Introduction to Healpix

IFT SCHOOL ON COSMOLOGY TOOLS

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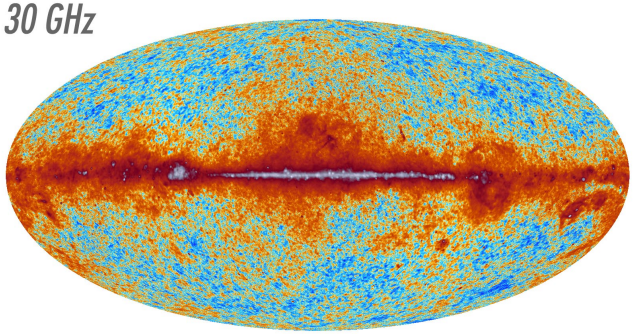
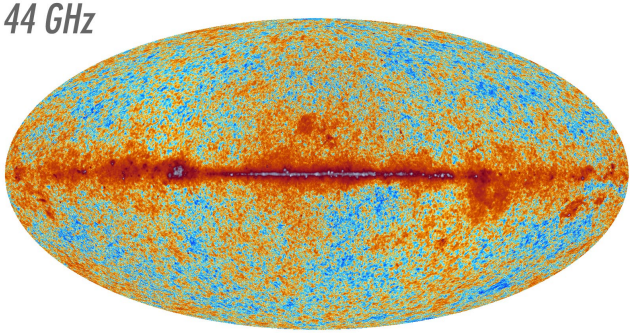
(**Hydra**: /home/prof7/HEALPIX/)

