On the double copy structure of soft gravitons

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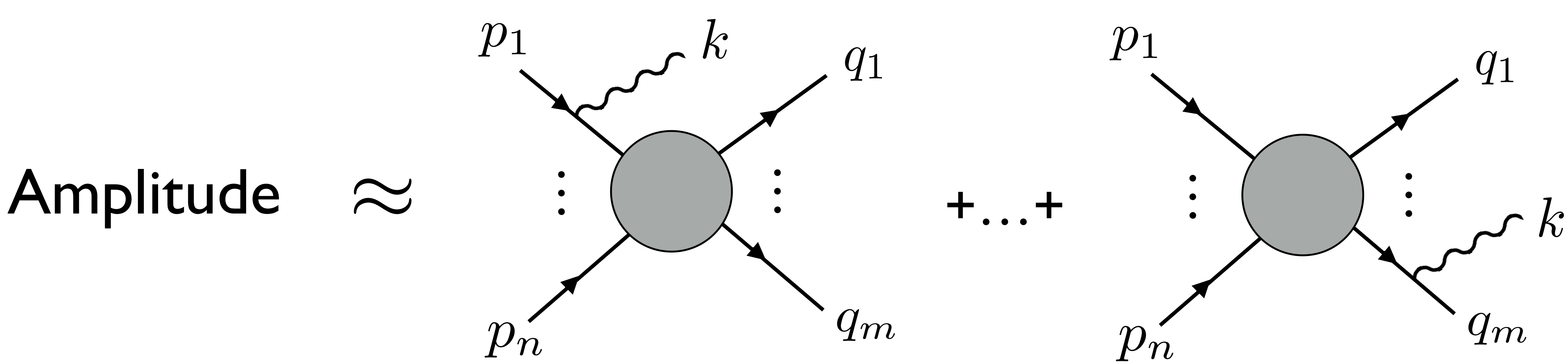
The revival of the soft theorems

Soft theorems in gauge theories and gravity are quite old stuff…

For **QED** it was formulated by **Low** in 1958. In the limit in which a photon is emitted with momentum $k \to 0$ the full amplitude factorizes as

$$\mathcal{A}_{n+1}(k;p_1,\ldots,p_n) \approx \left[\sum_{i=1}^{m} e_i \frac{q_i \cdot \epsilon(k)}{k \cdot q_i}\right] \mathcal{A}_n(p_1,\ldots,p_n)$$

This means that in the soft limit the amplitude is dominated by **bremsstrahlung from the external states**



In **gravity**, the soft theorem was formulated by Weinberg in 1964

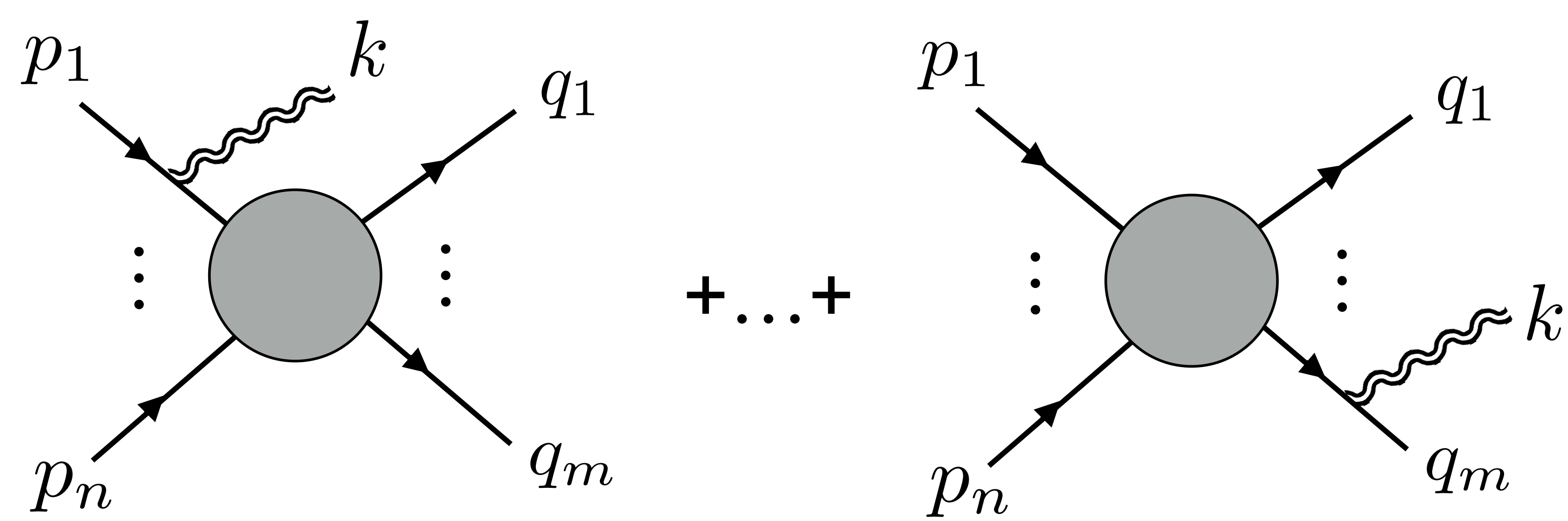
$$\mathcal{M}_{n+1}(k;p_1,\ldots,p_n) \approx \kappa \left[\sum_{i=1}^{n} \frac{p_i\cdot\varepsilon(k)\cdot p_i}{k\cdot p_i}\right]\mathcal{M}_n(p_1,\ldots,p_n)$$

where

$$p\cdot\varepsilon(k)\cdot p \equiv \varepsilon_{\mu\nu}(k)p^\mu p^\nu$$

Again, the dominant part of the amplitude is that in which the graviton is **bremsgestrahlt from the external states**

Amplitude $\approx$



Both Low's and Weinberg's theorems have **universal subleading corrections** in powers of the soft momentum

(Low 1958, Schwab & Volovich 2014, Casali 2014) (Cachazo & Strominger 2014)

For QED, we have

$$\mathcal{A}_{n+1}(k;p_1,\ldots,p_n)=\left(\sum_{i=1}^{n}e_i\frac{\epsilon\cdot p_i}{p_i\cdot k}+\sum_{i=1}^{n}e_i\frac{\epsilon^{\mu}k^{\nu}J^{(i)}_{\mu\nu}}{p_i\cdot k}\right)\mathcal{A}_n(p_1,\ldots,p_n)$$

whereas the NLO and NNLO corrections to Weinberg's theorem are

$$\mathcal{M}_{n+1}(k;p_1,\ldots,p_n)=\kappa\left[\sum_{i=1}^{n}\frac{\varepsilon^{\mu\nu}p_{i\mu}p_{i\nu}}{p_i\cdot k}+\sum_{i=1}^{n}\frac{\varepsilon^{\mu\nu}p_{i\mu}(k^{\alpha}J^{(i)}_{\nu\alpha})}{p_i\cdot k}\right.$$
$$\left.+\sum_{i=1}^{n}\frac{\varepsilon^{\mu\nu}(k^{\alpha}J^{(i)}_{\mu\alpha})(k^{\beta}J^{(i)}_{\nu\beta})}{p_i\cdot k}\right]\mathcal{M}_n(p_1,\ldots,p_n)$$

In both cases:

$$J^{(i)}_{\mu\nu}=p_{i,\mu}\frac{\partial}{\partial p_i^{\nu}}-p_{i,\nu}\frac{\partial}{\partial p_i^{\mu}}+\text{spin contribution}$$

Recently, soft-theorems became again fashionable because their relation to the **asymptotic symmetries**. (Strominger 2014)

In the case of (gauge-fixed) **gauge theories**, one considers “large” residual gauge transformations that do not approach the identity at infinity

Low’s theorem is a **Ward identity** associated to these transformations.

