Small-scale lensing as a diffusion process

Workshop on the large-scale structure, Madrid

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Based on [1508.07903]
with Julien Larena and Jean-Philippe Uzan

Introduction

Statement

Accurate interpretation of cosmological observations requires to account for **lensing** corrections.

Current situation

Cosmic lensing is generally described with the perturbation theory, in which matter is modelled as a **fluid**.

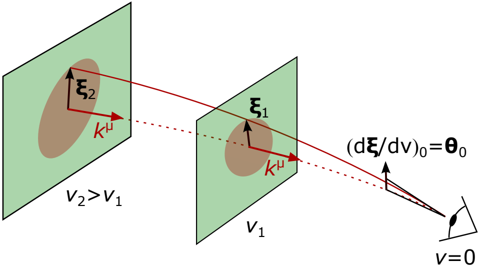
Problem

This approximation breaks down when very **narrow light beams** are involved (e.g. for SN observations).

The lensing Jacobi matrix

The Jacobi matrix $\mathcal{D}$ relates the morphology of a light beam to its observed angular aperture

$$\boxed{\xi^A(v) = \mathcal{D}^A{}_B(v)\theta_0^B}$$



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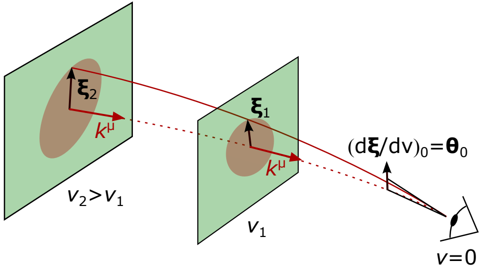

Its determinant defines the geometric distances

$$D_A = \sqrt{|\det \mathcal{D}(v_S)|} \qquad \text{and} \qquad D_L = (1+z)^2 D_A.$$

The Sachs equation

Evolution of $\boldsymbol{\mathcal{D}}$ ruled by the geodesic deviation equation

$$\boxed{\frac{\mathrm{d}^2\mathcal{D}^A{}_B}{\mathrm{d}v^2} = \mathcal{R}^A_C\mathcal{D}^C{}_B}$$

with the **optical tidal matrix** (projected Riemann tensor)

$$\mathcal{R}_{AB} \equiv R_{\mu\nu\rho\sigma} s^\mu_A k^\nu k^\rho s^\sigma_B,$$

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$$\mathcal{R}_{AB} \equiv R_{\mu\nu\rho\sigma} s^\mu_A k^\nu k^\rho s^\sigma_B,$$

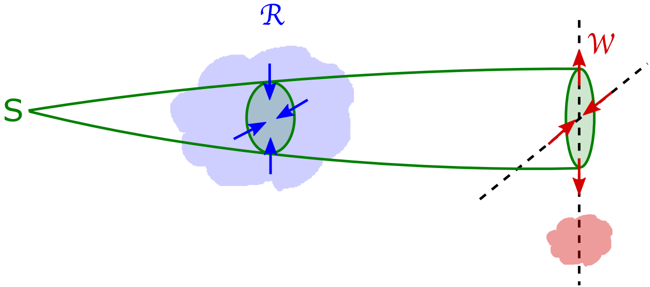
decomposed in to Ricci part and a Weyl part as

$$\boldsymbol{\mathcal{R}} = \underbrace{\begin{pmatrix} \mathscr{R} & 0 \\ 0 & \mathscr{R} \end{pmatrix}}_{\text{generates focusing}} + \underbrace{\begin{pmatrix} -\mathcal{W}_1 & \mathcal{W}_2 \\ \mathcal{W}_2 & \mathcal{W}_1 \end{pmatrix}}_{\text{generates distortions}}$$

