Conformal blocks, entanglement entropy and heavy states

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Higher-point conformal blocks & entanglement entropy of heavy states

with Pinaki Banerjee (IMSc) and Ritam Sinha (TIFR)

Why conformal blocks?

Much of the power and appeal of the AdS/CFT relies on universal features of CFTs which find natural analogues in gravity.
(eg. Cardy's formula, entanglement entropy)

[Cardy; Cardy-Calabrese; Ryu-Takayanagi;...]

Finding correlation functions by decomposition into conformal blocks is a powerful and minimalistic approach to extract information to study CFTs.

Conformal bootstrap

anomalous dimensions of operators, bounds on central charges, ...

[Rattazi-Rychkov-Tonni-Vichi; ElShowk-Paulos-Poland-Rychkov-SimmonsDuffin-Vichi;...]

AGT correspondence

Liouville and Toda conformal blocks are related to $\mathcal{N}=2$ 4d SCFTs

[Alday-Gaiotto-Tachikawa; Wyllard; Nekrasov;...]

Holography

bulk locality, graviton scattering, ...

[Heemskerk-Penedones-Polchinski-Sully; ElShowk-Papadodimas;
Fitzpartick-Kaplan-Walters; Jackson-McGough-Verlinde;...]

What are conformal blocks?

Consider a p-point correlation function

$$\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\cdots\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle$$

This can be rewritten as a sum of conformal blocks upon inserting $(p–3)$ resolutions of the identity.

$$\sum_{\alpha,\beta,\ldots,\zeta}\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle\langle\alpha|\mathcal{O}_3(z_3)|\beta\rangle\cdots\langle\zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle$$

[Ferrara-Grillo-Gatto; Zamolodchikov; Dolan-Osborne;…]

The sum is over all operators of the theory – complete set of states.

A typical term in the above sum is referred to as the conformal block

$$\mathcal{F}_{(p)}(z_i,h_i,\tilde{h}_j)=\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle\langle\alpha|\mathcal{O}_3(z_3)|\beta\rangle\cdots\langle\zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle$$

These are the building blocks of CFT correlators.

Conformal blocks and holography

Recently an intriguing relation has been found between conformal blocks and geodesics in AdS.

[Fitzpatrick-Kaplan-Walters; Asplund-Bernamonti-Galle-Hartman; Hijano-Kraus-Snively;…]

An important object, in this context, is the correlator between two heavy and two light operators.

The corresponding conformal block can be reproduced from the bulk from worldline configurations in a conical defect background.

This result is known for 4- and 5-point blocks.

[Fitzpatrick-Kaplan-Walters; Alkalaev-Belavin]

We generalise these results to blocks with an arbitrary number of light operator insertions, both from CFT and holography.

This result can be used to calculate entanglement entropy and mutual information of heavy states.

Heavy-light conformal blocks

Consider a 2d CFT at large central charge.

We shall focus on correlators with an arbitrary number of light operators and two heavy operators.

$$\langle \mathcal{O}_H(z_1, \bar{z}_1)\mathcal{O}_L(z_2, \bar{z}_2)\mathcal{O}_L(z_3, \bar{z}_3) \cdots \mathcal{O}_L(z_{m+1}, \bar{z}_{m+1})\mathcal{O}_H(z_{m+2}, \bar{z}_{m+2})\rangle$$

The conformal dimension of these operators are

$$\epsilon_H = \frac{6h_H}{c} = \mathrm{O}(1)$$
$$\epsilon_L = \frac{6h_L}{c} \ll 1$$

Example of light operators : twist operators to implement the replica trick while calculating entanglement entropy

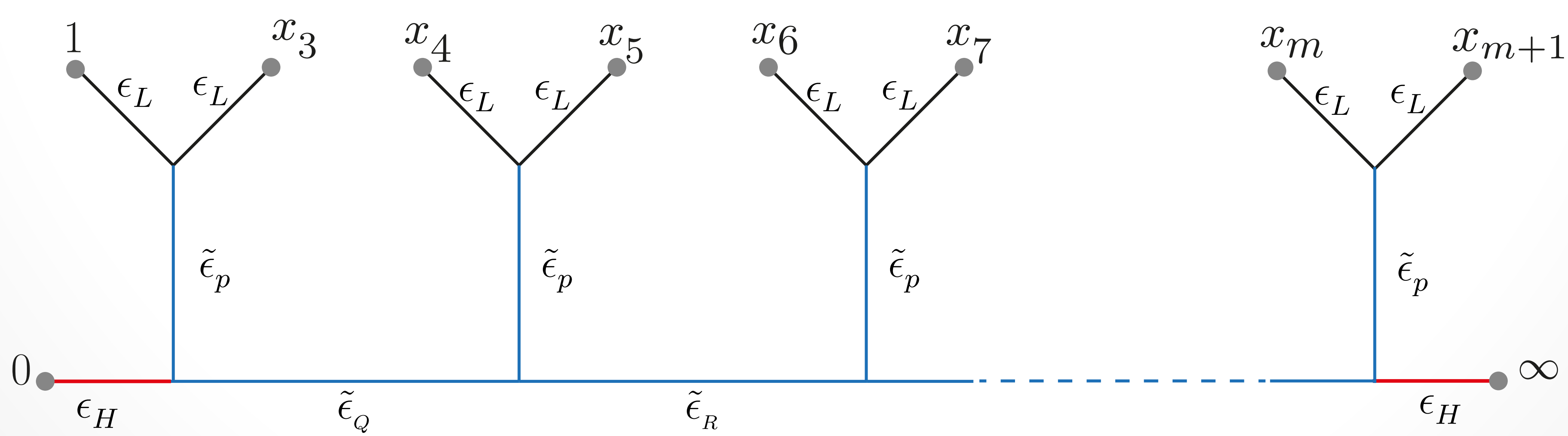
Heavy-light conformal blocks

Consider a 2d CFT at large central charge.

We shall focus on correlators with an arbitrary number of light operators and two heavy operators.

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(x_3,\bar{x}_3)\cdots\mathcal{O}_L(x_{m+1},\bar{z}_{x+1})\mathcal{O}_H(0)\rangle$$

We focus on OPE channels in which the light operators fuse in pairs.



