1)} \left(F_{1\mu} b^{\mu(1,-1)} + \rho_3 \pi^{(1,-1)}\right) + c^{\prime\ (0,0)} F_1^\mu f_\mu^{(0,0)}$$

$$f_\mu^{(0,0)}\left(F^\mu + \rho_1 f^{\mu(0,0)}\right) + f_\mu^{(0,0)}\bar{F}_1^\mu c'^{\,(0,0)}$$

Completing the square

$$\rho_1\left(f_\mu^{(0,0)} + \frac{1}{2\rho_1}(F_\mu + \bar{F}_1^\mu c'^{\,(0,0)})\right)^2 - \frac{1}{4\rho_1}(F_\mu + \bar{F}_1^\mu c'^{\,(0,0)})^2$$

$$S_{gf} = -\frac{1}{4\rho_1}\int d^n x\sqrt{|g|}\,(F_\mu + \bar{F}_1^\mu c'^{\,(0,0)})^2$$

(This is essentially Faddeev-Popov)

Consider now the fermionic terms

$$\pi^{(1,-1)}\left(F_2^\mu c_\mu^{(1,1)} + \rho_2 \pi'^{\,(1,1)}\right) + \pi'^{\,(1,1)}\left(F_1^\mu b_\mu^{(1,-1)} + \rho_3 \pi^{(1,-1)}\right) =$$

$$= \left(\pi^{(1,-1)} - F_1^\mu b_\mu^{(1,-1)}(\rho_2-\rho_3)^{-1}\right)(\rho_2-\rho_3)\left(\pi'^{\,(1,1)} + (\rho_2-\rho_3)^{-1}F_2^\mu c_\mu^{(1,1)}\right)$$

$$+ F_1^\mu b_\mu^{(1,-1)}(\rho_2-\rho_3)^{-1}F_2^\mu c_\mu^{(1,1)}$$

$$S_\pi + S^{bc}_{gf} = \int d^n x \, \left(\pi^{(1,-1)}(\rho_2-\rho_3)\pi'^{\,(1,1)} + F_1^\mu b_\mu^{(1,-1)}(\rho_2-\rho_3)^{-1} F_2^\mu c_\mu^{(1,1)}\right)$$

$$\int d^n x \; sX_{TD} = \int d^n x \, \Big(-b_\mu^{(1,-1)} sF^\mu + \bar{c}^{(0,-2)} F_2^\mu \nabla_\mu \phi^{(0,2)} + \pi^{(1,-1)}(\rho_2-\rho_3)\pi'^{\,(1,1)} + \\ + F_1^\mu b_\mu^{(1,-1)}(\rho_2-\rho_3)^{-1} F_2^\mu c_\mu^{(1,1)} - \frac{1}{4\rho_1}(F_\mu + \bar{F}_{1\mu} c'^{\,(0,0)})^2\Big)$$

The knack for choosing F_μ

- ☆ Cancel nonmimimal operators for graviton fluctuations
- ☆ Weyl invariant so that the gauge fixings decouple

$$F_\mu = \nabla^\nu h_{\mu\nu} - \frac{1}{n}\nabla_\mu h$$

$$sF_\mu = \Box c^T_\mu + \nabla^\nu \nabla_\mu c^T_\nu = \Box c^T_\mu + R^\nu_\mu c^T_\nu$$

In terms of unconstrained fields

$$\begin{aligned} sF_\mu &= \left(g_{\mu\alpha}\Box + R_{\mu\alpha}\right)\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu - R_{\mu\nu}\right)c^{\nu(1,1)} = \\ &= \Box^2 c^{(1,1)}_\mu - \nabla_\mu \Box \nabla_\nu c^{\nu(1,1)} - 2R_{\mu\rho}\nabla^\rho\nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} - \\ &- 2\nabla_\sigma R_{\mu\rho}\nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho}R^{\rho\nu} c^{(1,1)}_\nu \end{aligned}$$

$$\begin{aligned} S_{bc} = \int d^n x\, b^{\mu\,(1,-1)} \Big(\Box^2 c^{(1,1)}_\mu - \nabla_\mu \Box \nabla_\nu c^{\nu(1,1)} - 2R_{\mu\rho}\nabla^\rho\nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} - \\ -2\nabla_\sigma R_{\mu\rho}\nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho}R^{\rho\nu} c^{(1,1)}_\nu\Big) \end{aligned}$$

Choose

$$\begin{aligned} F^\mu_1 b^{(1,-1)}_\mu &= \nabla^\alpha b^{(1,-1)}_\alpha \\ F^\mu_2 c^{(1,1)}_\mu &= \nabla^\mu c^{(1,1)}_\mu \\ (\rho_2 - \rho_3)^{-1} &= -\Box \end{aligned}$$

Then

$$F^\mu_1 b^{(1,-1)}_\mu (\rho_2 - \rho_3)^{-1} F^\mu_2 c^{(1,1)}_\mu = -\left(\nabla^\nu b^{(1,-1)}_\nu\right) \Box \nabla^\mu c^{(1,1)}_\mu = b^{(1,-1)}\nu \nabla^\nu \Box \nabla^\mu c^{(1,1)}_\mu$$

$$S_{bc} + S^{bc}_{gf} = \int d^n x \sqrt{|g|}\, b^{\mu\,(1,-1)} \left(\Box^2 c^{(1,1)}_{\mu} - 2R_{\mu\rho}\nabla^{\rho}\nabla_{\nu} c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} - \right.$$

$$\left. -2\nabla_{\sigma} R_{\mu\rho} \nabla^{\sigma} c^{\rho(1,1)} - R_{\mu\rho} R^{\rho\nu} c^{(1,1)}_{\nu} \right)$$

$$S_{\pi} = \int d^n x\ \pi^{(1,-1)} \Box^{-1} \pi^{\prime\ (1,1)}$$

$$S_{hc'} = -\int d^n x\ \frac{1}{4\rho_1} \left[\bar{F}^{\mu}_{1} c^{\prime(0,0)} \bar{F}_{1\mu} c^{\prime(0,0)} + 2F_{\mu} \bar{F}^{\mu}_{1} c^{\prime\ (0,0)} \right] =$$