$$\mathscr{R} \equiv -\frac{1}{2} R_{\mu\nu} k^\mu k^\nu$$

$$\mathcal{W}_{AB} \equiv C_{\mu\nu\rho\sigma} s^\mu_A k^\nu k^\rho s^\sigma_B$$

Ricci and Weyl lensing

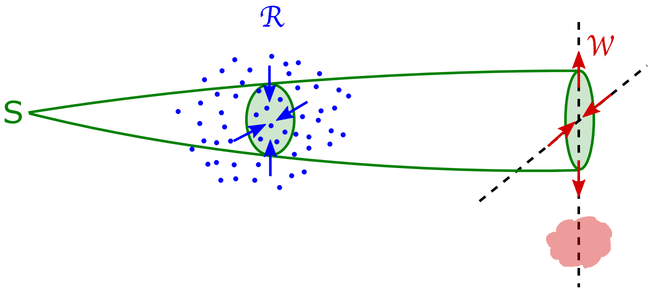


- Ricci focuses due to diffuse matter inside the beam, as

$$\mathcal{R} \equiv -\frac{1}{2} R_{\mu\nu} k^\mu k^\nu = -4\pi G T_{\mu\nu} k^\mu k^\nu = -4\pi G \omega^2 (\rho + p).$$

- Weyl distorts and focuses mostly due to matter outside the beam.

Ricci and Weyl lensing



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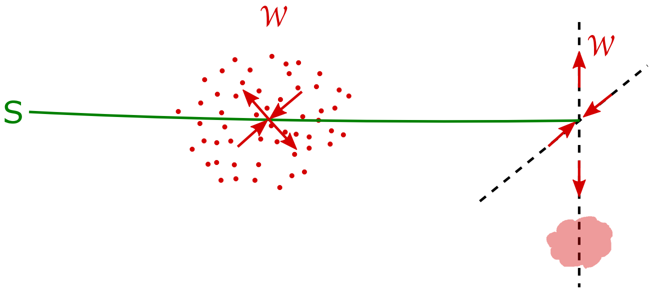
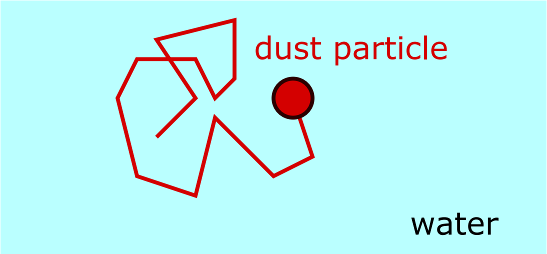


What is considered Ricci or Weyl depends on the beam's scale.

Question

How to efficiently model lensing due to very-small-scale structures?

Analogy with Brownian motion



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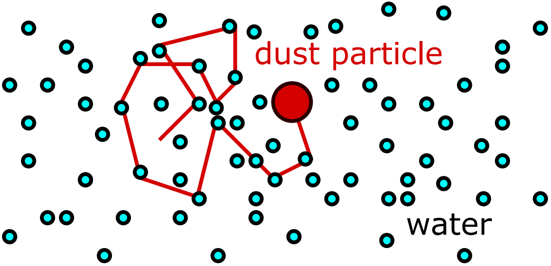

- Brownian motion due to a myriad of collisions between the particle and water molecules.
- It cannot be described with a fluid approach.
- Mathematical model: stochastic force.

The Sachs-Langevin equation

$$\frac{\mathrm{d}^2\boldsymbol{\mathcal{D}}}{\mathrm{d}v^2} = \left(\langle\boldsymbol{\mathcal{R}}\rangle + \delta\boldsymbol{\mathcal{R}}\right)\boldsymbol{\mathcal{D}}$$

- $\langle\boldsymbol{\mathcal{R}}\rangle$ (deterministic) encodes the large-scale structure;
- $\delta\boldsymbol{\mathcal{R}}$ (stochastic) models small-scale fluctuations.

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Hypotheses: statistical isotropy and white noises.

$$\begin{aligned}
\langle\mathcal{W}_A\rangle &= 0,\\
\langle\delta\mathcal{R}(v)\mathcal{W}_A(w)\rangle &= 0,\\
\langle\delta\mathcal{R}(v)\delta\mathcal{R}(w)\rangle &= C_{\mathcal{R}}(v)\delta(v-w)\\
\langle\mathcal{W}_A(v)\mathcal{W}_B(w)\rangle &= C_{\mathcal{W}}(v)\delta_{AB}\delta(v-w)
\end{aligned}$$

Covariance amplitude of X such that $C_X \sim (\delta X)^2 \Delta v_{\mathrm{coh}}$.