In **gravity**, the relevant asymptotic symmetry is the BMS group:

(Bondi, van der Burg, Metzner 1962; Sachs 1962)

$$\mathrm{BMS} = \mathcal{S} \times \mathrm{SL}(2,\mathbb{C})$$

Weinberg’s theorem is equivalent to the **Ward identity for supertranslations**.

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supertranslations — rotations of the transverse sphere

Weinberg's theorem is equivalent to the **Ward identity for supertranslations**.

On the other hand, there are a number of **hints** that, in an "on-shell" sense, **gravity = (gauge)²:**

- Kawai-Lewellen-Tye (KLT) tree-level identities, e.g.

$$A^{(5g)}(1,2,3,4,5) = -is_{12}s_{34}A_L^{(5\,\text{gauge})}(1,2,3,4,5)A_R^{(5\,\text{gauge})}(2,1,4,3,5)$$
$$-is_{13}s_{24}A_L^{(5\,\text{gauge})}(1,3,2,4,5)A_R^{(5\,\text{gauge})}(3,1,4,2,5)$$

- Bern-Carrasco-Johansson (BCJ) color-kinematics duality:

$$\mathcal{A}_{\text{gauge}} = g^{n-2}\sum_i \frac{c_i n_i'}{\prod_\alpha s_{\alpha,i}} \qquad \mathcal{M}_{\text{gravity}} = i\left(\frac{\kappa}{2}\right)^{n-2}\sum_i \frac{n_i'\widetilde{n}_i}{\prod_\alpha s_{\alpha,i}}$$

provided

$$c_i + c_j + c_k = 0 \qquad n_i' + n_j' + n_k' = 0$$

This can be extended to loop diagrams (before integration).

Does (soft graviton)=(soft gluon)²?

Let's focus now on the scattering of **two distinct scalar**

$\phi_i(x)$

$\widetilde{\phi}_i(x)$

transforming in representations T^a_{ij} and $\widetilde{T}^a_{ij}$ of the nonabelian gauge group and coupling according to

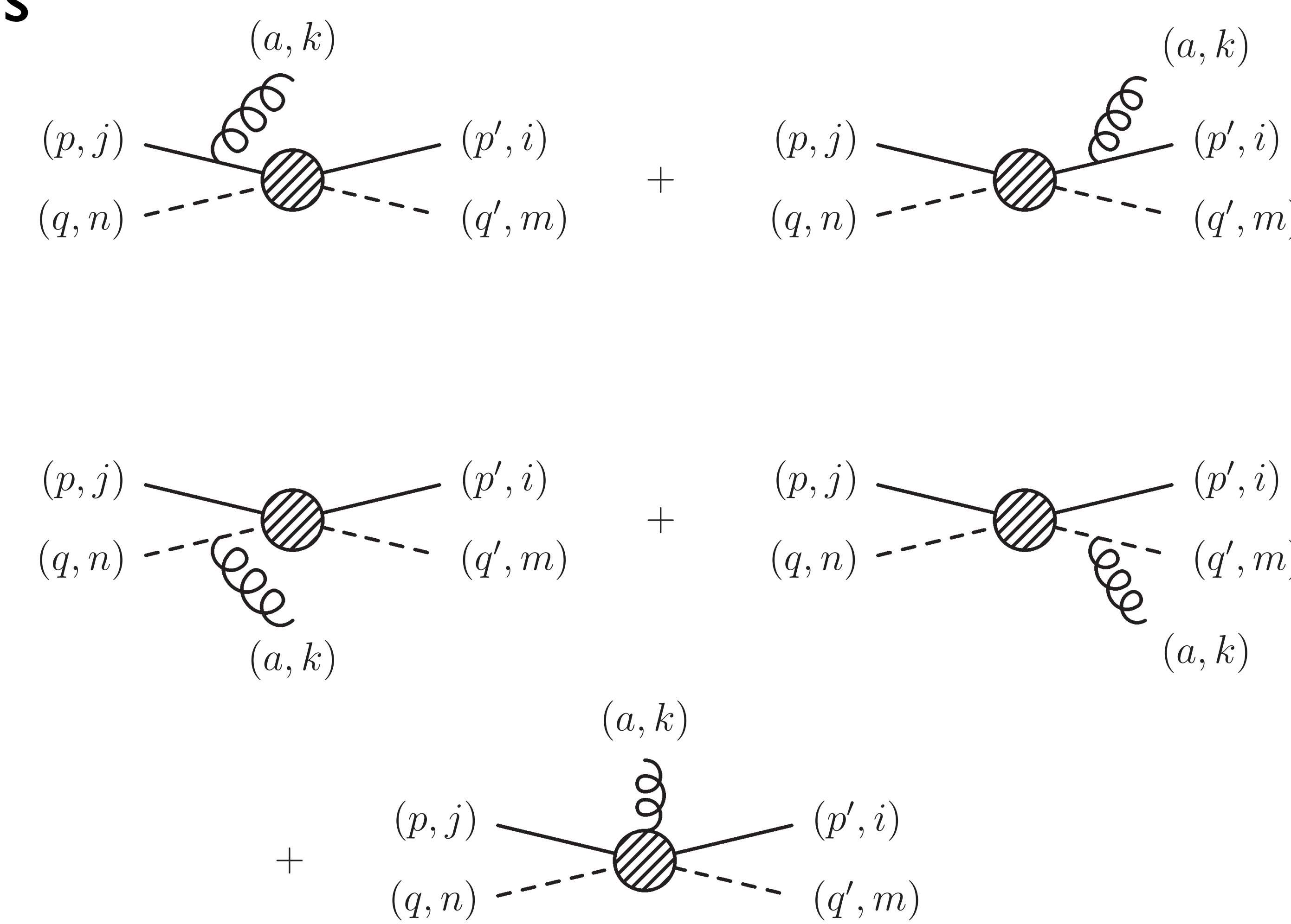
$$\mathcal{L} \supset \Big[(\partial^\mu - gA^{\mu\,a}T^a)\phi\Big]^\dagger \Big[(\partial^\mu - gA^{\mu\,a}T^a)\phi\Big] + \Big[(\partial^\mu - gA^{\mu\,a}\widetilde{T}^a)\widetilde{\phi}\Big]^\dagger \Big[(\partial^\mu - gA^{\mu\,a}\widetilde{T}^a)\widetilde{\phi}\Big]$$