$$= -\int d^n x\ \frac{1}{4\rho_1} \left[g^{\frac{2-n}{2n}} \nabla_{\mu} c^{\prime\ (0,0)} \nabla^{\mu} c^{\prime\ (0,0)} + 2g^{\frac{2-n}{4n}} F_{\mu} \nabla^{\mu} c^{\prime\ (0,0)} \right]$$

$$S_{\bar{c}\phi} = \int d^n x\ \bar{c}^{(0,-2)} \Box \phi^{(0,2)}$$

Summarizing

$$S^{TDiff}_{BRST} = \int d^n x \; b^\mu \Big(\Box^2 c^{(1,1)}_\mu - 2R_{\mu\rho}\nabla^\rho\nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} -$$

$$-2\nabla_\sigma R_{\mu\rho}\nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho}R^{\rho\nu} c^{(1,1)}_\nu \Big) + \bar{c}^{(0,-2)}\Box\phi^{(0,2)} + \pi^{(1,-1)}\Box^{-1}\pi'^{\,(1,1)} -$$

$$- \frac{1}{4\rho_1}\left(F_\mu F^\mu + g^{\frac{2-n}{2n}} \nabla_\mu c'^{\,(0,0)} \nabla^\mu c'^{\,(0,0)} + 2g^{\frac{2-n}{4n}} F_\mu \nabla^\mu c'^{\,(0,0)} \right) =$$

$$= S_{bc} + S^{bc}_{gf} + S_{\bar{c}\phi} + S_\pi + S_{hc'}$$

The Weyl sector

$$X_W = \nabla_\mu b^{(1,-1)} \nabla^\mu \left(f^{(0,0)} - \alpha\, g(h) \right)$$

$$S^{Weyl}_{BRST} = \int d^n x \, \left[\nabla_\mu f^{(0,0)} \nabla^\mu \left(f^{(0,0)} - \alpha\, g(h) \right) - \alpha \nabla_\mu b^{(1,-1)} \nabla^\mu \left(s g(h) \right) \right]$$

simplest choice $g(h) = h$

$$S_{Weyl} = \int d^n x \, \nabla_\mu f^{(0,0)} \nabla^\mu \left(f^{(0,0)} - \alpha\, h \right) - 2n\alpha \nabla_\mu b^{(1,-1)} \nabla^\mu c^{(1,1)} =$$
$$= \int d^n x \, \left(-f^{(0,0)} \Box f^{(0,0)} + \frac{\alpha}{2} f^{(0,0)} \Box h + \frac{\alpha}{2} h \Box f^{(0,0)} \right) + 2n\alpha\, b^{(1,-1)} \Box c^{(1,1)} = S_W + S_{hf}$$

Keeping the free parameter alpha until we put the result on shell yields a powerful check of our computations

One Loop Effective Action

$$W_\infty = W_\infty^{(UG)} + W_\infty^{(bc)} + W_\infty^{(\pi)} + W_\infty^{(\bar{c}\phi)} + W_\infty^{(W)}$$

$$S^{UG} = S_2 + S_{hc'} + S_{hf}$$

General one-loop structure

$$S = \int d^n x \; \Psi^A F_{AB} \Psi^B$$

For example

$$\Psi^A = \begin{pmatrix} b \\ c \end{pmatrix}$$

$$F_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times n\alpha\Box$$

Minimal operators $F_{AB} = \gamma_{AB}\Box^m + K_{AB}$

The computation of their determinant is a standard Schwinger-DeWitt calculation

Non-minimal operators need some care

$$\mathcal{L} = \frac{1}{4}h^{\mu\nu}\Box h_{\mu\nu} - \frac{1}{4n}h\Box h + \frac{1}{2}h^{\alpha\beta}h^{\mu}_{\beta}R_{\mu\alpha} + \frac{1}{2}h^{\mu\nu}h^{\alpha\beta}R_{\mu\alpha\nu\beta} - \frac{1}{n}hh^{\mu\nu}R_{\mu\nu} - \frac{1}{2n}h^{\mu\nu}h_{\mu\nu}R +$$
$$+ \left(-f\Box f + \frac{\alpha}{2}f\Box h + \frac{\alpha}{2}h\Box f\right) - \frac{1}{2}\left(\nabla_\mu c'^{\,(0,0)}\nabla^\mu c'^{\,(0,0)} + 2\left(\nabla_\nu h^\nu_\mu - \frac{1}{n}\nabla_\mu h\right)\nabla^\mu c'^{\,(0,0)}\right) +$$
$$+ \frac{1}{2n^2}h^2 R$$

$\rho_1 = \frac{1}{2}$

$$\Psi^A = \begin{pmatrix} h^{\mu\nu} \\ f \\ c' \end{pmatrix}$$

The nonminimal operator

$$F_{AB} = \gamma_{AB}\Box + J^{\mu\nu}_{AB}\nabla_\mu\nabla_\nu + M_{AB}$$

$$\gamma_{AB} = \begin{pmatrix} -\frac{1}{4}\left(\frac{1}{n}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma}\right) g_{\alpha\beta} & \frac{\alpha}{2} g_{\mu\nu} & -\frac{1}{2} g_{\mu\nu} \\ \frac{\alpha}{2} g_{\rho\sigma} & -1 & 0 \\ -\frac{1}{2} g_{\rho\sigma} & 0 & \frac{1}{2} \end{pmatrix}$$

$$M_{AB} = \begin{pmatrix} M_{hh} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J^{\alpha\beta}_{AB} = \begin{pmatrix} 0 & 0 & \frac{1}{4}\left(g^\alpha_\mu g^\beta_\nu + g^\alpha_\nu g^\beta_\mu\right) \\ 0 & 0 & 0 \\ \frac{1}{4}\left(g^\alpha_\mu g^\beta_\nu + g^\alpha_\nu g^\beta_\mu\right) & 0 & 0 \end{pmatrix}$$

$$M_{hh} = \left(\frac{1}{2}\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \frac{1}{n}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right) R_{\alpha\beta} - \frac{1}{2n}\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \frac{1}{n}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)\gamma_{\alpha\beta} R + \frac{1}{2} R_{(\mu\rho\nu\sigma)}$$

$$\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} = \frac{1}{4}\left(g_{\mu\rho}\delta^{(\alpha}_\nu\delta^{\beta)}_\sigma + g_{\mu\sigma}\delta^{(\alpha}_\nu\delta^{\beta)}_\rho + g_{\nu\rho}\delta^{(\alpha}_\mu\delta^{\beta)}_\sigma + g_{\nu\sigma}\delta^{(\alpha}_\mu\delta^{\beta)}_\rho\right)$$

$$\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\nu}\delta^{(\alpha}_\rho\delta^{\beta)}_\sigma + g_{\rho\sigma}\delta^{(\alpha}_\mu\delta^{\beta)}_\nu\right)$$