The monodromy method

At large central charge, the conformal blocks are expected to exponentiate

$$\mathcal{F}(z_i, h_i, \tilde{h}_i) = \exp\left[-\frac{c}{6} f(z_i, \epsilon_i, \tilde{\epsilon}_i)\right]$$

[Zamolodchikov^2]

One can insert a field which obeys the null-state condition at level 2 inside the conformal block.

$$\Psi(z, z_i, h_i, \tilde{h}_j) = \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle\langle\alpha|\hat{\psi}\mathcal{O}_3(z_3)|\beta\rangle \cdots \langle\zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle$$

Insertion of this multiples the conformal block by an overall wavefunction

$$\Psi(z, z_i) = \psi(z, z_i)\mathcal{F}(z_i, h_i, \tilde{h}_i)$$

[Fitzpatrick-Kaplan-Walters]

The null-state condition on $\hat{\psi}$ inserted within the conformal block can then be translated into an ODE.

The monodromy method

The ODE is

$$\frac{d^2\psi(z)}{dz^2} + T(z)\psi(z) = 0$$

where, $$T(z) = \sum_{i=1}^{p} \left[\frac{\epsilon_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} \right] .$$

$$\epsilon_i = 6h_i/c$$

$$c_i = -\frac{\partial f(z_i)}{\partial z_i}$$ ← Accessory parameters

Integrability condition : $$\frac{\partial c_i}{\partial z_j} = \frac{\partial c_j}{\partial z_i}$$

The monodromy method

The ODE is

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$$\epsilon_i = 6h_i/c$$

$$c_i = -\frac{\partial f(z_i)}{\partial z_i}$$ ← Accessory parameters

We need to determine these accessory parameters.

This can be done by studying monodromy properties of $\psi(z, z_i)$, around contours containing the operator insertions.

The monodromy method

The ODE is

$$\frac{d^2\psi(z)}{dz^2} + T(z)\psi(z) = 0$$

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$$\epsilon_i = 6h_i/c$$

$$c_i = -\frac{\partial f(z_i)}{\partial z_i}$$

← Accessory parameters

Monodromy around a contour γ_k = info about the resultant operator which arises upon fusing the operators within γ_k

$$\widetilde{\mathbb{M}}(\gamma_k) = -\begin{pmatrix} e^{+\pi i\Lambda} & 0 \\ 0 & e^{-\pi i\Lambda} \end{pmatrix} , \qquad \Lambda = \sqrt{1-4\tilde{\epsilon}_p}$$

The monodromy method

The ODE is

$$\frac{d^2\psi(z)}{dz^2} + T(z)\psi(z) = 0$$

where, $$T(z) = \sum_{i=1}^{p} \left[\frac{\epsilon_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} \right] \quad .$$

$$\epsilon_i = 6h_i/c$$

$$c_i = -\frac{\partial f(z_i)}{\partial z_i}$$

← Accessory parameters

Perturbation theory in
$\epsilon_L = 6h_L/c$
(heavy-light limit)

$$\psi(z) = \psi^{(0)}(z) + \psi^{(1)}(z) + \psi^{(2)}(z) + \cdots \,,$$

$$T(z) = T^{(0)}(z) + T^{(1)}(z) + T^{(2)}(z) + \cdots \,,$$

$$c_i(z) = c_i^{(0)}(z) + c_i^{(1)}(z) + c_i^{(2)}(z) + \cdots \,,$$

Choice of contours / OPE channels

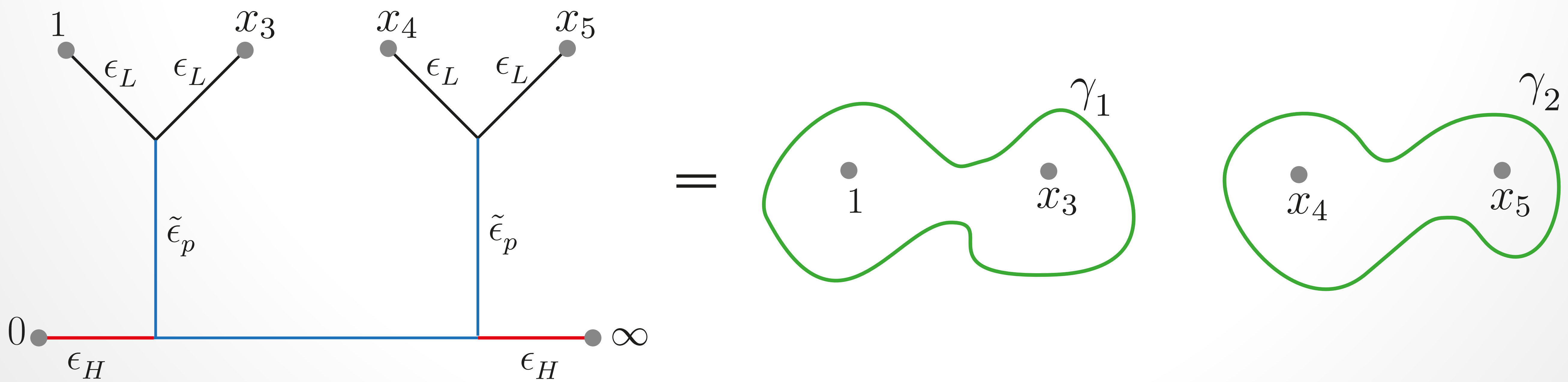
Choice of monodromy contour ⟷ Choice of OPE channel

We choose the contours such that
each of them contains a pair of light operators within.

This is equivalent to looking at
the OPE channel in which light operators fuse in pairs.

[Hartman; Faulkner]

This choice is geared towards entanglement entropy calculations.