The amplitude

$$\phi + \widetilde{\phi} \to \phi + \widetilde{\phi} + \text{gluon}$$

depends on seven color structures

$$c_1 = T^a_{ik}T^b_{kj}\widetilde{T}^b_{mn},$$
$$c_2 = T^b_{ik}T^a_{kj}\widetilde{T}^b_{mn},$$
$$c_3 = T^a_{ik}T^b_{kj}\widetilde{T}^b_{mn} + T^b_{ik}T^a_{kj}\widetilde{T}^b_{mn},$$
$$c_4 = T^b_{ij}\widetilde{T}^a_{mk}\widetilde{T}^b_{kn},$$
$$c_5 = T^b_{ij}\widetilde{T}^b_{m\ell}\widetilde{T}^a_{\ell n},$$
$$c_6 = T^b_{ij}\widetilde{T}^a_{m\ell}\widetilde{T}^b_{\ell n} + T^b_{ij}\widetilde{T}^b_{m\ell}\widetilde{T}^a_{\ell n},$$
$$c_7 = if^{abc}T^b_{ij}\widetilde{T}^c_{mn}$$



satisfying the Jacobi identities:

$$c_1 + c_2 - c_3 = 0, \qquad c_4 + c_5 - c_6 = 0,$$

$$c_1 - c_2 + c_7 = 0, \qquad c_4 - c_5 - c_7 = 0,$$

This structure is **general**: in principle, loop diagrams require **new vertices** (three- & four-gluon and "seagull" vertices) and one might encounter "new" color structures, e.g.

$$f^{abc}f^{cde}T^a_{i\ell}T^b_{\ell j}\widetilde{T}^d_{mn},\; f^{abc}f^{cde}T^b_{ij}\widetilde{T}^d_{mp}\widetilde{T}^e_{pn},\ldots$$

This, however can be reduced to c_3, c_6, and c_7 by using

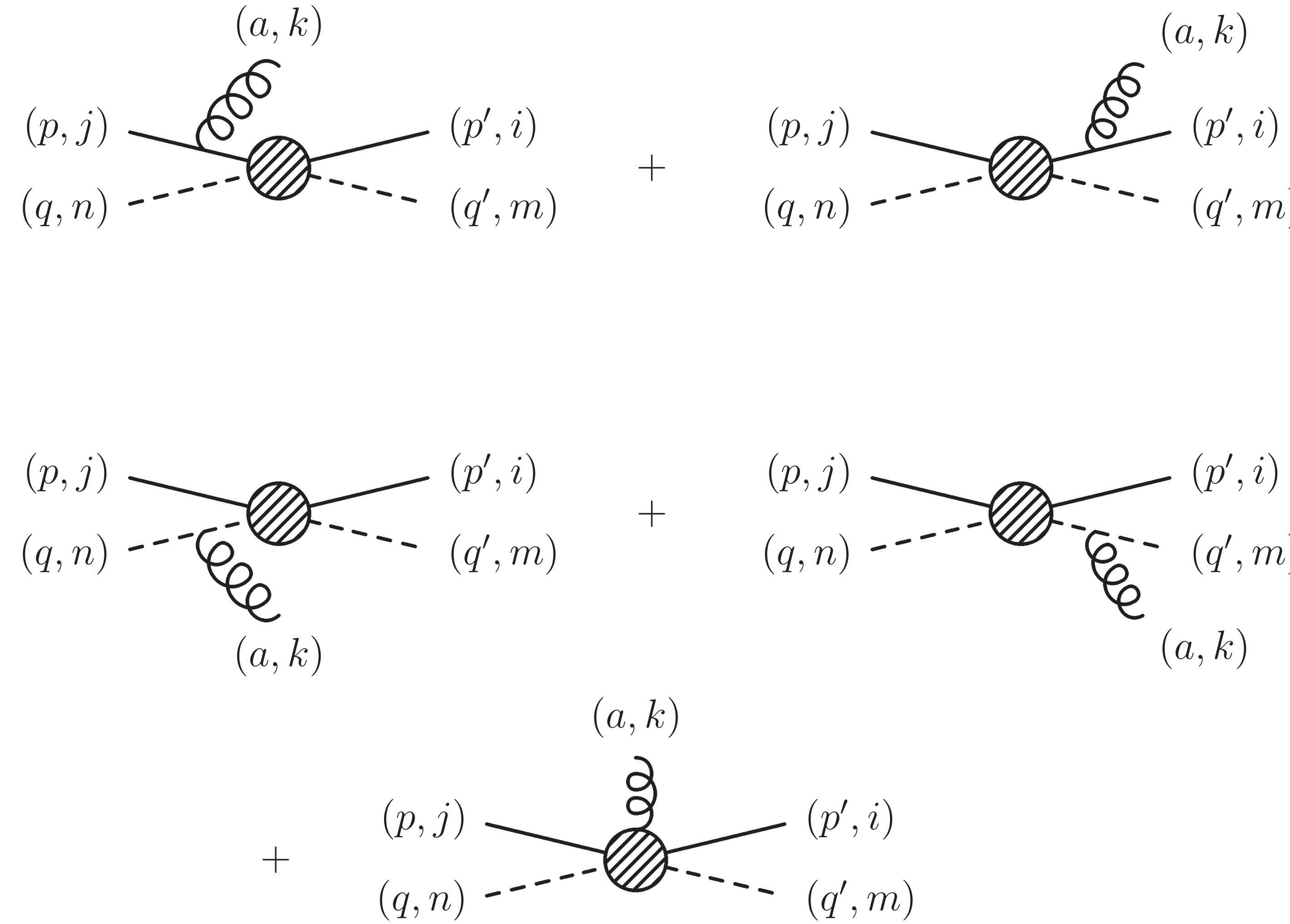
$$f^{abc}T^b_{ik} = i[T^a, T^c]_{ik}$$

together with closure relations

$$T^a_{ik}T^a_{\ell j} = \frac{1}{2}\left(\delta_{ij}\delta_{k\ell} - \frac{1}{N}\delta_{ik}\delta_{\ell j}\right). \qquad \text{for } \mathrm{SU}(N)$$