Barvinsky-Vilkovisky

$$D_{AB}(\nabla) = \gamma_{AB}\Box + J^{\alpha\beta}_{AB}\nabla_\alpha\nabla_\beta.$$

$$F_{AB}(\nabla|\lambda) = D_{AB}(\nabla|\lambda) + M_{AB}. \qquad 0 \le \lambda \le 1$$

$$F_{AB}(\nabla|\lambda) = \gamma_{AB}\Box + \lambda J^{\alpha\beta}_{AB}\nabla_\alpha\nabla_\beta + M_{AB} \qquad \hat{F}(\lambda)\hat{G}(\lambda) = \mathbb{I},$$

Schwinger

$$W(\lambda) = W(0) - \frac{1}{2}\int_0^\lambda d\lambda' Tr\left[\frac{d\hat{F}(\lambda)}{d\lambda'}\hat{G}(\lambda')\right]$$

$$\hat{D}(k) = D^A_B(k) = \gamma^{AC}D_{CB}(k) \qquad \hat{D}^{-1}(k) = \frac{\hat{K}(k)}{(k^2)^m}$$

$$\hat{D}(k)\hat{K}(k) = (k^2)^m\mathbb{I}. \qquad \hat{D}(\nabla)\hat{K}(\nabla) = \Box^m + \hat{K}_1(\nabla)$$

$$\hat{F}(\nabla)\hat{K}(\nabla) = \Box^m + \hat{M}(\nabla)$$

$$\hat{G} = -\hat{K}\frac{\mathbb{I}}{\Box^m}\sum_{p=0}^{4}\left(-M\frac{\mathbb{I}}{\Box^m}\right)^p + O\left(\mathfrak{m}^5\right)$$

$$\hat{G} = -\hat{K}\sum_{p=0}^{4}(-1)^p\,\hat{M}_p\frac{\mathbb{I}}{\Box^{m(p+1)}} + O\left(\mathfrak{m}^5\right)$$

$$\hat{M}_0 = \mathbb{I}$$
$$\hat{M}_{p+1} = \hat{M}\hat{M}_p + [\Box^m, \hat{M}_p]$$

In our case the computation simplifies somewhat

$$W(\lambda) = W(0) - \frac{1}{2}\int_0^\lambda d\lambda' Tr\left[\hat{J}^{\alpha\beta}\nabla_\alpha\nabla_\beta\left\{\hat{K}\left(-\frac{\mathbb{I}}{\Box^3} + \hat{M}\frac{\mathbb{I}}{\Box^6} - 3[\Box, \hat{M}]\frac{\mathbb{I}}{\Box^7} - \hat{M}^2\frac{\mathbb{I}}{\Box^9}\right)\right\}\right]$$

The minimal piece is a standard computation

$$W(0) = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x\left\{\frac{16}{15}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \left(\frac{2}{8\alpha^2-1} - \frac{46}{15}\right)R_{\mu\nu}R^{\mu\nu} + \left(\frac{13}{24} + \frac{1}{2-16\alpha^2}\right)R^2\right\} \tag{3.30}$$

The nonminimal part is reduced through the Barvinsky-Vilkovisky technique to the computation of functional traces

$$Tr\left(\mathcal{O}_{\nu_1\nu_2\ldots\nu_j}\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_p}\frac{\mathbb{I}}{\Box^n}\right)$$

$$-\frac{1}{2}\int_0^1 d\lambda' Tr\left[\hat{J}^{\alpha\beta}\nabla_\alpha\nabla_\beta\left\{\hat{K}\left(-\frac{\mathbb{I}}{\Box^3}+\hat{M}\frac{\mathbb{I}}{\Box^6}-3[\Box,\hat{M}]\frac{\mathbb{I}}{\Box^7}-\hat{M}^2\frac{\mathbb{I}}{\Box^9}\right)\right\}\right]=$$

$$=\frac{1}{16\pi^2}\frac{1}{n-4}\int d^nx\left\{\left(\frac{1}{6\alpha^2}+\frac{2}{1-8\alpha^2}\right)R_{\mu\nu}R^{\mu\nu}+\frac{1}{24}\left(\frac{12}{8\alpha^2-1}-\frac{1}{\alpha^2}-5\right)R^2\right\}$$

$$W_\infty^{UG}=\frac{1}{16\pi^2}\frac{1}{n-4}\int d^nx\left\{\frac{16}{15}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}+\left(\frac{1}{6\alpha^2}-\frac{46}{15}\right)R_{\mu\nu}R^{\mu\nu}+\left(\frac{1}{3}-\frac{1}{24\alpha^2}\right)R^2\right\}$$

Effective Action= sum of gauge piece plus BRST exact (ghosts)

$$W_\infty=W_\infty^{UG}+W_\infty^{bc}+W_\infty^{\pi}+W_\infty^{\bar{c}\phi}+W_\infty^{W}$$

$$W_\infty=\frac{1}{16\pi^2}\frac{1}{n-4}\int d^nx\left(\frac{119}{90}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}+\left(\frac{1}{6\alpha^2}-\frac{359}{90}\right)R_{\mu\nu}R^{\mu\nu}+\frac{1}{72}\left(22-\frac{3}{\alpha^2}\right)R^2\right)$$

Christensen & Duff $$W_\infty^{GR}\equiv\frac{1}{16\pi^2(n-4)}\int\sqrt{|g|}d^4x\left(\frac{53}{45}W_4-\frac{1142}{135}\Lambda^2\right)$$

On-shell divergences

$$R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} = 0$$

$$R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = E_4$$

$$R_{\mu\nu} R^{\mu\nu} = \frac{1}{4} R^2$$

$$R = \text{constant}$$

$$W_{\infty}^{\text{on-shell}} = \frac{1}{16\pi^2} \frac{1}{n-4} \int d^n x \left(\frac{119}{90} E_4 - \frac{83}{120} R^2 \right)$$

$$\frac{1}{M^{2\beta - n}} \int d^n x \; R^{\beta}$$

Conclusions

- Unimodular gravity solves one aspect of the cosmological constant problem, namely, why the vacuum energy does not generate a huge value for it.