The lensing Fokker-Planck equation

The probability density function $p(\boldsymbol{\mathcal{D}}, \dot{\boldsymbol{\mathcal{D}}}; v)$ satisfies

$$\frac{\partial p}{\partial v} = -\dot{\mathcal{D}}_{AB} \frac{\partial p}{\partial \mathcal{D}_{AB}} - \langle \mathcal{R} \rangle \mathcal{D}_{AB} \frac{\partial p}{\partial \dot{\mathcal{D}}_{AB}} + \frac{1}{2} \left(C_{\mathcal{R}} \, \delta_{AE} \delta_{CF} + C_{\mathcal{W}} \, \delta_{AC} \delta_{EF} - C_{\mathcal{W}} \, \varepsilon_{AC} \varepsilon_{EF} \right) \mathcal{D}_{EB} \mathcal{D}_{FD} \frac{\partial^2 p}{\partial \dot{\mathcal{D}}_{AB} \partial \dot{\mathcal{D}}_{CD}},$$

with a drift term and a diffusion term.

- It generates evolutions equations for the moments of $p(\boldsymbol{\mathcal{D}}, \dot{\boldsymbol{\mathcal{D}}}; v)$.
- Order-n moments form a closed system (no hierarchy).
- Everything is contained in the functions $\langle \mathcal{R} \rangle (v)$, $C_{\mathcal{R}}(v)$, $C_{\mathcal{W}}(v)$.

General results

- **Correction to the mean angular distance**. Let D_0 be such that $\ddot{D}_0 = \langle \mathcal{R} \rangle D_0$ (e.g. FLRW distance, or Kantowki-Dyer-Roeder distance), then

$$\delta^{(1)}_{D_\mathrm{A}} \equiv \frac{\langle D_\mathrm{A} \rangle - D_0}{D_0} \qquad \text{at first order in Weyl fluctuations}$$

$$= -\int_0^v \mathrm{d}v_1 \int_0^{v_1} \mathrm{d}v_2 \int_0^{v_2} \mathrm{d}v_3 \left[\frac{D_0^2(v_3)}{D_0(v_1) D_0(v_2)} \right]^2 2 C_\mathcal{W}(v_3)$$

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- **Dispersion of the angular distance** $\sigma^2_{D_\mathrm{A}} \equiv \left\langle D_\mathrm{A}^2 \right\rangle - \left\langle D_\mathrm{A} \right\rangle^2$,

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3} \left(\frac{\sigma_{D_\mathrm{A}}}{D_0} \right)^2 - 2 D_0^6 (C_\mathcal{R} - 2 C_\mathcal{W}) \left(\frac{\sigma_{D_\mathrm{A}}}{D_0} \right)^2 \approx 2 C_\mathcal{R} D_0^6 + 6 \int \mathrm{d}x \left[\frac{\mathrm{d}^2 \delta^{(1)}_{D_\mathrm{A}}}{\mathrm{d}x^2} \right]^2 ,$$

with $\mathrm{d}x \equiv \mathrm{d}v / D_0^2$.

Application to a Swiss-cheese model

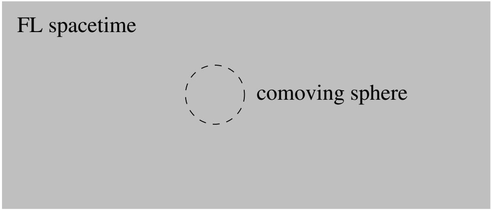
The Einstein-Straus method in brief

Construction

1. start from a homogeneous and isotropic model;

Application to a Swiss-cheese model

The Einstein-Straus method in brief



Construction

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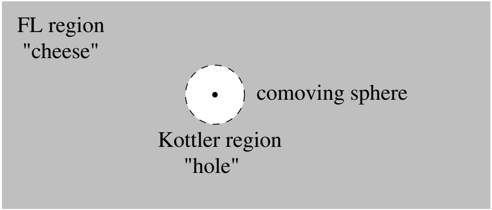

Construction

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3. concentrate the matter it contains at the center;

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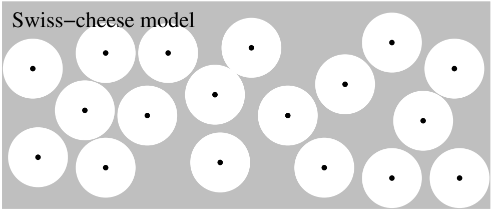

Construction

1. start from a homogeneous and isotropic model;
2. pick a comoving sphere;
3. concentrate the matter it contains at the center;
4. do it again, without overlapping holes.