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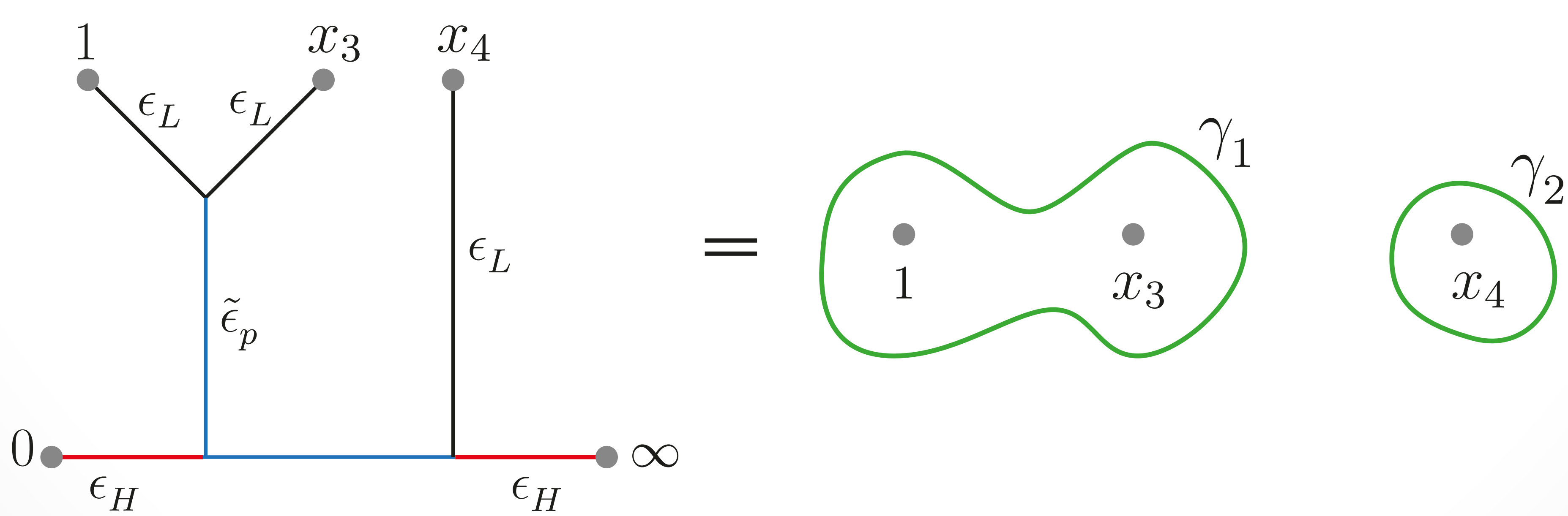


Seed solutions

[Alkalaev-Belavin]

Idea : Use the accessory parameters of the lower point blocks as zeroth order solutions for the higher-point ones.

We work to the leading order in the light-parameter $\epsilon_L = 6h_L/c$.

Solutions to the ODE to the linear order in ϵ_L can be found by the method of variation of parameters.

Accessory parameters

The monodromy conditions for all the contours form a coupled system of equations for the accessory parameters.

Performing the exercise for 5- and 6-point blocks provides sufficient intuition to guess the solutions.

For light operators located at x_p and x_q living with a contour, the corresponding accessory parameters are

$$c_p = \frac{-\epsilon_L(x_q^\alpha(\alpha-1)+x_p^\alpha(\alpha+1))+(x_p x_q)^{\alpha/2}\alpha\tilde{\epsilon}_a}{x_p(x_p^\alpha - x_q^\alpha)}$$

$$c_q = \frac{-\epsilon_L(x_p^\alpha(\alpha-1)+x_q^\alpha(\alpha+1))+(x_q x_p)^{\alpha/2}\alpha\tilde{\epsilon}_a}{x_q(x_q^\alpha - x_p^\alpha)}$$

These obey the integrability conditions.

Factorisation of higher-point blocks

The accessory parameters can now be used to obtain the conformal block

$$c_i = -\frac{\partial f_{(p)}(z_i,\epsilon_i,\tilde{\epsilon}_j)}{\partial z_i} \qquad \mathcal{F}_{(p)}(z_i,h_i,\tilde{h}_j) = \exp\left[-\frac{c}{6} f_{(p)}(z_i,\epsilon_i,\tilde{\epsilon}_j)\right]$$

Even-point conformal blocks

The $(m+2)$-point block factorises

into a product of $m/2$ 4-point conformal blocks

$$\mathcal{F}_{(m+2)}(\{x_i\};\epsilon_L,\epsilon_H;\tilde{\epsilon}_a) = \prod_{\boldsymbol{\Omega}_i \mapsto \{(p,q)\}} \exp\left[-\frac{c}{6} f_{(4)}(x_p,x_q;\epsilon_L,\epsilon_H;\tilde{\epsilon}_a)\right]$$

$$= \prod_{\boldsymbol{\Omega}_i \mapsto \{(p,q)\}} \mathcal{F}_{(4)}(x_p,x_q;\epsilon_L,\epsilon_H;\tilde{\epsilon}_a)$$

OPE channel (pairings of the light operators)

Factorisation of higher-point blocks

The accessory parameters can now be used to obtain the conformal block

$$c_i = -\frac{\partial f_{(p)}(z_i, \epsilon_i, \tilde{\epsilon}_j)}{\partial z_i} \qquad \mathcal{F}_{(p)}(z_i, h_i, \tilde{h}_j) = \exp\left[-\frac{c}{6} f_{(p)}(z_i, \epsilon_i, \tilde{\epsilon}_j)\right]$$

Odd-point conformal blocks

The $(m+2)$-point block factorises

into a product of $(m-1)/2$ 4-point conformal blocks and a 3-point function

$$\mathcal{F}_{(m+2)}(\{x_i\}; \epsilon_L, \epsilon_H; \tilde{\epsilon}_a) = (x_s)^{-\epsilon_L} \prod_{\mathbf{\Omega}_i^A \mapsto \{(p,q)\}} \exp\left[-\frac{c}{6} f_{(4)}(x_p, x_q; \epsilon_L, \epsilon_H; \tilde{\epsilon}_a)\right]$$

$$= (x_s)^{-\epsilon_L} \prod_{\mathbf{\Omega}_i^A \mapsto \{(p,q)\}} \mathcal{F}_{(4)}(x_p, x_q; \epsilon_L, \epsilon_H; \tilde{\epsilon}_a)$$