- It is a Wilsonian solution to the problem of weighing vacuum energy
(No need to modify UV behavior)

In spite of the fact that UG is quite close to GR, which can be worked out in the gauge g=-1, this mechanism does not work in GR because there is no Weyl symmetry there

EM should be derived BEFORE gauge fixing!

$$\sqrt{|g|}\,(\text{Something}) = \sqrt{|g|}\;8\pi G\;\;(\text{Something Else})$$

Incidentally, the low energy limit of string theory could as well be UG instead of GR.

With on-shell amplitudes only it is not possible to discriminate the two.

Backup

Schwinger- DeWitt

$$W = -\frac{1}{2}\log\,\det\,D \qquad K(s,f,D) = Tr\left(f\;e^{-sD}\right)$$

$$\zeta(s,D) = \Gamma(s)^{-1}\int_0^\infty dt\;t^{s-1}\left\langle\psi_i\right|e^{-tD}\left|\psi_i\right\rangle = \int_0^\infty dt\;t^{s-1}K(t,D)$$

$$\log\left(det(D)\right) = -\int_0^\infty \frac{dt}{t}\left\langle\psi_i\right|e^{-tD}\left|\psi_i\right\rangle = -\int_0^\infty \frac{dt}{t}K(t,D)$$

$$W = -\frac{1}{2}\int_0^\infty \frac{dt}{t}K(t,D)$$

Regularized Effective Action

$$W_{reg} = -\frac{1}{2}\mu^{2s}\int_0^\infty \frac{dt}{t^{1-s}}K(t,D) = -\frac{1}{2}\mu^{2s}\Gamma(s)\zeta(s,D)$$

$$W_{reg} = -\frac{1}{2}\left(\frac{1}{s} - \gamma_E + \log\left(\mu^2\right)\right)\zeta(0,D) - \frac{1}{2}\zeta'(0,D)$$

$$W_{ren} = -\frac{1}{2}\zeta'(0,D) - \frac{1}{2}\log\left(\mu^2\right)\zeta(0,D)$$

Short time expansion

$$K(s,D) = \frac{1}{(4\pi s)^{n/2}}\sum_{i=0} s^{i/2} a_i(D)$$

Zeta Function Out of Heat Kernel

$$\zeta(0,D) = a_n(D)$$

Pole Part of the Effective Action

$$W_\infty = \frac{1}{(4\pi)^{n/2}}\frac{1}{n-4}a_n(D)$$

Laplace-type Operator

$$D = -\gamma_{AB}\Box + N^{\mu}_{AB}\nabla_{\mu} + M_{AB}$$

$$\mathcal{D} = \nabla + \omega, \qquad D = -\gamma_{AB}\mathcal{D}^2 - E_{AB}$$

$$\omega^{A}_{\mu\ B} = \frac{1}{2}\gamma^{AC}N_{\mu CB}$$

$$E^{A}_{B} = \gamma^{AC}(-M_{CB} - \omega_{\mu CF}\omega^{\mu\ F}_{B} - \nabla_{\mu}\omega^{\mu}_{CB})$$

$$a_4 = \frac{1}{360}\int d^n x\sqrt{|g|}\, Tr(60\Box E + 60RE + 180E^2 + 12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 30\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu}) \tag{A.15}$$

$$[\mathcal{D}_{\mu}, \mathcal{D}_{\nu}]V_A = \hat{\mathcal{R}}_{\mu\nu A}{}^{B}V_B$$

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu] h^{\rho\sigma} &= \hat{\mathcal{R}}^{\rho\sigma}{}_{\mu\nu\alpha\beta} h^{\alpha\beta} \\
\hat{\mathcal{R}}^{\rho\sigma}_{\mu\nu\alpha\beta} &= \frac{1}{2}\left(R^{\rho}_{\alpha\mu\nu} g^{\sigma}_{\beta} + R^{\sigma}_{\alpha\mu\nu} g^{\rho}_{\beta} + R^{\rho}_{\beta\mu\nu} g^{\sigma}_{\alpha} + R^{\sigma}_{\beta\mu\nu} g^{\rho}_{\alpha}\right)
\end{aligned}
$$

Starting Point: The d'Alembert Operator

$$D = -\Box$$

$$a_4(\Box) = \frac{1}{360} \int d^n x \sqrt{|g|} \; \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)$$

Quartic Operator

$$D = \gamma_{AB}\Box^2 + \Omega^{\mu\nu\alpha}_{AB}\nabla_\mu\nabla_\nu\nabla_\alpha + J^{\mu\nu}_{AB}\nabla_\mu\nabla_\nu + H^{\mu}_{AB}\nabla_\mu + P_{AB}$$

$$W_\infty = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x\sqrt{|g|}\,Tr\left(\frac{1}{90}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - \frac{1}{90}R_{\mu\nu}R^{\mu\nu} + \frac{1}{36}R^2\mathbb{I} - \hat{P} + \frac{1}{6}\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu} - \right.$$
$$\left. -\frac{1}{6}J^{(\mu\nu)}R_{\mu\nu} + \frac{1}{12}J^{\mu}_{\mu}R + \frac{1}{48}(J^{\mu}_{\mu})^2 + \frac{1}{24}J_{(\mu\nu)}J^{(\mu\nu)} - \frac{1}{2}J^{[\mu\nu]}\hat{\mathcal{R}}_{\mu\nu}\right) \qquad (A$$