$$s_1 = 2k \cdot p, \qquad s_2 = 2k \cdot q$$

$$s_{1'} = 2k \cdot p', \qquad s_{2'} = 2k \cdot q'$$

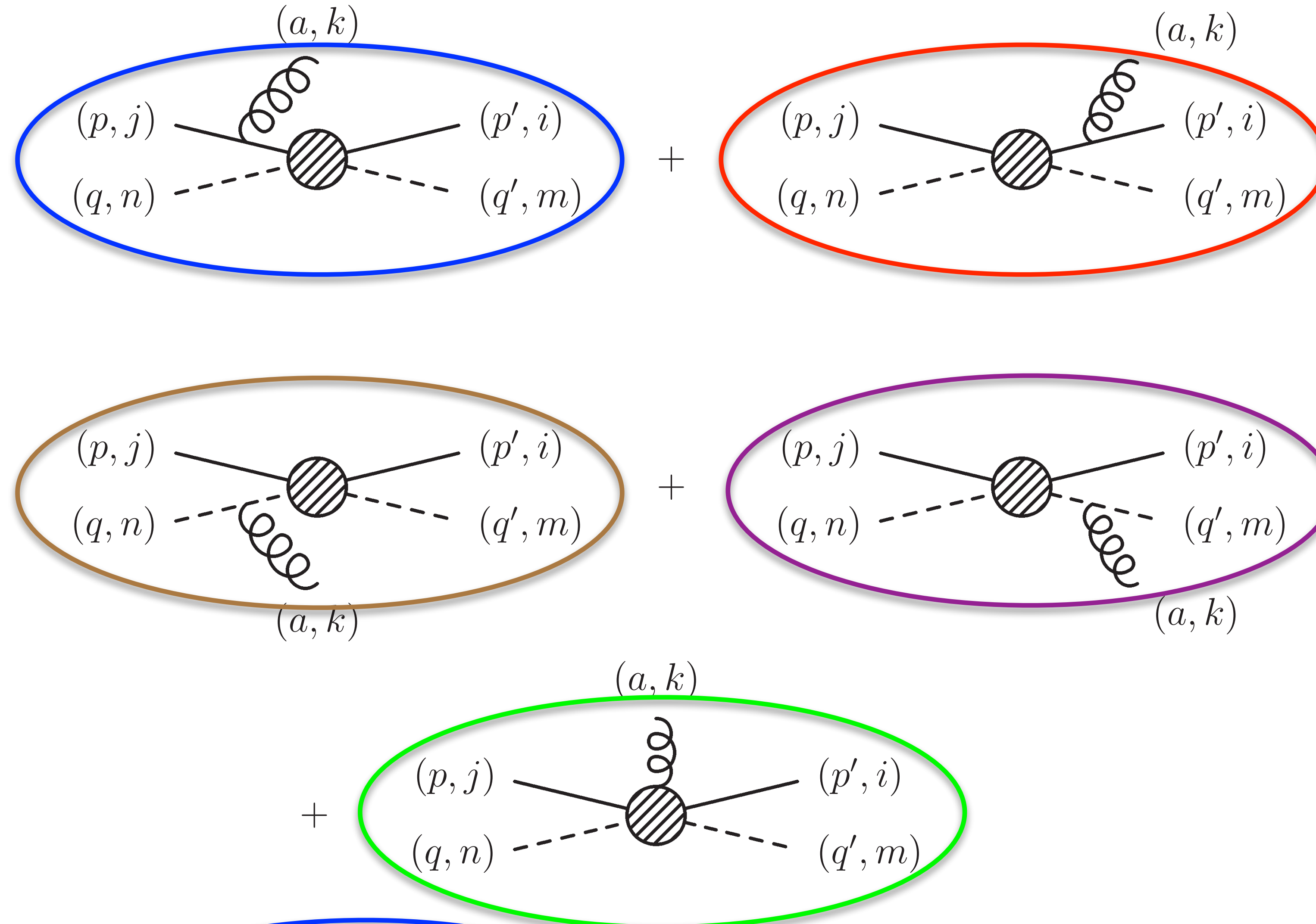


Using these color factors and the kinematic invariants, the **full amplitude** can be written as

$$\begin{aligned}\mathcal{A}_5 &= 2g\left[c_1\frac{p'\cdot\epsilon}{s_{1'}}\mathcal{A}_4(p,q,p'+k,q') - c_2\frac{p\cdot\epsilon}{s_1}\mathcal{A}_4(p-k,q,p',q')\right.\\ &+ \left. c_4\frac{q'\cdot\epsilon}{s_{2'}}\mathcal{A}_4(p,q,p',q'+k) - c_5\frac{q\cdot\epsilon}{s_2}\mathcal{A}_4(p,q-k,p',q')\right]\\ &+ \frac{g}{2}\Big[c_3\,\epsilon_\mu\mathcal{B}_1^\mu(k;p,q,p',q') + c_6\,\epsilon_\mu\mathcal{B}_2^\mu(k;p,q,p',q') + c_7\,\epsilon_\mu\mathcal{B}_3^\mu(k;p,q,p',q')\Big].\end{aligned}$$

$$s_1 = 2k \cdot p, \qquad s_2 = 2k \cdot q$$

$$s_{1'} = 2k \cdot p', \qquad s_{2'} = 2k \cdot q'$$



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$$\begin{aligned} \mathcal{A}_5 \;=\; & 2g\Big[c_1 \frac{p'\cdot\epsilon}{s_{1'}}\mathcal{A}_4(p,q,p'+k,q') - c_2\frac{p\cdot\epsilon}{s_1}\mathcal{A}_4(p-k,q,p',q') \\ & + c_4\frac{q'\cdot\epsilon}{s_{2'}}\mathcal{A}_4(p,q,p',q'+k) - c_5\frac{q\cdot\epsilon}{s_2}\mathcal{A}_4(p,q-k,p',q')\Big] \\ & + \frac{g}{2}\Big[c_3\,\epsilon_\mu\mathcal{B}_1^\mu(k;p,q,p',q') + c_6\,\epsilon_\mu\mathcal{B}_2^\mu(k;p,q,p',q') + c_7\,\epsilon_\mu\mathcal{B}_3^\mu(k;p,q,p',q')\Big]. \end{aligned}$$

To find the soft-gluon theorem, we implement gauge invariance

$$\mathcal{A}_5\Big|_{\epsilon_\mu\to k_\mu} = 0$$

and expand in powers of the gluon momentum.

This alone fixes $\mathcal{B}_i$ in terms of the four-point scalar amplitudes $\mathcal{A}_4(p,q,p',q')$ to leading order in the gluon momentum

$$\begin{aligned}\mathcal{A}_5 &= 2g\Bigg(c_1\frac{p'\cdot\epsilon}{s_{1'}} - c_2\frac{p\cdot\epsilon}{s_1} + c_4\frac{q'\cdot\epsilon}{s_{2'}} - c_5\frac{q\cdot\epsilon}{s_2} \\ &+ c_1\frac{\epsilon^\mu k^\nu J^{(1')}_{\mu\nu}}{s_{1'}} + c_2\frac{\epsilon^\mu k^\nu J^{(1)}_{\mu\nu}}{s_1} + c_4\frac{\epsilon^\mu k^\nu J^{(2')}_{\mu\nu}}{s_{2'}} + c_5\frac{\epsilon^\mu k^\nu J^{(2)}_{\mu\nu}}{s_2}\Bigg)\mathcal{A}_4(p,q,p',q').\end{aligned}$$

where, since we are scattering scalar particles

$$J^{(i)}_{\mu\nu} = p_{i,\mu}\frac{\partial}{\partial p_i^\nu} - p_{i,\nu}\frac{\partial}{\partial p_j^\mu}$$

(only "orbital" angular momentum)

$$\mathcal{A}_5 = 2g\Bigg(c_1 \frac{p'\cdot\epsilon}{s_{1'}} - c_2\frac{p\cdot\epsilon}{s_1} + c_4\frac{q'\cdot\epsilon}{s_{2'}} - c_5\frac{q\cdot\epsilon}{s_2}$$
$$+ \; c_1\frac{\epsilon^\mu k^\nu J^{(1')}_{\mu\nu}}{s_{1'}} + c_2\frac{\epsilon^\mu k^\nu J^{(1)}_{\mu\nu}}{s_1} + c_4\frac{\epsilon^\mu k^\nu J^{(2')}_{\mu\nu}}{s_{2'}} + c_5\frac{\epsilon^\mu k^\nu J^{(2)}_{\mu\nu}}{s_2}\Bigg)\mathcal{A}_4(p,q,p',q').$$

In this form, the tensor structure of the soft gluon prefactor is entangled with the derivatives of the four-point amplitude.