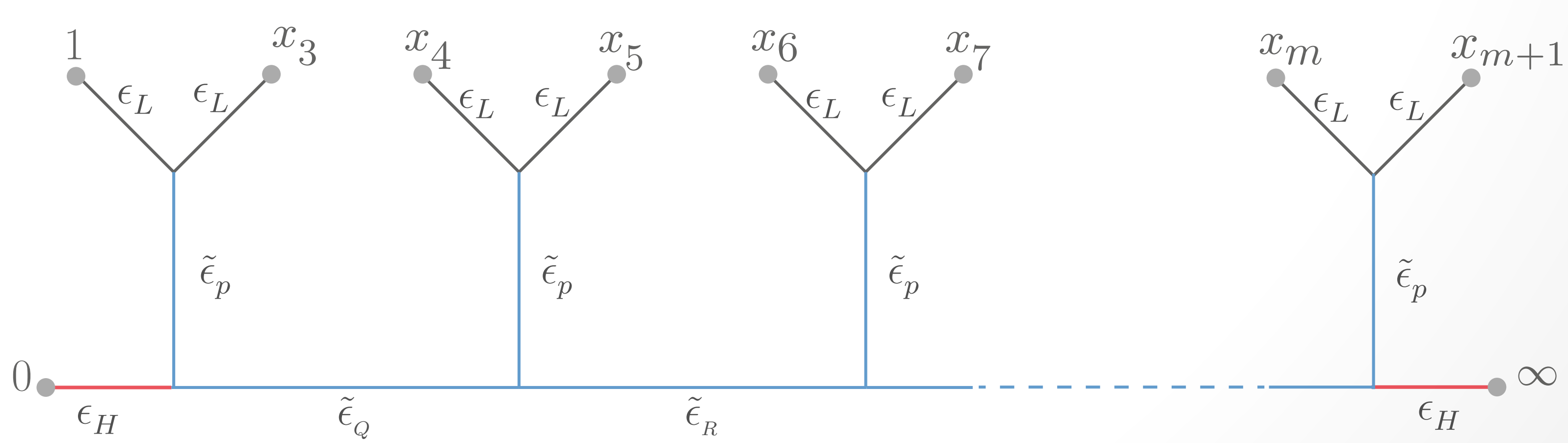
where, $f_{(4)}(x_i, x_j; \epsilon_L, \epsilon_H; \epsilon_p) = \epsilon_L \left((1-\alpha)\log x_i x_j + 2\log\frac{x_i^\alpha - x_j^\alpha}{\alpha}\right) + 2\tilde{\epsilon}_p \log\left[4\alpha \frac{x_j^{\alpha/2} + x_i^{\alpha/2}}{x_j^{\alpha/2} - x_i^{\alpha/2}}\right]$

Caveats

This factorization is true only …

- for this specific choice of OPE channels
- at large central charge
- in the heavy-light limit
- $\tilde{\epsilon}_p \ll \epsilon_L$

$$\mathcal{F}_{(m+2)}(\{x_i\};\epsilon_L,\epsilon_H;\tilde{\epsilon}_a) = \prod_{\Omega_i \mapsto \{(p,q)\}} \mathcal{F}_{(4)}(x_p,x_q;\epsilon_L,\epsilon_H;\tilde{\epsilon}_a)$$



Entanglement entropy of heavy states

Entanglement entropy is defined as

$$S_{\mathcal{A}} \;=\; -\mathrm{tr}_{\mathcal{A}}\,\rho_{\mathcal{A}}\log\rho_{\mathcal{A}}$$

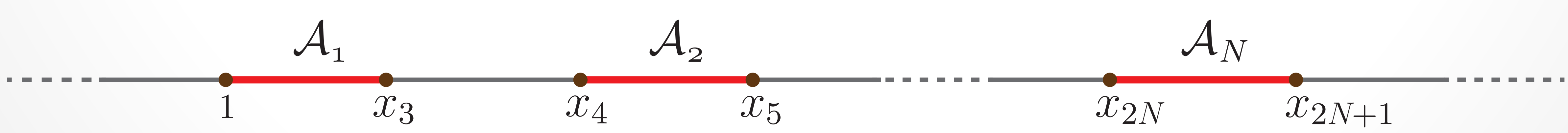
It is often calculated using the replica trick via the Rényi entropy

$$S_{\mathcal{A}}^{(n)} = \frac{1}{1-n}\log\,\mathrm{tr}_{\mathcal{A}}\,(\rho_{\mathcal{A}})^n$$

Density matrix

$$\rho = |\psi\rangle\langle\psi| \qquad |\psi\rangle = \mathcal{O}_H(0)|0\rangle \quad \langle\psi| = \lim_{z,\bar{z}\to\infty}\langle 0|\mathcal{O}_H(\infty)$$

State-operator correspondence



Entanglement entropy of heavy states

Replica trick : $\mathrm{tr}_{\mathcal{A}}\,(\rho_{\mathcal{A}})^n$ from partition function of n-sheeted Riemann surface glued along the cuts.

This is given by the correlation function of twist and anti-twist operators.

$$\mathrm{tr}_{\mathcal{A}}\,(\rho_{\mathcal{A}})^n = \langle\Psi|\sigma_n(1)\bar{\sigma}_n(x_3)\sigma_n(x_4)\bar{\sigma}_n(x_5)\sigma_n(x_6)\bar{\sigma}_n(x_7)\ldots\sigma_n(x_{2N-2})\bar{\sigma}_n(x_{2N-1})|\Psi\rangle$$

Furthermore, these are sandwiched between the heavy states.

[Asplund-Bernamonti-Galle-Hartman]

Conformal dimensions of the (anti-)twist operators are

$$h_{\sigma_n} = h_{\bar{\sigma}_n} = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

This is therefore a correlation function of 2 heavy operators and an even number of light operators.

Entanglement entropy of heavy states

Twist operators fuse into the identity.
We therefore need to consider contribution from the identity block.

Entanglement entropy of *N* disjoint intervals in a heavy exited state is

$$S_{\mathcal{A}} = \lim_{n\to 1} S_{\mathcal{A}}^{(n)} = \frac{c}{3} \min_i \left\{ \sum_{\boldsymbol{\Omega}_i \mapsto \{(p,q)\}} \log \frac{(x_p^\alpha - x_q^\alpha)}{\alpha (x_p x_q)^{\frac{\alpha-1}{2}}} \right\}$$

$$\alpha = \sqrt{1 - 24 h_H / c}$$

This generalises …

- the vacuum result for multiple intervals to excited states
- the excited state result of a single interval to multiple intervals