The (b c) System

$$\int d^n x\, b^\mu \left(\Box^2 c_\mu^{(1,1)} - 2R_{\mu\rho}\nabla^\rho\nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho}c^{\rho(1,1)} - 2\nabla_\sigma R_{\mu\rho}\nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho}R^{\rho\nu}c_\nu^{(1,1)}\right)$$

$$J^{\mu\nu}_{\alpha\beta} = -2R^\mu_\alpha \delta^\nu_\beta$$

$$H^\mu_{\alpha\beta} = -2\nabla^\mu R_{\alpha\beta}$$

$$P_{\alpha\beta} = -\Box R_{\alpha\beta} - R_{\alpha\rho}R^\rho_\beta$$

$$[\nabla_\mu, \nabla_\nu]c^\alpha = \hat{\mathcal{R}}_{\mu\nu}{}^{\alpha\beta}c_\beta = R_{\mu\nu}{}^{\alpha\beta}c_\beta$$

$$W^{bc}_\infty = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x \left(\frac{11}{45}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - \frac{41}{45}R_{\mu\nu}R^{\mu\nu} - \frac{1}{18}R^2\right)$$

The (π ,$\pi^{'}$) System

$$S_\pi = \int d^n x \; \pi^{(1,-1)} \Box^{-1} \pi'^{\,(1,1)}$$

$$a_4(\Box) = -\frac{1}{360} \int d^n x \; \left(12 \Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)$$

The ($\bar{c}\phi$) System

$$\int d^n x\ \bar{c}^{(0,-2)} \Box \phi^{(0,2)}$$

$$W_{\infty}^{\bar{c}\phi} = \frac{1}{16\pi^2}\frac{1}{n-4}\frac{1}{180}\int d^n x\ \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)$$

The (w) System

$$2n\alpha \int d^n x \; b\Box c$$

$$W_\infty^W = -\frac{1}{16\pi^2}\frac{1}{n-4}\frac{1}{180}\int d^n x \; \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)$$

Functional Traces

$$Tr\left(\mathcal{O}_{\nu_1\nu_2\ldots\nu_j}\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_p}\frac{\mathbb{I}}{\Box^n}\right)$$

$$(\hat{F}(\nabla))^{-n} = \frac{1}{(n-1)!}\left[\left(\frac{d}{dm^2}\right)^{n-1} G(m^2)\right]_{m^2=0}$$

$$G(m^2) = \int_0^\infty \exp\left(-sm^2\right)\exp\left(-s\hat{F}(\nabla)\right)$$

$$\exp(-s\hat{F}(\nabla))\delta(x,x') = \frac{1}{(4\pi)^{n/2}}\frac{\mathcal{D}^{1/2}(x,x')}{s^{n/2}}\exp\left(-\frac{\sigma(x,x')}{2s}\right)\hat{\Omega}(s|x,x')$$

$$\hat{\Omega}(s|x,x') = \sum_{n=0}^{\infty} s^n \hat{a}_n(x,x')$$

$$\mathcal{D}(x,x') = \left| det\left(-\frac{\partial\sigma}{\partial x^{\mu}\partial x'^{\nu}}\right)\right|$$

$$\mathcal{D}(x,x') = g^{1/2}(x)g^{1/2}(x')\Delta(x,x')$$

$$\frac{\mathbb{I}}{\Box^n} = \frac{1}{(n-1)!}\int_0^{\infty} ds\ s^{n-1}\exp\left(-s\hat{\Box}\right)$$

$$\int_0^{\infty}\frac{ds}{s^{n/2+k}},\ \ \text{with}\ \ \ k=-1,0,1$$

$$\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta\frac{\mathbb{I}}{\Box^2} =$$

$$= \frac{\sqrt{g}}{8(n-4)\pi^2} \Big\{ \Big[\frac{1}{36}\left(R_{\mu\nu}R_{\alpha\beta}+R_{\mu\alpha}R_{\nu\beta}+R_{\mu\beta}R_{\nu\alpha}\right) + \frac{1}{180}\left(R^\lambda_\mu\left(11R_{\nu\alpha\beta\lambda}-R_{\beta\alpha\nu\lambda}\right) + R^\lambda_\nu\left(11R_{\mu\alpha\beta\lambda}-R_{\beta\alpha\mu\lambda}\right) +\right.$$

$$\left. +R^\lambda_\alpha\left(11R_{\mu\nu\beta\lambda}-R_{\beta\nu\mu\lambda}\right) + R^\lambda_\beta\left(11R_{\mu\nu\alpha\lambda}-R_{\alpha\nu\mu\lambda}\right)\right) + \frac{1}{90}\left(R_\mu{}^\lambda{}_\nu{}^\sigma\left(R_{\lambda\alpha\sigma\beta}+R_{\lambda\beta\sigma\alpha}\right) +\right.$$

$$\left. +R_\mu{}^\lambda{}_\alpha{}^\sigma\left(R_{\lambda\nu\sigma\beta}+R_{\lambda\beta\sigma\nu}\right) + R_\mu{}^\lambda{}_\beta{}^\sigma\left(R_{\lambda\nu\sigma\alpha}+R_{\lambda\alpha\sigma\nu}\right)\right) + \frac{1}{20}\left(\nabla_\mu\nabla_\nu R_{\alpha\beta}+\nabla_\mu\nabla_\alpha R_{\nu\beta}+\nabla_\mu\nabla_\beta R_{\nu\alpha}+\right.$$

$$\left. +\nabla_n\nabla_\alpha R_{\mu\beta}+\nabla_\nu\nabla_\beta R_{\mu\alpha}+\nabla_\alpha\nabla_\beta R_{\mu\nu}\right)\Big]\mathbb{I} + \frac{1}{12}\Big[R_{\mu\nu}\hat{\mathcal{R}}_{\alpha\beta}+R_{\mu\alpha}\hat{\mathcal{R}}_{\nu\beta}+R_{\mu\beta}\hat{\mathcal{R}}_{\nu\alpha}+R_{\nu\alpha}\hat{\mathcal{R}}_{\mu\beta}+$$

$$+R_{\nu\beta}\hat{\mathcal{R}}_{\mu\alpha}+R_{\alpha\beta}\hat{\mathcal{R}}_{\mu\nu}\Big] + \frac{1}{2}\Big[\nabla_\mu\nabla_\nu\hat{\mathcal{R}}_{\alpha\beta}+\nabla_\mu\nabla_\alpha\hat{\mathcal{R}}_{\nu,b}+\nabla_\nu\nabla_\alpha\hat{\mathcal{R}}_{\mu\beta}\Big] + \frac{1}{8}\Big[\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}_{\alpha\beta}+\hat{\mathcal{R}}_{\alpha\beta}\hat{\mathcal{R}}_{\mu\nu}+$$