However, using the four-particle invariants s and t we can write

$$\begin{aligned}
\epsilon^\mu k^\nu J^{(1')}_{\mu\nu} &= A_{1'}\frac{\partial}{\partial s} + B_{1'}\frac{\partial}{\partial t},\\
\epsilon^\mu k^\nu J^{(1)}_{\mu\nu} &= A_{1}\frac{\partial}{\partial s} + B_{1}\frac{\partial}{\partial t},\\
\epsilon^\mu k^\nu J^{(2')}_{\mu\nu} &= A_{2'}\frac{\partial}{\partial s} + B_{2'}\frac{\partial}{\partial t},\\
\epsilon^\mu k^\nu J^{(2)}_{\mu\nu} &= A_{2}\frac{\partial}{\partial s} + B_{2}\frac{\partial}{\partial t},
\end{aligned}$$

where

$$\begin{aligned}
A_{1'} &= (\epsilon\cdot p')(q'\cdot k) - (\epsilon\cdot q')(p'\cdot k),\\
A_{1} &= (\epsilon\cdot p)(q\cdot k) - (\epsilon\cdot q)(p\cdot k),\\
A_{2'} &= (\epsilon\cdot q')(p'\cdot k) - (\epsilon\cdot p')(q'\cdot k),\\
A_{2} &= (\epsilon\cdot q)(p\cdot k) - (\epsilon\cdot p)(q\cdot k),\\
B_{1'} &= (\epsilon\cdot p)(p'\cdot k) - (\epsilon\cdot p')(p\cdot k),\\
B_{1} &= (\epsilon\cdot p')(p\cdot k) - (\epsilon\cdot p)(p'\cdot k),\\
B_{2'} &= (\epsilon\cdot q)(q'\cdot k) - (\epsilon\cdot q')(q\cdot k),\\
B_{2} &= (\epsilon\cdot q')(q\cdot k) - (\epsilon\cdot q)(q'\cdot k).
\end{aligned}$$

The advantage of this expression is that now the whole tensor structure is confined to the coefficients of the derivatives

$$\mathcal{A}_5 = 2g\left[c_1\frac{p'\cdot\epsilon}{s_{1'}} - c_2\frac{p\cdot\epsilon}{s_1} + c_4\frac{q'\cdot\epsilon}{s_{2'}} - c_5\frac{q\cdot\epsilon}{s_2} + \left(c_1\frac{A_{1'}}{s_{1'}} + c_2\frac{A_1}{s_1} + c_4\frac{A_{2'}}{s_{2'}} + c_5\frac{A_2}{s_2}\right)\frac{\partial}{\partial s} + \left(c_1\frac{B_{1'}}{s_{1'}} + c_2\frac{B_1}{s_1} + c_4\frac{B_{2'}}{s_{2'}} + c_5\frac{B_2}{s_2}\right)\frac{\partial}{\partial t}\right]\mathcal{A}_4(s,t).$$

Besides, **due to the Jacobi identities** it enjoys a "generalized invariance"

$$A_{1'} \longrightarrow A_{1'} + s_{1'}\,\alpha(p,q,p',q'), \qquad B_{1'} \longrightarrow B_{1'} + s_{1'}\,\beta(p,q,p',q'),$$

$$A_1 \longrightarrow A_1 - s_1\,\alpha(p,q,p',q'), \qquad B_1 \longrightarrow B_1 - s_1\,\beta(p,q,p',q'),$$

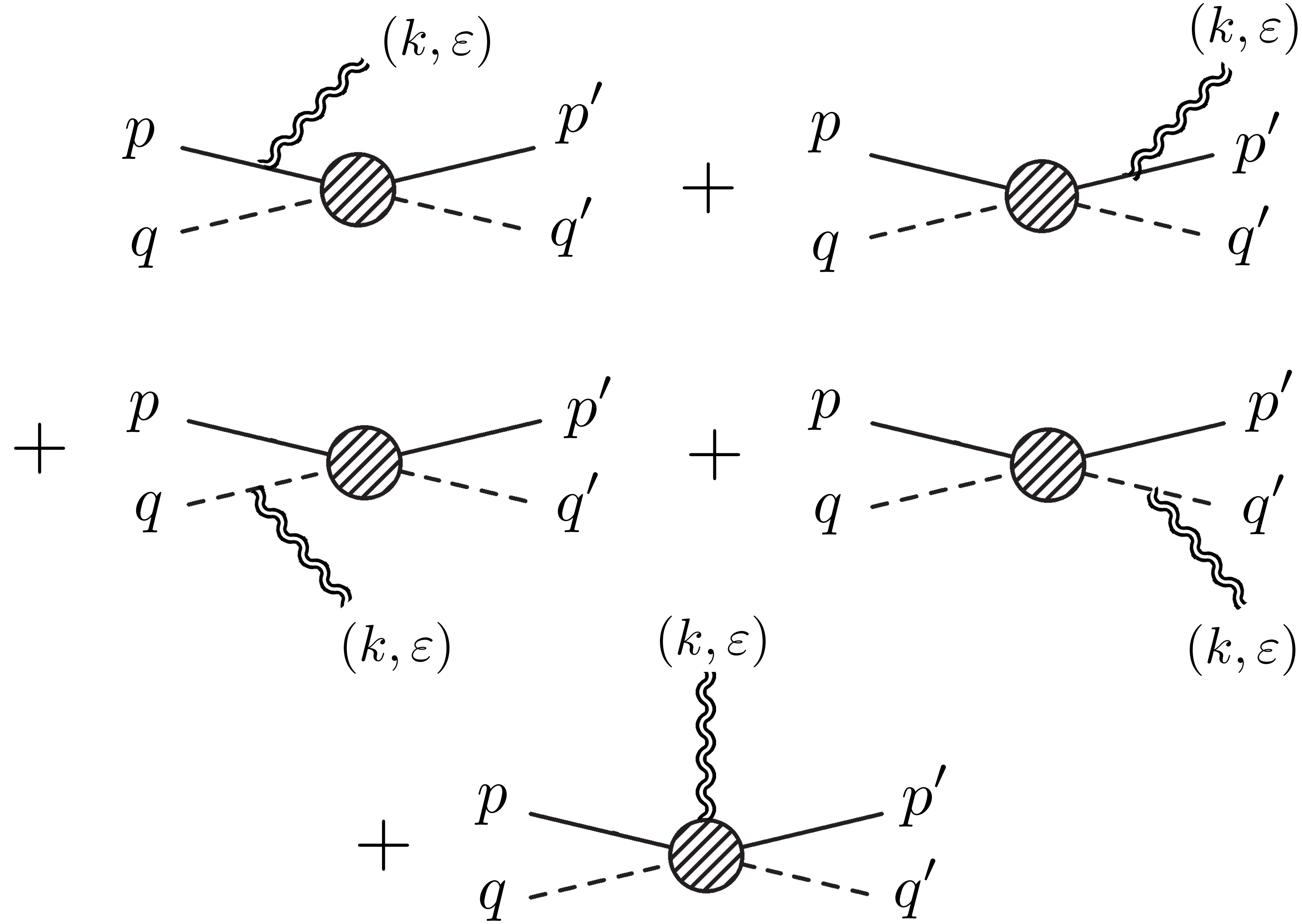
$$A_{2'} \longrightarrow A_{2'} + s_{2'}\,\alpha(p,q,p',q'), \qquad B_{2'} \longrightarrow B_{2'} + s_{2'}\,\beta(p,q,p',q'),$$

$$A_2 \longrightarrow A_2 - s_2\,\alpha(p,q,p',q'), \qquad B_2 \longrightarrow B_2 - s_2\,\beta(p,q,p',q'),$$

for arbitrary functions $\alpha(p,q,p',q')$ and $\beta(p,q,p',q')$

Let us move to **gravity**.