Conformal blocks from holography

The heavy excited state is dual to the conical defect geometry

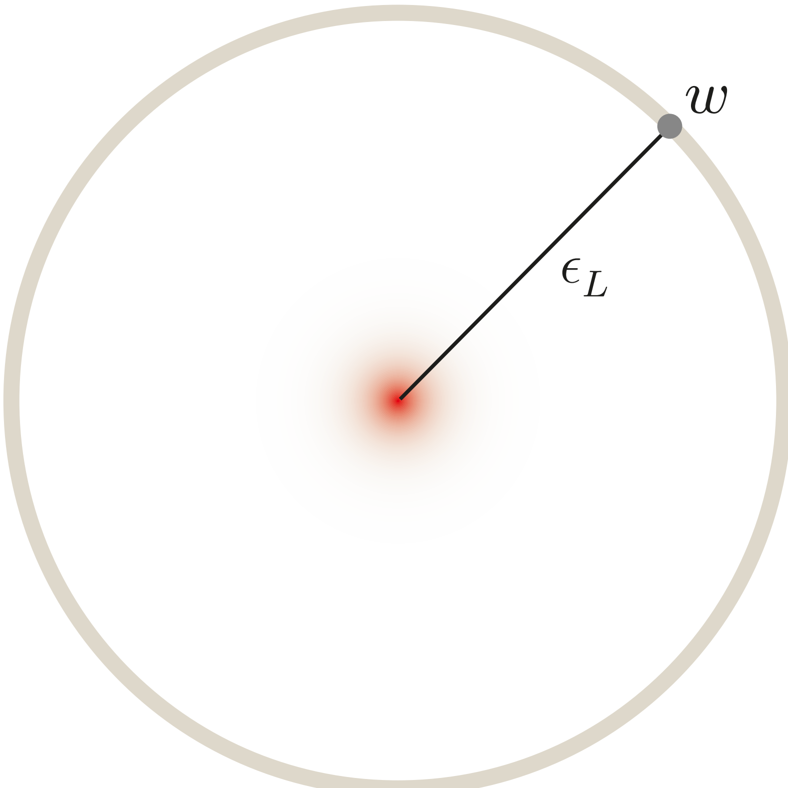
$$ds^2 = \frac{\alpha^2}{\cos^2\rho}\left(-dt^2 + \frac{1}{\alpha^2}d\rho^2 + \sin^2\rho\, d\phi^2\right), \quad \text{with } \alpha = \sqrt{1 - 24h_H/c}.$$

The light operators are dual to bulk scalars of masses of $\mathrm{O}(c)$ and can be approximated by worldlines.

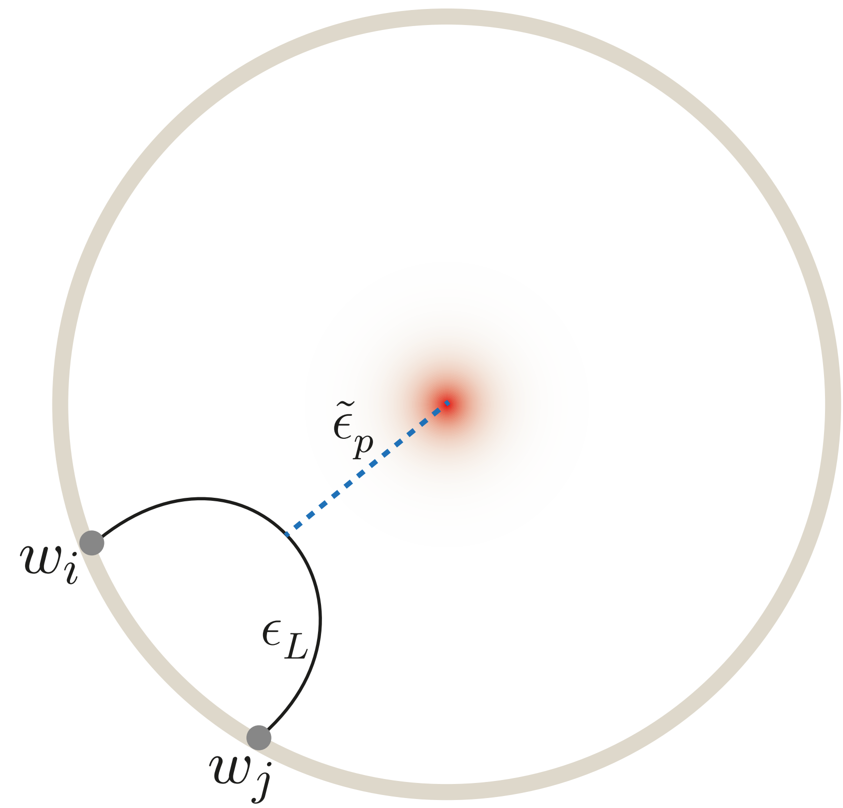
Conformal blocks from holography

The conformal blocks can be reproduced by considering lengths of suitable worldline configurations in the bulk.

[Hijano-Kraus-Snively-Perlmutter]