$$+\hat{\mathcal{R}}_{\mu\alpha}\hat{\mathcal{R}}_{\nu\beta}+\hat{\mathcal{R}}_{\nu\beta}\hat{\mathcal{R}}_{\mu\alpha}+\hat{\mathcal{R}}_{\mu\beta}\hat{\mathcal{R}}_{\nu\alpha}+\hat{\mathcal{R}}_{\nu\alpha}\hat{\mathcal{R}}_{\mu\beta}\Big] - \frac{1}{12}\Big[\hat{\mathcal{R}}_{\mu\lambda}\left(R^\lambda_{\alpha\nu\beta}+R^\lambda_{\beta\nu\alpha}\right)+\hat{\mathcal{R}}_{\nu\lambda}\left(R^\lambda_{\alpha\mu\beta}+R^\lambda_{\beta\mu\alpha}\right)+$$

$$+\hat{\mathcal{R}}_{\alpha\lambda}\left(R^\lambda_{\nu\mu\beta}+R^\lambda_{\beta\mu\nu}\right)+\hat{\mathcal{R}}_{\beta\lambda}\left(R^\lambda_{\mu\nu\alpha}+R^\lambda_{\alpha\nu\mu}\right)\Big] - \frac{1}{2}\Big[-\frac{1}{9}\left(R_{\alpha\mu\beta\nu}+R_{\beta\mu\alpha\nu}\right)R\mathbb{I}+$$

$$+g_{\mu\nu}\Big[\Big(\frac{1}{36}R_{\alpha\beta}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\alpha\sigma\beta}+\frac{1}{90}R_{\rho\sigma\lambda\alpha}R^{\rho\sigma\lambda}{}_\beta-\frac{1}{45}R_{\alpha\lambda}R^\lambda_\beta+\frac{1}{60}\Box R_{\alpha\beta}+\frac{1}{20}\nabla_\alpha\nabla_\beta R\Big)\mathbb{I}+$$

$$+\frac{1}{12}\left(\hat{\mathcal{R}}_{\alpha\lambda}\hat{\mathcal{R}}^\lambda_\beta+\hat{\mathcal{R}}_{\beta\lambda}\hat{\mathcal{R}}^\lambda_\alpha\right)-\frac{1}{12}\left(\nabla_\alpha\nabla^\lambda\hat{\mathcal{R}}_{\lambda\beta}+\nabla_\beta\nabla^\lambda\hat{\mathcal{R}}_{\lambda\alpha}\right)+\frac{1}{12}R\hat{\mathcal{R}}_{\alpha\beta}\Big]+$$

$$+g_{\mu\alpha}\Big[\Big(\frac{1}{36}R_{\nu\beta}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\nu\sigma\beta}+\frac{1}{90}R_{\rho\sigma\lambda\nu}R^{\rho\sigma\lambda}{}_\beta-\frac{1}{45}R_{\nu\lambda}R^\lambda_\beta+\frac{1}{60}\Box R_{\nu\beta}+\frac{1}{20}\nabla_\nu\nabla_\beta R\Big)\mathbb{I}+$$

$$+\frac{1}{12}\left(\hat{\mathcal{R}}_{\nu\lambda}\hat{\mathcal{R}}^\lambda_\beta+\hat{\mathcal{R}}_{\beta\lambda}\hat{\mathcal{R}}^\lambda_\nu\right)-\frac{1}{12}\left(\nabla_\nu\nabla^\lambda\hat{\mathcal{R}}_{\lambda\beta}+\nabla_\beta\nabla^\lambda\hat{\mathcal{R}}_{\lambda\nu}\right)+\frac{1}{12}R\hat{\mathcal{R}}_{\nu\beta}\Big]+$$

$$+g_{\mu\beta}\Big[\Big(\frac{1}{36}R_{\nu\alpha}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\nu\sigma\alpha}+\frac{1}{90}R_{\rho\sigma\lambda\nu}R^{\rho\sigma\lambda}{}_\alpha-\frac{1}{45}R_{\nu\lambda}R^\lambda_\alpha+\frac{1}{60}\Box R_{\nu\alpha}+\frac{1}{20}\nabla_\nu\nabla_\alpha R\Big)\mathbb{I}+$$

$$+\frac{1}{12}\left(\hat{\mathcal{R}}_{\nu\lambda}\hat{\mathcal{R}}^\lambda_\alpha+\hat{\mathcal{R}}_{\alpha\lambda}\hat{\mathcal{R}}^\lambda_\nu\right)-\frac{1}{12}\left(\nabla_\nu\nabla^\lambda\hat{\mathcal{R}}_{\lambda\alpha}+\nabla_\alpha\nabla^\lambda\hat{\mathcal{R}}_{\lambda\nu}\right)+\frac{1}{12}R\hat{\mathcal{R}}_{\nu\alpha}\Big]+$$

$$+g_{\nu\alpha}\Big[\Big(\frac{1}{36}R_{\mu\beta}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\mu\sigma\beta}+\frac{1}{90}R_{\rho\sigma\lambda\mu}R^{\rho\sigma\lambda}{}_\beta-\frac{1}{45}R_{\mu\lambda}R^\lambda_\beta+\frac{1}{60}\Box R_{\mu\beta}+\frac{1}{20}\nabla_\mu\nabla_\beta R\Big)\mathbb{I}+$$

$$+\frac{1}{12}\left(\hat{\mathcal{R}}_{\mu\lambda}\hat{\mathcal{R}}^\lambda_\beta+\hat{\mathcal{R}}_{\beta\lambda}\hat{\mathcal{R}}^\lambda_\mu\right)-\frac{1}{12}\left(\nabla_\mu\nabla^\lambda\hat{\mathcal{R}}_{\lambda\beta}+\nabla_\beta\nabla^\lambda\hat{\mathcal{R}}_{\lambda\mu}\right)+\frac{1}{12}R\hat{\mathcal{R}}_{\mu\beta}\Big]+$$