In the soft limit, we know that the first correction to Weinberg's theorem can be written in terms of the **angular momentum** operators of the four scalars



$$
\begin{aligned}
\mathcal{M}_5 &= \kappa\left(-\frac{p\cdot\varepsilon\cdot p}{s_1}+\frac{p'\cdot\varepsilon\cdot p'}{s_{1'}}-\frac{q\cdot\varepsilon\cdot q}{s_2}+\frac{q'\cdot\varepsilon\cdot q'}{s_{2'}}\right.\\
&+\left.\frac{p'_\mu\varepsilon^{\mu\nu}k^\alpha J^{(1')}_{\nu\alpha}}{s_{1'}}+\frac{p_\mu\varepsilon^{\mu\nu}k^\alpha J^{(1)}_{\nu\alpha}}{s_1}+\frac{q'_\mu\varepsilon^{\mu\nu}k^\alpha J^{(2')}_{\nu\alpha}}{s_{2'}}+\frac{q_\mu\varepsilon^{\mu\nu}k^\alpha J^{(2)}_{\nu\alpha}}{s_2}\right)\mathcal{M}_4(p,q,p',q'),
\end{aligned}
$$

Using again the kinematic invariants, the angular momentum terms have the form

$$
\begin{aligned}
p'_\mu \varepsilon^{\mu\nu} k^\alpha J^{(1')}_{\nu\alpha} &= \widetilde{A}_{1'}\frac{\partial}{\partial s} + \widetilde{B}_{1'}\frac{\partial}{\partial t}, \\
p_\mu \varepsilon^{\mu\nu} k^\alpha J^{(1)}_{\nu\alpha} &= \widetilde{A}_{1}\frac{\partial}{\partial s} + \widetilde{B}_{1}\frac{\partial}{\partial t}, \\
q'_\mu \varepsilon^{\mu\nu} k^\alpha J^{(2')}_{\nu\alpha} &= \widetilde{A}_{2'}\frac{\partial}{\partial s} + \widetilde{B}_{2'}\frac{\partial}{\partial t}, \\
q_\mu \varepsilon^{\mu\nu} k^\alpha J^{(2)}_{\nu\alpha} &= \widetilde{A}_{2}\frac{\partial}{\partial s} + \widetilde{B}_{2}\frac{\partial}{\partial t},
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{A}_{1'} &= (p'\cdot\varepsilon\cdot p')(q'\cdot k) - (p'\cdot\varepsilon\cdot q')(p'\cdot k), \\
\widetilde{A}_{1} &= (p\cdot\varepsilon\cdot p)(q\cdot k) - (p\cdot\varepsilon\cdot q)(p\cdot k), \\
\widetilde{A}_{2'} &= (q'\cdot\varepsilon\cdot q')(p'\cdot k) - (q'\cdot\varepsilon\cdot p')(q'\cdot k), \\
\widetilde{A}_{2} &= (q\cdot\varepsilon\cdot q)(p\cdot k) - (q\cdot\varepsilon\cdot p)(q\cdot k), \\
\\
\widetilde{B}_{1'} &= (p'\cdot\varepsilon\cdot p)(p'\cdot k) - (p'\cdot\varepsilon\cdot p')(p\cdot k), \\
\widetilde{B}_{1} &= (p\cdot\varepsilon\cdot p')(p\cdot k) - (p\cdot\varepsilon\cdot p)(p'\cdot k), \\
\widetilde{B}_{2'} &= (q'\cdot\varepsilon\cdot q)(q'\cdot k) - (q'\cdot\varepsilon\cdot q')(q\cdot k), \\
\widetilde{B}_{2} &= (q\cdot\varepsilon\cdot q')(q\cdot k) - (q\cdot\varepsilon\cdot q)(q'\cdot k).
\end{aligned}
$$

and the amplitude reads

$$
\begin{aligned}
\mathcal{M}_5 &= \kappa\Bigg[-\frac{p\cdot\varepsilon\cdot p}{s_1} + \frac{p'\cdot\varepsilon\cdot p'}{s_{1'}} - \frac{q\cdot\varepsilon\cdot q}{s_2} + \frac{q'\cdot\varepsilon\cdot q'}{s_{2'}} \\
&+ \left(\frac{\widetilde{A}_{1'}}{s_{1'}} + \frac{\widetilde{A}_{1}}{s_{1}} + \frac{\widetilde{A}_{2'}}{s_{2'}} + \frac{\widetilde{A}_{2}}{s_{2}}\right)\frac{\partial}{\partial s} + \left(\frac{\widetilde{B}_{1'}}{s_{1'}} + \frac{\widetilde{B}_{1}}{s_{1}} + \frac{\widetilde{B}_{2'}}{s_{2'}} + \frac{\widetilde{B}_{2}}{s_{2}}\right)\frac{\partial}{\partial t}\Bigg]\mathcal{M}_4(s,t).
\end{aligned}
$$

$$\mathcal{M}_5 = \kappa\Bigg[-\frac{p\cdot\varepsilon\cdot p}{s_1}+\frac{p'\cdot\varepsilon\cdot p'}{s_{1'}}-\frac{q\cdot\varepsilon\cdot q}{s_2}+\frac{q'\cdot\varepsilon\cdot q'}{s_{2'}}$$
$$+\left(\frac{\widetilde{A}_{1'}}{s_{1'}}+\frac{\widetilde{A}_{1}}{s_{1}}+\frac{\widetilde{A}_{2'}}{s_{2'}}+\frac{\widetilde{A}_{2}}{s_{2}}\right)\frac{\partial}{\partial s}+\left(\frac{\widetilde{B}_{1'}}{s_{1'}}+\frac{\widetilde{B}_{1}}{s_{1}}+\frac{\widetilde{B}_{2'}}{s_{2'}}+\frac{\widetilde{B}_{2}}{s_{2}}\right)\frac{\partial}{\partial t}\Bigg]\mathcal{M}_4(s,t).$$

As in the gauge case, the tensor structure is not disentangled from the derivatives acting on the four-point amplitude.