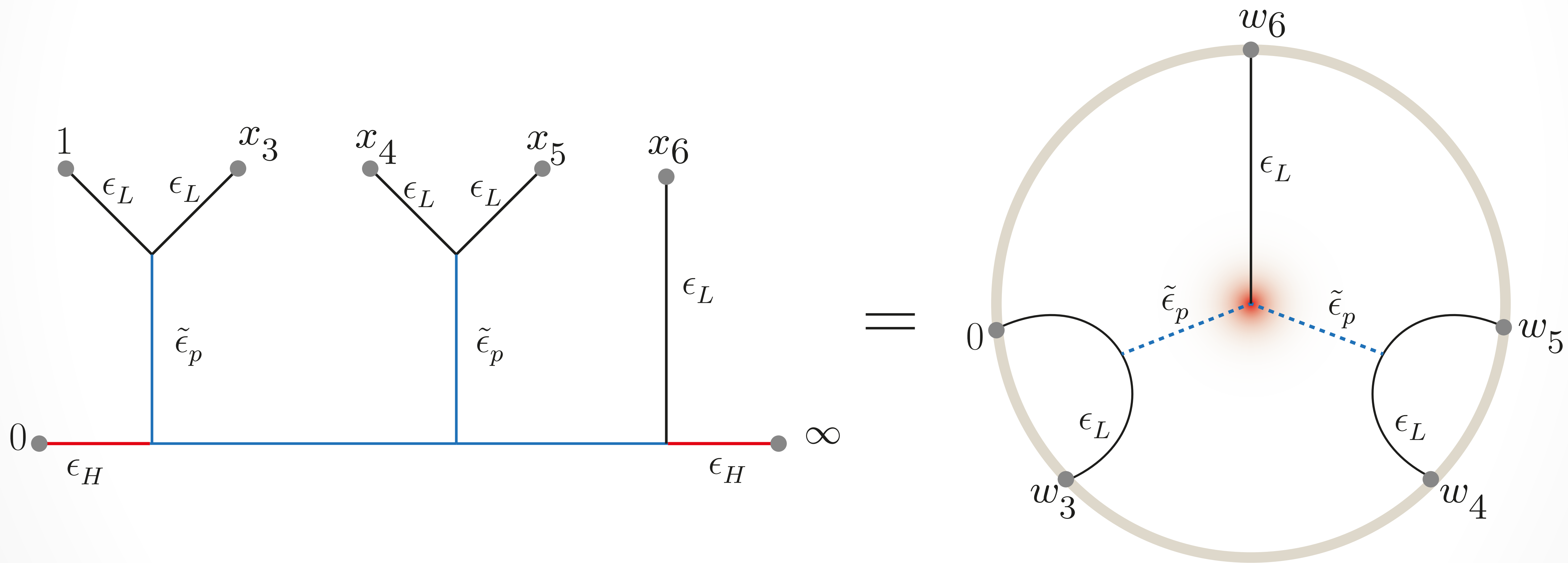
3-point function



4-point block

Conformal blocks from holography

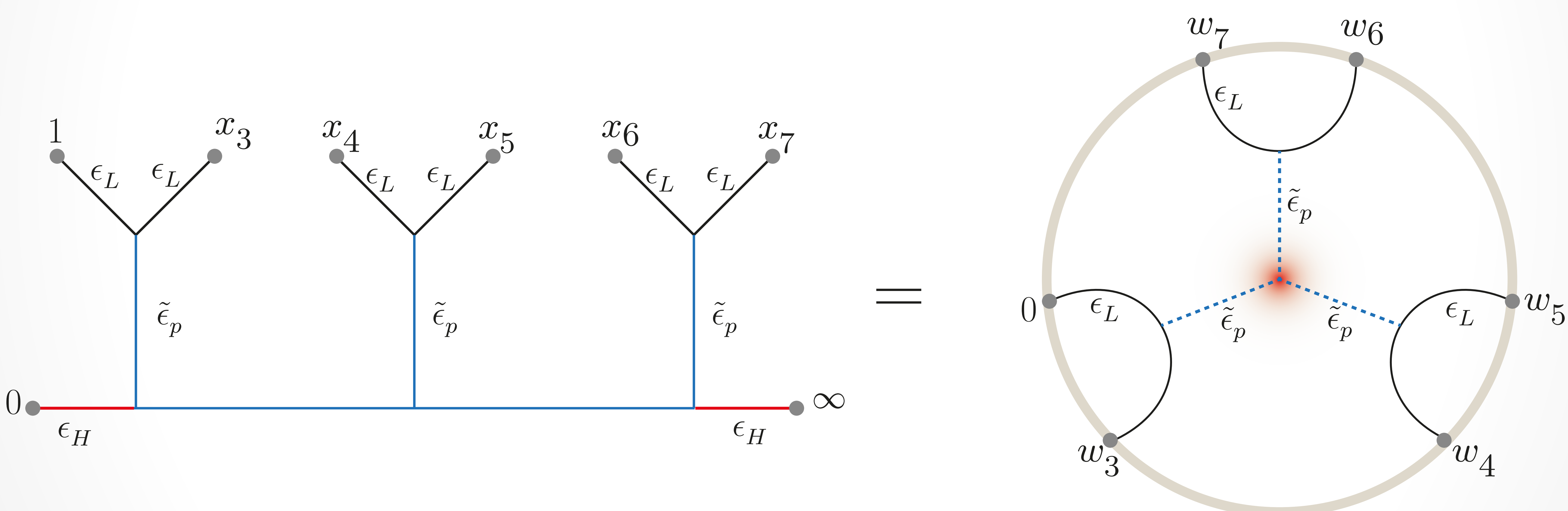
This can be suitably generalised to higher point conformal blocks



7-point conformal block

Conformal blocks from holography

This can be suitably generalised to higher point conformal blocks



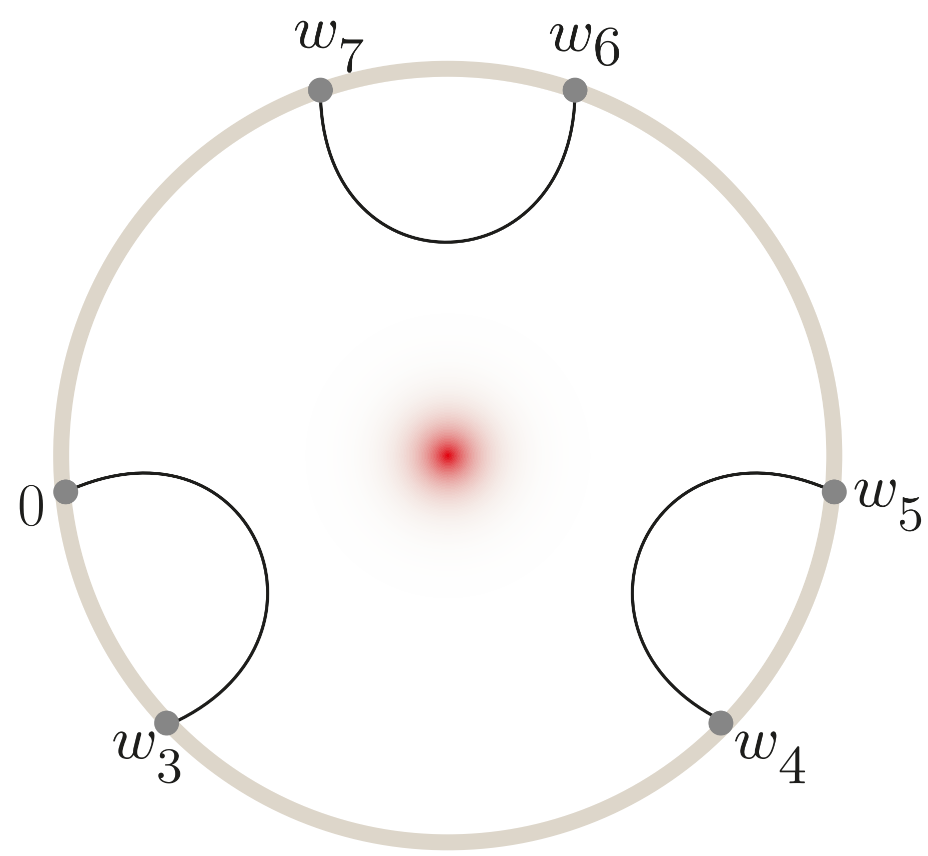
8-point conformal block

Precisely matches with CFT results.