$$+g_{\nu\beta}\Big[\Big(\frac{1}{36}R_{\mu\alpha}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\mu\sigma\alpha}+\frac{1}{90}R_{\rho\sigma\lambda\mu}R^{\rho\sigma\lambda}{}_\alpha-\frac{1}{45}R_{\mu\lambda}R^\lambda_\alpha+\frac{1}{60}\Box R_{\mu\alpha}+\frac{1}{20}\nabla_\mu\nabla_\alpha R\Big)\mathbb{I}+$$

$$+\frac{1}{12}\left(\hat{\mathcal{R}}_{\mu\lambda}\hat{\mathcal{R}}^\lambda_\alpha+\hat{\mathcal{R}}_{\alpha\lambda}\hat{\mathcal{R}}^\lambda_\mu\right)-\frac{1}{12}\left(\nabla_\mu\nabla^\lambda\hat{\mathcal{R}}_{\lambda\alpha}+\nabla_\alpha\nabla^\lambda\hat{\mathcal{R}}_{\lambda\mu}\right)+\frac{1}{12}R\hat{\mathcal{R}}_{\mu\alpha}\Big]+$$

$$+g_{\alpha\beta}\Big[\Big(\frac{1}{36}R_{\mu\nu}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\mu\sigma\nu}+\frac{1}{90}R_{\rho\sigma\lambda\mu}R^{\rho\sigma\lambda}{}_\nu-\frac{1}{45}R_{\mu\lambda}R^\lambda_\nu+\frac{1}{60}\Box R_{\mu\nu}+\frac{1}{20}\nabla_\mu\nabla_\nu R\Big)\mathbb{I}+$$

$$+\frac{1}{12}\left(\hat{\mathcal{R}}_{\mu\lambda}\hat{\mathcal{R}}^\lambda_\nu+\hat{\mathcal{R}}_{\nu\lambda}\hat{\mathcal{R}}^\lambda_\mu\right)-\frac{1}{12}\left(\nabla_\mu\nabla^\lambda\hat{\mathcal{R}}_{\lambda\nu}+\nabla_\nu\nabla^\lambda\hat{\mathcal{R}}_{\lambda\mu}\right)+\frac{1}{12}R\hat{\mathcal{R}}_{\mu\nu}\Big]\Big]+$$

$$+\frac{1}{4}\left(g_{\mu\nu}g_{\alpha\beta}+g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha}\right)\Big[\Big[\frac{1}{180}\left(R_{\lambda\sigma\rho\gamma}R^{\lambda\sigma\rho\gamma}-R_{\lambda g}R^{\lambda\sigma}\right)+\frac{1}{30}\Box R-\frac{1}{72}R^2\Big]\mathbb{I}+\frac{1}{12}\hat{\mathcal{R}}_{\lambda\sigma}\hat{\mathcal{R}}^{\lambda\sigma}\Big]\Big\}$$

$$\nabla_\mu \nabla_\nu \frac{\mathbb{I}}{\Box} = \frac{\sqrt{g}}{8(n-4)\pi^2} \frac{1}{2} \left\{ \left[-g_{\mu\nu} \left(\frac{1}{180} R_{\alpha\beta\lambda\sigma} R^{\alpha\beta\lambda\sigma} - \frac{1}{180} R_{\alpha\beta} R^{\alpha\beta} + \frac{1}{72} R^2 + \frac{1}{30} \Box R \right) \mathbb{I} + \frac{1}{45} R^{\alpha\beta} R_{\alpha\mu\beta\nu} + \right.\right.$$

$$+ \frac{1}{45} R_{\alpha\beta\lambda\mu} R_\nu^{\alpha\beta\lambda} - \frac{2}{45} R_{\mu\alpha} R_\nu^\alpha + \frac{1}{18} R R_{\mu\nu} + \frac{1}{30} \Box R_{\mu\nu} + \left. \frac{1}{10} \nabla_\mu \nabla_\nu R \right] - \frac{1}{12} g_{\mu\nu} \hat{\mathcal{R}}_{\alpha\beta} \hat{\mathcal{R}}^{\alpha\beta} + \frac{1}{6} R \hat{\mathcal{R}}_{\mu\nu} +$$

$$\left. + \frac{1}{6} \hat{\mathcal{R}}_{\mu\alpha} \hat{\mathcal{R}}_\nu^\alpha + \frac{1}{6} \hat{\mathcal{R}}_{\nu\alpha} \hat{\mathcal{R}}_\mu^\alpha - \frac{1}{6} \nabla_\mu \nabla^\alpha \hat{\mathcal{R}}_{\alpha\nu} - \frac{1}{6} \nabla_\nu \nabla^\alpha \hat{\mathcal{R}}_{\alpha\mu} \right\}$$

$$p = 2n - 1$$

$$\nabla_\mu \frac{\mathbb{I}}{\Box} = \frac{\sqrt{g}}{8(n-4)\pi^2} \left(\frac{1}{12} \nabla_\mu R \mathbb{I} - \frac{1}{6} \nabla^\nu \hat{\mathcal{R}}_{\nu\mu} \right)$$

$$p = 2n - 2$$

$$\frac{\mathbb{I}}{\Box} = \frac{\sqrt{g}}{8(n-4)\pi^2}\frac{1}{6}R\mathbb{I}$$

$$\nabla_\mu\nabla_\nu\frac{\mathbb{I}}{\Box^2} = -\frac{\sqrt{g}}{8(n-4)\pi^2}\left[\frac{1}{6}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\mu\nu}\right]$$

$$\begin{aligned}\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu\frac{\mathbb{I}}{\Box^3} = -\frac{\sqrt{g}}{8(n-4)\pi^2}\frac{1}{4}\Big\{ & -\frac{2}{6}R_{\mu\beta\nu\alpha} - \frac{2}{6}R_{\nu\beta\mu\alpha}\mathbb{I} + g_{\mu\nu}\left(\frac{1}{6}R_{\alpha\beta}\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\alpha\beta}\right) + g_{\beta\nu}\left(\frac{1}{6}R_{\alpha\mu}\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\alpha\mu}\right) + \\ & + g_{\alpha\nu}\left(\frac{1}{6}R_{\mu\beta}\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\mu\beta}\right) + g_{\beta\mu}\left(\frac{1}{6}R_{\alpha\nu}\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\alpha\nu}\right) + g_{\alpha\mu}\left(\frac{1}{6}R_{\beta\nu}\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\beta\nu}\right) + \\ & + g_{\alpha\beta}\left(\frac{1}{6}R_{\mu\nu}\mathbb{I} + \frac{1}{2}\hat{\mathcal{R}}_{\mu\nu}\right) - \frac{1}{12}g^{(2)}_{\mu\nu\alpha\beta}\mathbb{I}\Big\}\end{aligned}$$

