Spinning strings with mixed fluxes

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Type IIB string theory with mixed R-R and NS-NS fluxes

AdS_3 **string backgrounds come from the D1-D5 system**

The corresponding near-horizon geometries with 16 supersymmetries are

$$AdS_3 \times S^3 \times T^4 \qquad \text{and} \qquad AdS_3 \times S^3 \times S^3 \times S^1$$

The dual gauge theory is a SCFT with $\mathcal{N} = (4, 4)$ symmetry

$\rightarrow$ **Type IIB string backgrounds on** AdS_3 **backgrounds can be supported by a mixture of R-R and NS-NS fluxes**

We will focur on type IIB strings on $AdS_3 \times S^3 \times T^4$,
with no dynamics along T^4 Thus

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\phi^2 + d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2 \ ,$$

with

$$b_{t\phi} = q \sinh^2\rho \ , \quad b_{\phi_1\phi_2} = -q\cos^2\theta \ ,$$

where $0 \leq q \leq 1$

When $q = 0$ we have **pure R-R flux** $\rightarrow$ **Green-Schwarz coset**
(**integrable**, both at classical and quantum level)

The value $q = 1$ is the limit of **pure NS-NS flux**

$\downarrow$

Supersymmetric WZW model

$$S = -\frac{1}{2}\Big[\int d^2\sigma\,\frac{1}{2}\mathrm{tr}(\mathcal{J}_+\mathcal{J}_-) - q\int d^3\sigma\,\frac{1}{3}\epsilon^{abc}\mathrm{tr}(\mathcal{J}_a\mathcal{J}_b\mathcal{J}_c)\Big]$$

with the currents $\mathcal{J}_a = g^{-1}\partial_a g$

[Maldacena,Ooguri]

Type IIB string theory on $AdS_3 \times S^3$with mixed R-R and NS-NS fluxes
is also **integrable**

[Cagnazo,Zarembo]

(Lax operator, nested Bethe ansatz equations, scattering matrix, dressing phase factor, ...)

An integrable deformation of the Neumann-Rosochatius system

Spinning string ansatz
(for the simpler case of rotation on S^3)

$$X_1 + iX_2 = r_1(\sigma)\, e^{i\varphi_1(\tau,\sigma)} \;, \quad X_3 + iX_4 = r_2(\sigma)\, e^{i\varphi_2(\tau,\sigma)}$$

with the angles are chosen as

$$\varphi_i(\tau,\sigma) = \omega_i \tau + \alpha_i(\sigma)$$

When we enter the spinning ansatz in the world sheet action we find

$$L_{S^3} = \frac{\sqrt{\lambda}}{2\pi}\Big[\sum_{i=1}^{2} \frac{1}{2}\left[(r_i')^2 + r_i^2(\alpha_i')^2 - r_i^2\omega_i^2\right] - \frac{\Lambda}{2}(r_1^2 + r_2^2 - 1)$$

$$+ q r_2^2\,(\omega_1\alpha_2' - \omega_2\alpha_1')\Big]$$

where Λ is a Lagrange multiplier to impose the condition that the solutions live on S^3

The equations for the **angles** can be easily integrated once

$$\alpha_i' = \frac{v_i + q r_2^2 \epsilon_{ij} \omega_j}{r_i^2} \,, \quad i = 1, 2$$

(v_i are some integration constants)

while the **radial coordinates** lead to

$$r_1'' = -r_1 \omega_1^2 + r_1 \alpha_1'^2 - \Lambda r_1$$

$$r_2'' = -r_2 \omega_2^2 + r_2 \alpha_2'^2 - \Lambda r_2 + 2 q r_2 (\omega_1 \alpha_2' - \omega_2 \alpha_1')$$

Furthermore, from the **isometries** of the problem

$$E = \sqrt{\lambda}\, w_0$$

$$J_1 = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left(r_1^2 \omega_1 - q r_2^2 \alpha_2' \right)$$

$$J_2 = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left(r_2^2 \omega_2 + q r_2^2 \alpha_1' \right)$$

The resulting system is a **deformation of the Neumann-Rosochatius** integrable system

(system of **oscillators on a sphere**, with a centrifugal barrier and a deformation coming from the mixture of fluxes)

Integrals of motion

Integrability of the Neumann-Rosochatius system follows from the existence of a set of integrals of motion in involution, **the Uhlenbeck constants**

For the case of a closed string rotating in S^3 there are two integrals I_1 and I_2, but they must satisfy the constraint $I_1 + I_2 = 1$ and we are left with a single independent constant

Integrability of the deformation by fluxes of the Neumann-Rosochatius system also follows from a **deformation of the Uhlenbeck constant**

$$\bar{I}_1 = r_1^2(1-q^2) + \frac{1}{\omega_1^2 - \omega_2^2}\left[(r_1 r_2' - r_1' r_2)^2 + \frac{(v_1 + q\omega_2)^2}{r_1^2} r_2^2 + \frac{v_2^2}{r_2^2} r_1^2\right]$$