Entanglement entropy from holography

The Ryu-Takayanagi minimal area proposal reproduces our CFT results for entanglement entropy.

Since twist operators fuse into the vacuum, one just needs to consider the vacuum block.



Geodesic configuration in the bulk ⟷ OPE channel in the CFT

To summarize …

Higher point conformal blocks are tractable in the heavy-light limit.

These conformal blocks can be reproduced precisely
from the holographic dual.

This is applied to find entanglement entropy
of disjoint intervals in heavy states.

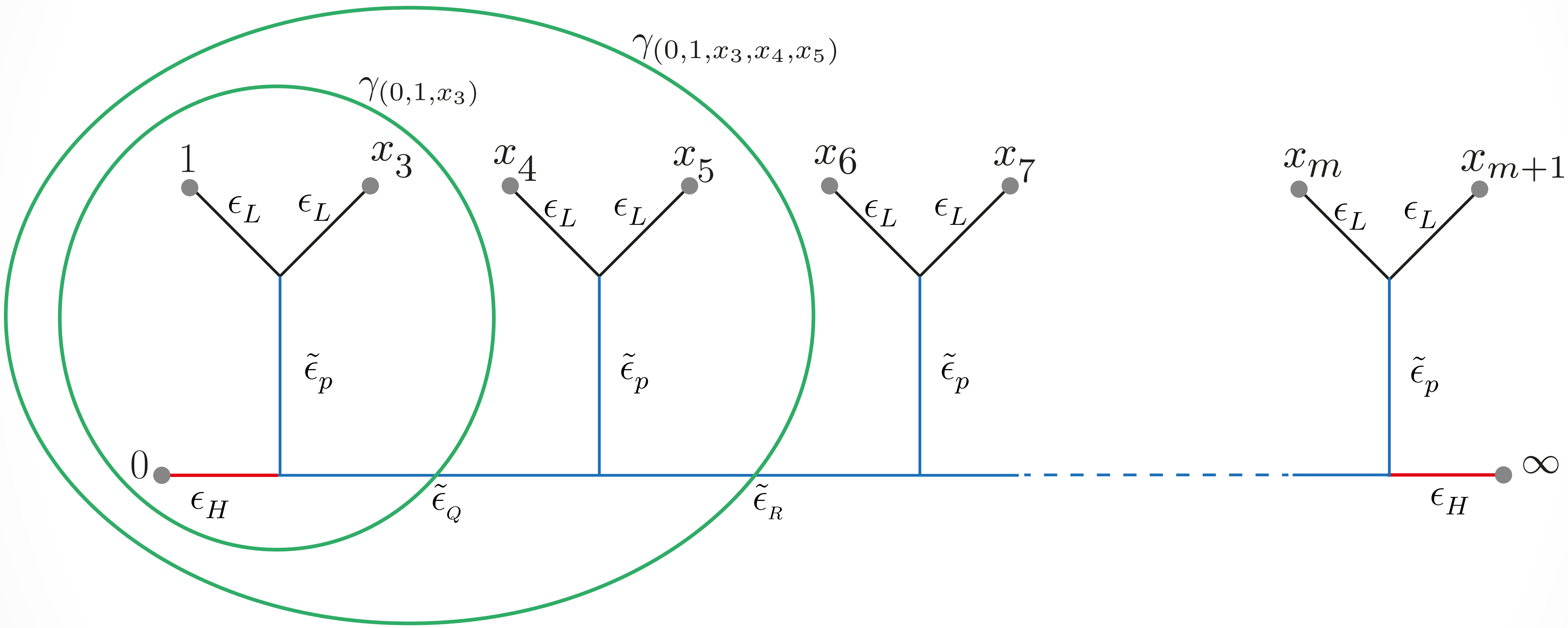
This conformal block can be rewritten in terms of geodesic lengths
(bulk locality?)

Applications – mutual information in local quenches, scrambling, chaos, …

Extensions – higher spin holography, one-loop corrections,
higher dimensions, …

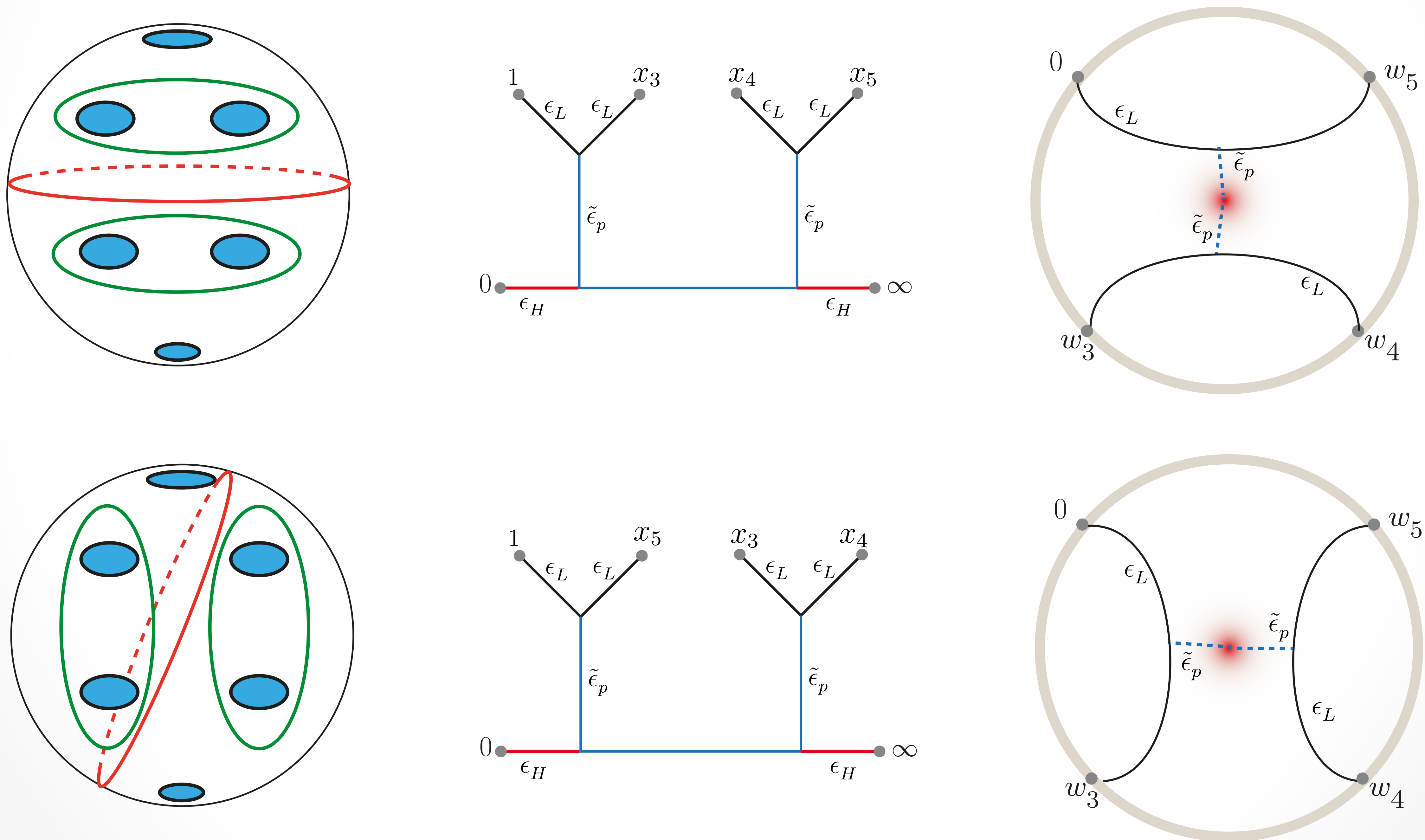
Thank you!