$$\begin{aligned}\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu\nabla_\sigma\nabla_\lambda\frac{\mathbb{I}}{\Box^3} = -\frac{\sqrt{g}}{8(n-4)\pi^2}\frac{1}{6}\Big\{ & g^{(2)}_{\mu\nu\alpha\beta}\hat{B}_{\sigma\lambda} + g^{(2)}_{\mu\nu\alpha\sigma}\hat{B}_{\beta\lambda} + g^{(2)}_{\mu\nu\beta\sigma}\hat{B}_{\alpha\lambda} + g^{(2)}_{\mu\alpha\beta\sigma}\hat{B}_{\nu\lambda} + g^{(2)}_{\mu\nu\alpha\lambda}\hat{B}_{\beta\sigma} + \\ & + g^{(2)}_{\mu\nu\beta\lambda}\hat{B}_{\alpha\sigma} + g^{(2)}_{\mu\alpha\beta\lambda}\hat{B}_{\nu\sigma} + g^{(2)}_{\nu\alpha\beta\lambda}\hat{B}_{\mu\sigma} + g^{(2)}_{\mu\nu\sigma\lambda}\hat{B}_{\alpha\beta} + g^{(2)}_{\mu\alpha\sigma\lambda}\hat{B}_{\nu\beta} + g^{(2)}_{\nu\alpha\sigma\lambda}\hat{B}_{\mu\beta} + \\ & + g^{(2)}_{\mu\beta\sigma\lambda}\hat{B}_{\nu\alpha} + g^{(2)}_{\nu\beta\sigma\lambda}\hat{B}_{\mu\alpha} + g^{(2)}_{\alpha\beta\sigma\lambda}\hat{B}_{\mu\nu} - \frac{1}{12}\Big[g_{\sigma\lambda}\left(R_{\beta\nu\alpha\mu} + R_{\alpha\nu\beta\mu}\right) + \\ & + g_{\beta\lambda}\left(R_{\sigma\nu\alpha\mu} + R_{\alpha\nu\sigma\mu}\right) + g_{\alpha\lambda}\left(R_{\sigma\nu\beta\mu} + R_{\beta\nu\sigma\mu}\right) + g_{\nu\lambda}\left(R_{\sigma\alpha\beta\mu} + R_{\beta\alpha\sigma\mu}\right) + \\ & + g_{\mu\lambda}\left(R_{\sigma\alpha\beta\nu} + R_{\beta\alpha\sigma\nu}\right) + g_{\beta\sigma}\left(R_{\lambda\nu\alpha\mu} + R_{\alpha\nu\lambda\mu}\right) + g_{\alpha\sigma}\left(R_{\lambda\nu\beta\mu} + R_{\beta\nu\lambda\mu}\right) + \\ & + g_{\mu\sigma}\left(R_{\lambda\alpha\beta\nu} + R_{\beta\alpha\lambda\nu}\right) + g_{\alpha\beta}\left(R_{\lambda\nu\sigma\mu} + R_{\sigma\nu\lambda\mu}\right) + g_{\nu\beta}\left(R_{\lambda\alpha\sigma\mu} + R_{\sigma\alpha\lambda\mu}\right) + \\ & + g_{\mu\beta}\left(R_{\lambda\alpha\sigma\nu} + R_{\sigma\alpha\lambda\nu}\right) + g_{\nu\alpha}\left(R_{\lambda\beta\sigma\mu} + R_{\sigma\beta\lambda\mu}\right) + g_{\mu\alpha}\left(R_{\lambda\beta\sigma\nu} + R_{\sigma\beta\lambda\nu}\right) + \\ & + g_{\mu\nu}\left(R_{\lambda\beta\sigma\alpha} + R_{\sigma\beta\lambda\alpha}\right) + \frac{1}{8}g^{(3)}_{\mu\nu\alpha\beta\sigma\lambda}R\Big]\mathbb{I}\Big\}\end{aligned}$$

Symmetrized outer products of metric tensors

$$g^{(0)} = 1$$
$$g^{(1)}_{\mu\nu} = g_{\mu\nu}$$
$$g^{(2)}_{\mu\nu\alpha\beta} = g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} + g_{\mu\nu}g_{\alpha\beta}$$
$$g^{(3)}_{\mu\nu\alpha\beta\sigma\lambda} = g_{\mu\nu}g^{(2)}_{\alpha\beta\sigma\lambda} + g_{\mu\alpha}g^{(2)}_{\nu\beta\sigma\lambda} + g_{\mu\beta}g^{(2)}_{\nu\alpha\sigma\lambda} + g_{\mu\sigma}g^{(2)}_{\nu\alpha\beta\lambda} + g_{\mu\lambda}g^{(2)}_{\nu\alpha\beta\sigma}$$
$$g^{(n+1)}_{\mu_1\ldots\mu_{2n+2}} = \sum_{i=2}^{2n+2} g_{\mu_1\mu_i}g^{(n)}_{\mu_2\ldots\mu_{i-1}\mu_{i+1}\mu_{2n+2}}$$
$$\hat{B}_{\alpha\beta} = \frac{1}{24}R_{\alpha\beta}\mathbb{I} + \frac{1}{8}\hat{\mathcal{R}}_{\alpha\beta}$$

p=2n-4

$$\nabla_{\mu_1}.....\nabla_{\mu_{2n-4}}\frac{\mathbb{I}}{\Box^n} = -\frac{\sqrt{g}}{8(n-4)\pi^2}\frac{g^{(n-2)}_{\mu_1....\mu_{2n-4}}}{2^{n-2}(n-1)!}$$