Amount of holes quantified by the smoothness parameter

$$\bar{\alpha} \equiv \frac{V_{\mathrm{FL}}}{V}$$

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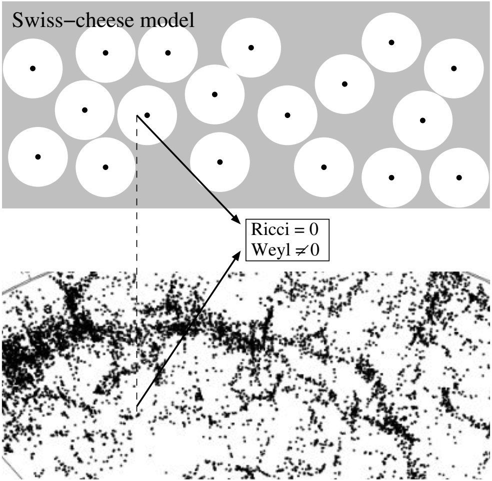

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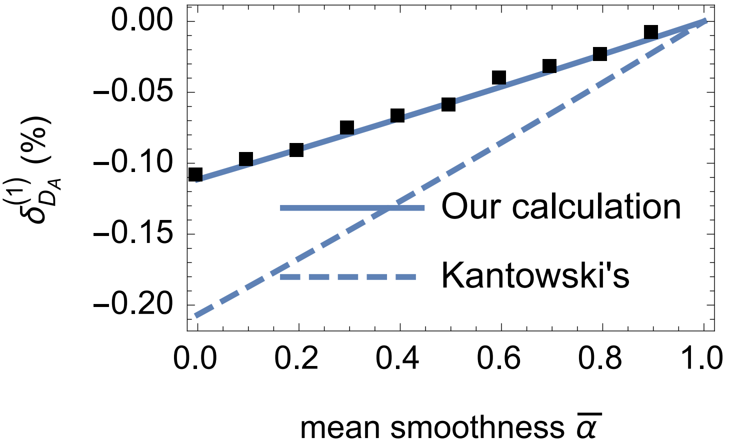

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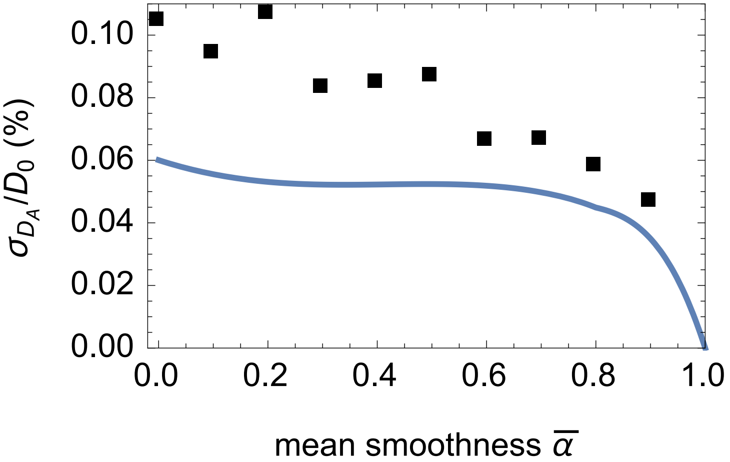
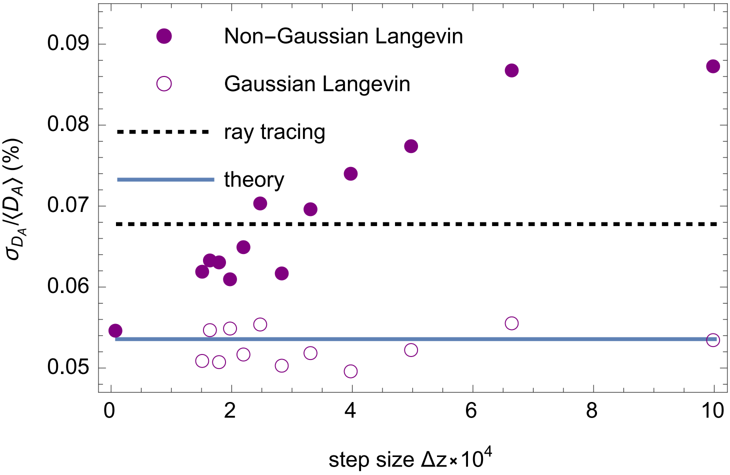


Non-gaussianity: a limitation



Conclusion

The idea

Modelling small-scale lensing as a diffusion process.