Spinning string solutions in $AdS_3 \times S^3$ with mixed fluxes

We can find the most general kind of solutions by introducing an **ellipsoidal coordinate** defined through

$$\frac{r_1^2}{\zeta - \omega_1^2} + \frac{r_2^2}{\zeta - \omega_2^2} = 0$$

The equation of motion for ζ is

$$\zeta'^2 = -4P_3(\zeta)$$

where

$$P_3(\zeta) = (1-q^2)(\zeta-\omega_1^2)^2(\zeta-\omega_2^2) + (\zeta-\omega_1^2)(\zeta-\omega_2^2)(\omega_1^2-\omega_2^2)\bar{I}_1$$

$$+(\zeta-\omega_1^2)^2 v_2^2 + (\zeta-\omega_2^2)^2(v_1+q\omega_2)^2 = (1-q^2)\prod_{i=1}^{3}(\zeta-\zeta_i)$$

The polynomial $P_3(\zeta)$ defines an elliptic curve, $s^2 + P_3(\zeta) = 0$

If we change variables to

$$\zeta = \zeta_2 + (\zeta_3 - \zeta_2)\eta^2 \ ,$$

we are left with the differential equation for the Jacobian elliptic cosine,

$$\eta'^2 = (1 - q^2)(\zeta_3 - \zeta_1)(1 - \eta^2)(1 - \kappa + \kappa\eta^2)$$

($\kappa = (\zeta_3 - \zeta_2)/(\zeta_3 - \zeta_1)$ is the elliptic modulus)

We conclude that the **radial coordinate** is solved by the elliptic sine

$$r_1^2(\sigma) = \frac{\zeta_3 - \omega_1^2}{\omega_2^2 - \omega_1^2} + \frac{\zeta_2 - \zeta_3}{\omega_2^2 - \omega_1^2}\, \mathrm{sn}^2\left(\sigma\sqrt{(1 - q^2)(\zeta_3 - \zeta_1)}, \kappa\right)$$

Now we could find E, J_1 and J_2 and solve for

$$E = E(\sqrt{\lambda}, J_1, J_2)$$

The result is lengthy and cumbersome in the most general case

Instead we can look at the problem when $q = 1$
(pure NR-NS regime, or WZW limit)

$\rightarrow$ The case where $q = 0$ corresponds to the (undeformed)
Neumann-Rosochatius system

[Arutyunov,Frolov,Russo,Tseytlin]

Solutions with pure NS-NS flux

In the limit where $q = 1$ the elliptic surface degenerates

$$\left(P_3(\zeta) = (1 - q^2)(\zeta - \omega_1^2)^2(\zeta - \omega_2^2) + (\zeta - \omega_1^2)(\zeta - \omega_2^2)(\omega_1^2 - \omega_2^2)\bar{I}_1 \right)$$

The problem reduces to

$$\zeta'^2 = -4P_2(\zeta)$$

The elliptic sine becomes a trigonometric sine and thus

$$r_1^2(\sigma) = \frac{\tilde{\zeta}_2 - \omega_1^2}{\omega_2^2 - \omega_1^2} + \frac{\tilde{\zeta}_1 - \tilde{\zeta}_2}{\omega_2^2 - \omega_1^2}\ \sin^2(\omega\sigma)$$

where

$$\omega^2 = (\omega_1^2 - \omega_2^2)\bar{I}_1 + (v_1 + \omega_2)^2 + v_2^2$$

Now the energy and the angular momenta can be expressed in a very compact form

$$E^2 = \lambda\big(\omega^2 + \omega_1^2 - \omega_2^2 - 2v_1\omega_2 - 2v_2\omega_1\big)$$

$$\frac{J_1}{\sqrt{\lambda}} = \omega_1 - v_2 \ , \qquad \frac{J_2}{\sqrt{\lambda}} = \bar{m}_1 - v_1$$

($\bar{m}_i$ are some winding numbers)

Furthermore we can use the Virasoro constraints to write

$$E^2 = \lambda\big(\bar{m}_1^2 - \bar{m}_2^2 + 4\omega\bar{m}_2 - 3\omega^2\big) - 2\sqrt{\lambda}J(\bar{m}_1 + \bar{m}_2 - 2\omega)$$

with $J = J_1 + J_2$

$\rightarrow$ In an identical way we can treat solutions spinning in AdS_3

Solutions rotating in $AdS_3 \times S^3$ can be solved similarly,
but the solution needs to be written in terms of theta functions
(the diffferential equation involves an hyperelliptic surface)

The (less general) case of **constant radii solutions** also leads to rather compact expressions

(constant radii can be achieved by **colliding two roots** in the elliptic curve)

In the presence of mixed fluxes

$$E^2 = J^2 - 2\sqrt{\lambda}q\bar{m}_1 J + \frac{\lambda}{J}\left[(\bar{m}_1^2 J_1 + \bar{m}_2^2 J_2)(1-q^2) + q^2\bar{m}_1^2 J\right] + \cdots$$

which in the case of pure NS-NS flux becomes

$$E = J - \sqrt{\lambda}\bar{m}_1$$

Conclusions

- Type IIB string theory on $AdS_3 \times S^3$ with mixed R-R and NS-NS fluxes can be solved using a deformation of a classical integrable system
- We can reproduce solutions in $\mathrm{AdS}_3/\mathrm{CFT}_2$ that are obtained by solving the WZW model (using a pulsating string) [Maldacena,Ooguri]
- We can study quadratic fluctuations around our solutions
- Other deformations of the Neumann-Rosochatius system can be constructed to describe for instance $(AdS_5 \times S^5)_\eta$