The gravitational amplitude has the "generalized invariance"

$$\widetilde{A}_i \longrightarrow \widetilde{A}_i + s_i\,\widetilde{\alpha}_i(p,q,p',q'),$$
$$\widetilde{B}_i \longrightarrow \widetilde{B}_i + s_i\,\widetilde{\beta}_i(p,q,p',q'),$$

where

$$\sum_i \widetilde{\alpha}_i(p,q,p',q') = 0, \qquad \sum_i \widetilde{\beta}_i(p,q,p',q') = 0.$$

We exploit this invariance to write gravity coefficients can be written as **product** of the gauge theory coefficients!

$$\widetilde{A}'_{1'} = \frac{2\varepsilon_{\mu\nu}A^{\mu}_{1'}A^{\nu}_{1'}}{s_{1'}+s_{2'}}, \qquad \widetilde{A}'_{1} = \frac{2\varepsilon_{\mu\nu}A^{\mu}_{1}A^{\nu}_{1}}{s_{1}+s_{2}}, \qquad \widetilde{A}'_{2'} = \frac{2\varepsilon_{\mu\nu}A^{\mu}_{2'}A^{\nu}_{2'}}{s_{1'}+s_{2'}}, \qquad \widetilde{A}'_{2} = \frac{2\varepsilon_{\mu\nu}A^{\mu}_{2}A^{\nu}_{2}}{s_{1}+s_{2}}$$

and

$$\widetilde{B}'_{1'} = -\frac{2\varepsilon_{\mu\nu}B^{\mu}_{1'}B^{\nu}_{1'}}{t_1-t_2}, \qquad \widetilde{B}'_{1} = \frac{2\varepsilon_{\mu\nu}B^{\mu}_{1}B^{\nu}_{1}}{t_1-t_2}, \qquad \widetilde{B}'_{2'} = \frac{2\varepsilon_{\mu\nu}B^{\mu}_{2'}B^{\nu}_{2'}}{t_1-t_2}, \qquad \widetilde{B}'_{2} = -\frac{2\varepsilon_{\mu\nu}B^{\mu}_{2}B^{\nu}_{2}}{t_1-t_2}.$$

where

$$t_1 = (p-p')^2, \qquad t_2 = (q-q')^2$$

Apart from the kinematic denominator, we have found a double copy structure for the coefficients of the derivatives in the soft limit.

Comparing the gauge theory and the gravity amplitude we can do better:

$$\begin{aligned}\mathcal{A}_5 &= 2\,g\,\epsilon_\mu\left[c_1\frac{p'^\mu}{s_{1'}} - c_2\frac{p^\mu}{s_1} + c_4\frac{q'^\mu}{s_{2'}} - c_5\frac{q^\mu}{s_2}\right.\\
&+ \left(c_1\frac{A^\mu_{1'}}{s_{1'}+s_{2'}}\frac{1}{s_{1'}} + c_2\frac{A^\mu_{1}}{s_1+s_2}\frac{1}{s_1} + c_4\frac{A^\mu_{2'}}{s_{1'}+s_{2'}}\frac{1}{s_{2'}} + c_5\frac{A^\mu_{2}}{s_1+s_2}\frac{1}{s_2}\right)(s_1+s_2)\frac{\partial}{\partial s}\\
&+ \left.\left(c_1\frac{B^\mu_{1'}}{t_1-t_2}\frac{1}{s_{1'}} + c_2\frac{B^\mu_{1}}{t_1-t_2}\frac{1}{s_1} + c_4\frac{B^\mu_{2'}}{t_1-t_2}\frac{1}{s_{2'}} + c_5\frac{B^\mu_{2}}{t_1-t_2}\frac{1}{s_2}\right)(t_1-t_2)\frac{\partial}{\partial t}\right]\mathcal{A}_4(s,t)\end{aligned}$$

$$\begin{aligned}\mathcal{M}_5 &= \kappa\,\varepsilon_{\mu\nu}\left\{\frac{p'^\mu p'^\nu}{s_{1'}} - \frac{p^\mu p^\nu}{s_1} + \frac{q'^\mu q'^\nu}{s_{2'}} - \frac{q^\mu q^\nu}{s_2}\right.\\
&+ 2\left[\frac{A^\mu_{1'}A^\nu_{1'}}{(s_{1'}+s_{2'})^2}\frac{1}{s_{1'}} + \frac{A^\mu_{1}A^\nu_{1}}{(s_1+s_2)^2}\frac{1}{s_1} + \frac{A^\mu_{2'}A^\nu_{2'}}{(s_{1'}+s_{2'})^2}\frac{1}{s_{2'}} + \frac{A^\mu_{1}A^\nu_{1}}{(s_1+s_2)^2}\frac{1}{s_2}\right](s_1+s_2)\frac{\partial}{\partial s}\\
&+ \left.2\left[-\frac{B^\mu_{1'}B^\nu_{1'}}{(t_1-t_2)^2}\frac{1}{s_{1'}} + \frac{B^\mu_{1}B^\nu_{1}}{(t_1-t_2)^2}\frac{1}{s_1} + \frac{B^\mu_{2'}B^\nu_{2'}}{(t_1-t_2)^2}\frac{1}{s_{2'}} - \frac{B^\mu_{1}B^\nu_{1}}{(t_1-t_2)^2}\frac{1}{s_2}\right](t_1-t_2)\frac{\partial}{\partial t}\right\}\mathcal{M}_4(s,t).\end{aligned}$$

where, we have to take into account that

$$t_1 - t_2 = k\cdot(p - p' + q' - q) \qquad s_1 + s_2 = s_{1'} + s_{2'} = k\cdot(p + q - p' - q')$$

This structure is very similar to the "double copy" of KLT and BCJ

color factors → second copy of the numerator

However, there are very important differences:

- It only affects the "soft prefactor", not the full amplitude.
- The double copy structure does not require the gauge theory numerators to satisfy any "Jacobi-like" identities.
- Moreover, our five-point scalar amplitude **does not** satisfy BCJ duality.

Our result can be interpreted as

$$\text{(soft graviton)} = \text{(soft gluon)}^2$$

Our results have been derived in the soft gluon/graviton limit…

However, in 1967 **Gribov** pointed out that the factorization is valid in a larger kinematic domain. For two colliding hadrons of mass μ with momenta p and q this regime is:

$$2\,p\cdot k,\ 2\,q\cdot k \ll s \qquad \mathbf{k}_\perp^2 \approx \frac{(2\,p\cdot k)(2\,q\cdot k)}{s} \ll \mu^2$$

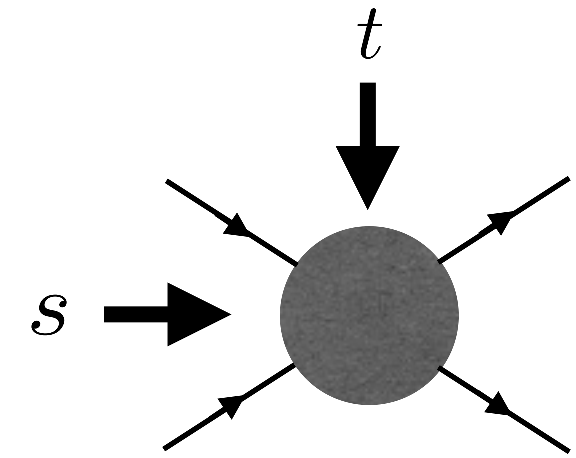
whereas Low's theorem is valid when

$$2p\cdot k \ll \mu^2,\ 2q\cdot k \ll \mu^2$$

In our notation, Gribov's limit corresponds to

$$s_1, s_2 \ll s, \qquad \mathbf{k}_\perp^2 \ll \mu^2 \ll s, \qquad |t_1 - t_2| \ll \mu\sqrt{-t_1} \approx \mu\sqrt{-t_2}.$$