Backup slides



$\gamma_{(0,1,x_3,x_4,x_5)}$
$\gamma_{(0,1,x_3)}$
1
x_3
x_4
x_5
x_6
x_7
x_m
x_{m+1}
ϵ_L
$\tilde{\epsilon}_p$
0
∞
ϵ_H
$\tilde{\epsilon}_Q$
$\tilde{\epsilon}_R$

Conformal blocks, Riemann surfaces and holography

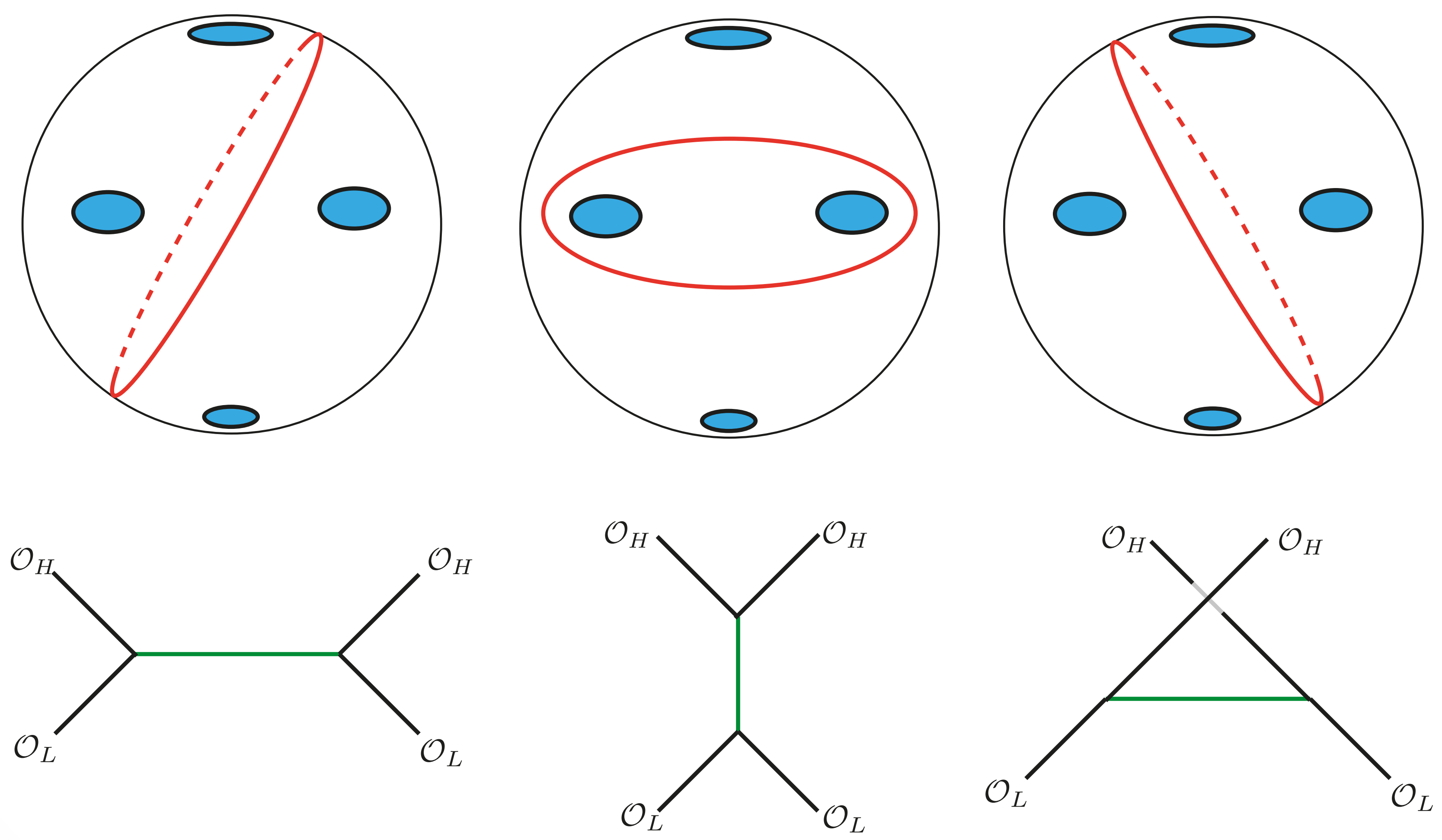


Conformal blocks and Riemann surfaces

n-point correlation functions of a CFT are associated with
a Riemann surface with n-punctures.

[Moore-Seiberg;...]

Decomposition of the correlator into conformal blocks
= Decomposition of the Riemann surface into 3-holed-spheres.



Our results on conformal blocks describe
specific regions of the moduli space of the associated Riemann surface.