Advantages

- Simple and flexible approach.
- Paves the way to a multiscale treatment of cosmic lensing.
- Potential applications: accurate estimation of SN lensing; lensing by a stochastic background of gravitational waves, etc.

To be done

- Apply the formalism to realistic cosmological models, and compare with observations.
- Merge with the standard treatment of lensing by the large-scale structure.
- Address the problem of non-Gaussianity.

The (Kantowski-)Dyer-Roeder approximation

[Kantowski 69, Dyer & Roeder 72,73,74]

Effective distance-redshift relation in a clumpy universe.

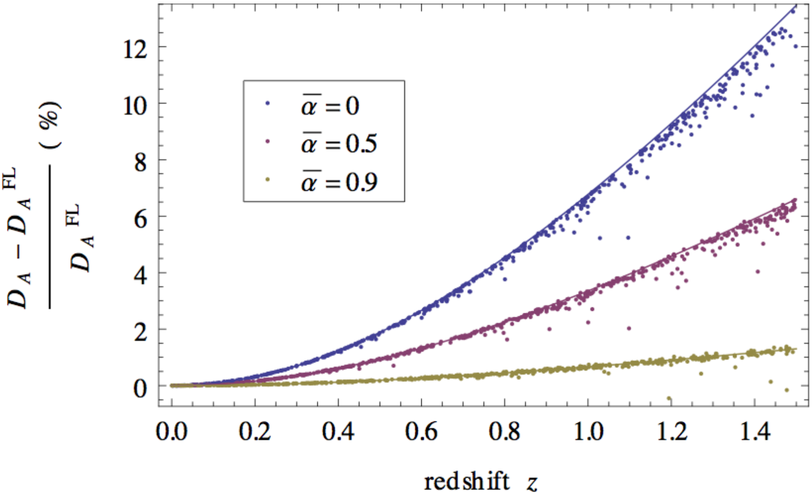
Hypotheses:

1. Affine parameter-redshift relation identical to FL.
2. Neglected Weyl focusing.
3. Reduced Ricci focusing wrt FL: $\mathcal{R}_{\text{eff}} = \alpha \mathcal{R}_{\text{FL}}$

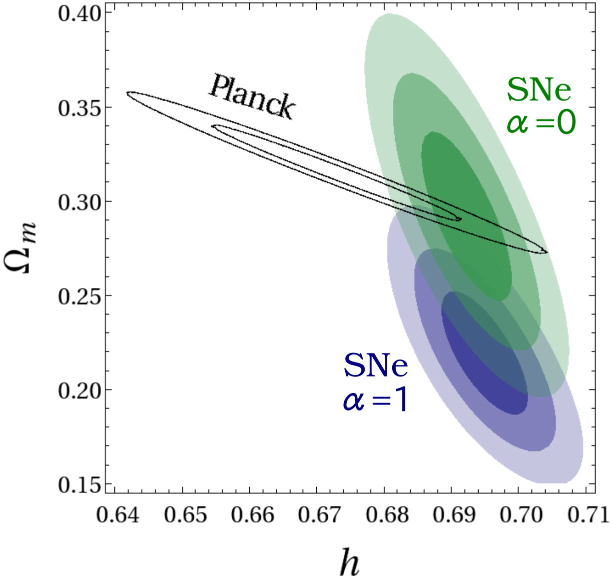
Leads to the KDR equation

$$\frac{\mathrm{d}^2 D_{\mathrm{A}}}{\mathrm{d}z^2} + \left(\frac{2}{1+z} + \frac{\mathrm{d} \ln H}{\mathrm{d}z} \right) \frac{\mathrm{d} D_{\mathrm{A}}}{\mathrm{d}z} + \frac{3\alpha\Omega_{\mathrm{m0}}}{2} \left[\frac{H_0}{H(z)} \right]^2 (1+z) D_{\mathrm{A}}(z) = 0.$$

Swiss-cheese models and the (Kantowski-)Dyer-Roeder approximation



CMB VS SNe



0.40
0.35
0.30
0.25
0.20
0.15
Ω_m
0.64
0.65
0.66
0.67
0.68
0.69
0.70
0.71
h
Planck
SNe
$\alpha = 0$
SNe
$\alpha = 1$