In the four-point function, the Gribov limit implies



$$s \gg t \sim \mu^2$$

Now, the four point gauge amplitude is a **homogeneous function** of degree 0

$$\left(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\right)\mathcal{A}_4(s,t) = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial s}\mathcal{A}_4(s,t) = -\frac{t}{s}\frac{\partial}{\partial t}\mathcal{A}_4(s,t).$$

Thus, in the Gribov limit derivatives with respect to s are suppressed. In practical terms, we can consider the four-point amplitude to be constant in s

$$\begin{aligned}\mathcal{A}_5 &= 2g\left[c_1\frac{p'\cdot\epsilon}{s_{1'}}\mathcal{A}_4(s,t_2) - c_2\frac{p\cdot\epsilon}{s_1}\mathcal{A}_4(s,t_2) + c_4\frac{q'\cdot\epsilon}{s_{2'}}\mathcal{A}_4(s,t_1) - c_5\frac{q\cdot\epsilon}{s_2}\mathcal{A}_4(s,t_1)\right] \\ &+ \frac{g}{2}\epsilon_\mu\Big[c_3\,\mathcal{B}_1^\mu(k;p,q,p',q') + c_6\,\mathcal{B}_2^\mu(k;p,q,p',q') + c_7\,\mathcal{B}_3^\mu(k;p,q,p',q')\Big].\end{aligned}$$

where

$$\frac{t_1 - t_2}{2} \ll \frac{t_1 + t_2}{2} \equiv t.$$

Again, gauge invariant fixes the first correction in the

$$\mathcal{A}_5 = 2g\Bigg[c_1\frac{p'\cdot\epsilon}{s_{1'}} - c_2\frac{p\cdot\epsilon}{s_1} + c_4\frac{q'\cdot\epsilon}{s_{2'}} - c_5\frac{q\cdot\epsilon}{s_2} + \left(c_1\frac{B_{1'}}{s_{1'}} + c_2\frac{B_1}{s_1} + c_4\frac{B_{2'}}{s_{2'}} + c_5\frac{B_2}{s_2}\right)\frac{\partial}{\partial t}\Bigg]\mathcal{A}_4(s,t).$$

On the gravity side, at fixed order in perturbation theory the amplitude has the structure

$$\mathcal{M}_4(s,t) = (\kappa^2 s)^{\frac{n}{2}} f\left(\frac{s}{t}\right)$$

If at large energies $f(s/t) \sim (s/t)^\alpha$

$$\frac{\partial}{\partial s}\mathcal{M}_4(s,t) \ll \frac{\partial}{\partial t}\mathcal{M}_4(s,t)$$

and we can drop s-derivatives in the amplitude.

The **gravitational amplitude** in the Gribov limit is

$$\begin{aligned}\mathcal{M}_5 &= \kappa\left[\frac{p'\cdot\varepsilon\cdot p'}{s_{1'}} - \frac{p\cdot\varepsilon\cdot p}{s_1} + \frac{q'\cdot\varepsilon\cdot q'}{s_{2'}} - \frac{q\cdot\varepsilon\cdot q}{s_2}\right.\\ &+ \left.\left(\frac{\widetilde{B}_{1'}}{s_{1'}} + \frac{\widetilde{B}_1}{s_1} + \frac{\widetilde{B}_{2'}}{s_{2'}} + \frac{\widetilde{B}_2}{s_2}\right)\frac{\partial}{\partial t}\right]\mathcal{M}_4(s,t).\end{aligned}$$

Comparing with the gauge theory amplitude

$$\begin{aligned}\mathcal{A}_5 &= 2g\left[c_1\frac{p'\cdot\epsilon}{s_{1'}} - c_2\frac{p\cdot\epsilon}{s_1} + c_4\frac{q'\cdot\epsilon}{s_{2'}} - c_5\frac{q\cdot\epsilon}{s_2}\right.\\ &+ \left.\left(c_1\frac{B_{1'}}{s_{1'}} + c_2\frac{B_1}{s_1} + c_4\frac{B_{2'}}{s_{2'}} + c_5\frac{B_2}{s_2}\right)\frac{\partial}{\partial t}\right]\mathcal{A}_4(s,t).\end{aligned}$$

We find that the double copy structure of the graviton **survives** in the **Gribov limit**

Conclusions

- Using a five-point scalar identity we have found a "moral" identity

$$(\text{soft graviton}) = (\text{soft gluon})^2$$

- The gravitational amplitude is obtained by **replacing** color factors by a **second copy** of the kinematic numerator.

- This double-copy **only** affects the contribution of the **soft gluon/graviton**, not the "hard piece of the amplitude.

- This might be **reminiscent** of BCJ but very it is quite **different** in other aspects: no need to implement Jacobi identities.

Some work in progress: Does the double copy structure survives for higher-point scattering amplitudes and higher loops?

- For **higher-point amplitudes** we have to work with a larger number of redundant kinematic invariants

$$s_{ij} = (p_i + p_j)^2 \qquad\qquad (i < j)$$

For gauge theories we find

$$\mathcal{A}_{n+1}(k; p_1, \ldots, p_n) = g \sum_{i=1}^{n} \frac{\epsilon \cdot p_i}{p_i \cdot k} \mathcal{A}_n(s_{ab}) + 2g \sum_{i<j} \frac{(\epsilon \cdot p_i)(k \cdot p_j) - (\epsilon \cdot p_j)(k \cdot p_i)}{p_i \cdot k} \frac{\partial}{\partial s_{ij}} \mathcal{A}_n(s_{ab})$$

whereas in gravity we have

$$\begin{aligned}\mathcal{M}_{n+1}(k; p_1, \ldots, p_n) &= \kappa \sum_{i=1}^{n} \frac{\varepsilon^{\mu\nu} p_{i\mu} p_{i\nu}}{p_i \cdot k} \mathcal{M}_n(s_{ab}) \\ &+ 2\kappa \sum_{j<i} \frac{(p_i \cdot \varepsilon \cdot p_i)(k \cdot p_j) - (p_i \cdot \varepsilon \cdot p_j)(k \cdot p_i)}{p_i \cdot k} \frac{\partial}{\partial s_{ij}} \mathcal{M}_n(s_{ab})\end{aligned}$$

One needs to find an **efficient parametrization** of the "kinematic submanifold" in order to uncover any double copy structure.

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?

One needs to find an **efficient parametrization** of the "kinematic submanifold" in order to uncover any double copy structure.

- Do "Gribov gravitons/photons" have anything to say about about **asymptotic symmetries** at null infinity?

Gribov gravitons are not soft, still Low's factorization holds

Can it still be interpreted as a Ward identity?

THANK YOU