The 2015 Planck view of the sky

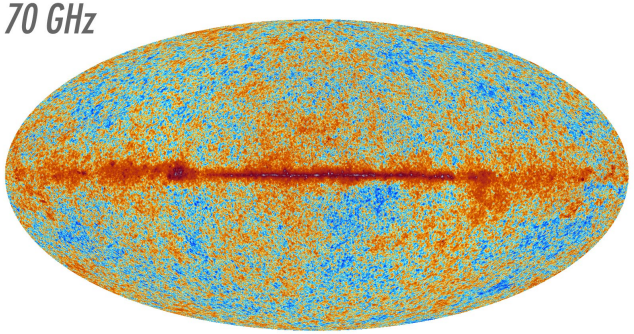
30 GHz



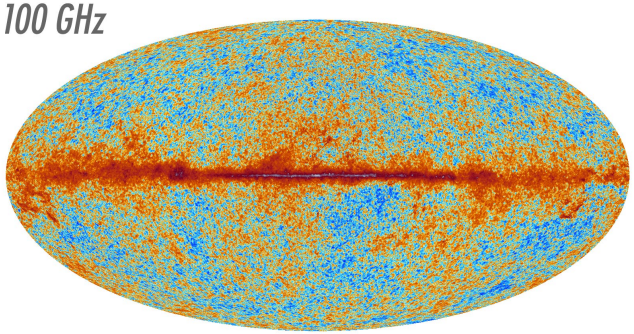
44 GHz



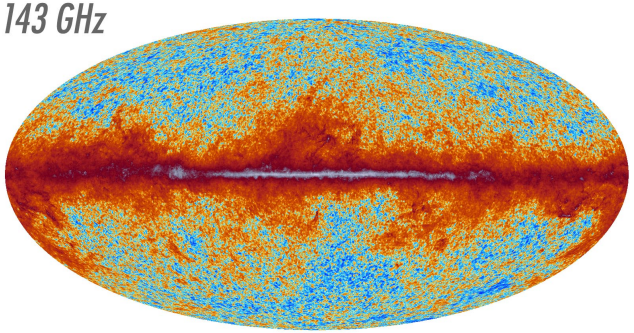
70 GHz



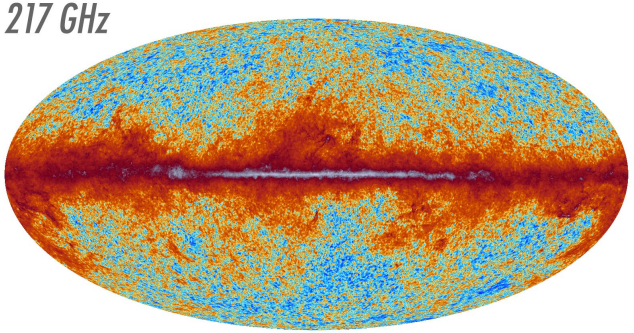
100 GHz



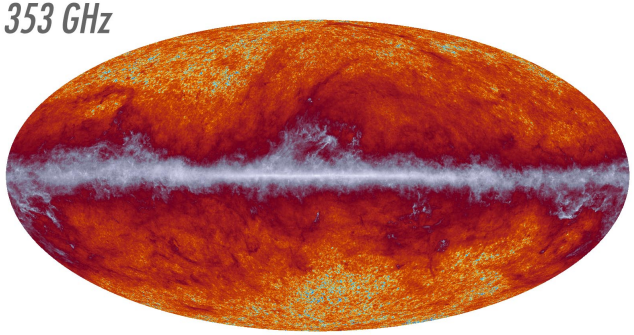
143 GHz



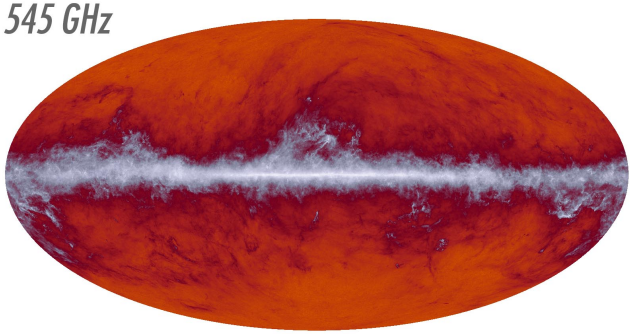
217 GHz



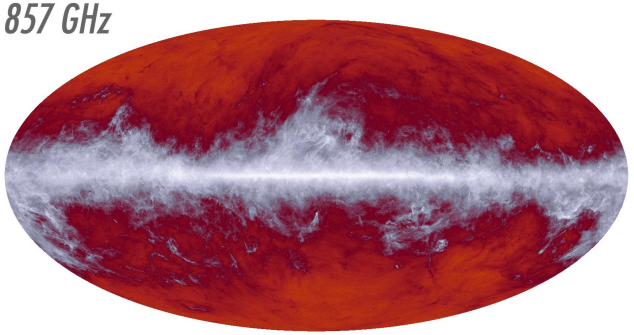
353 GHz

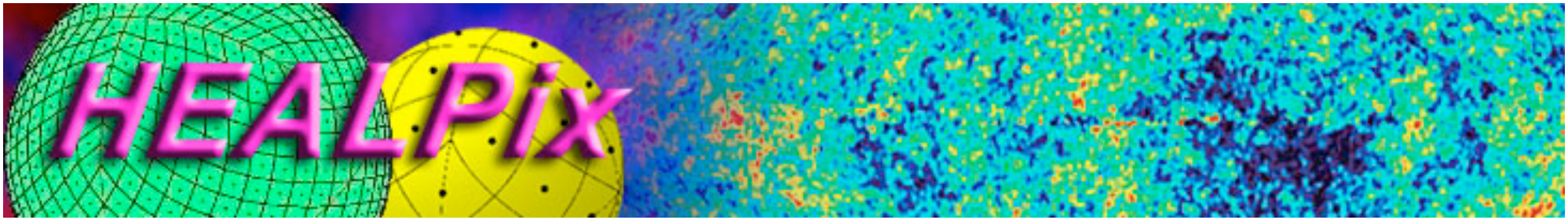


545 GHz



857 GHz



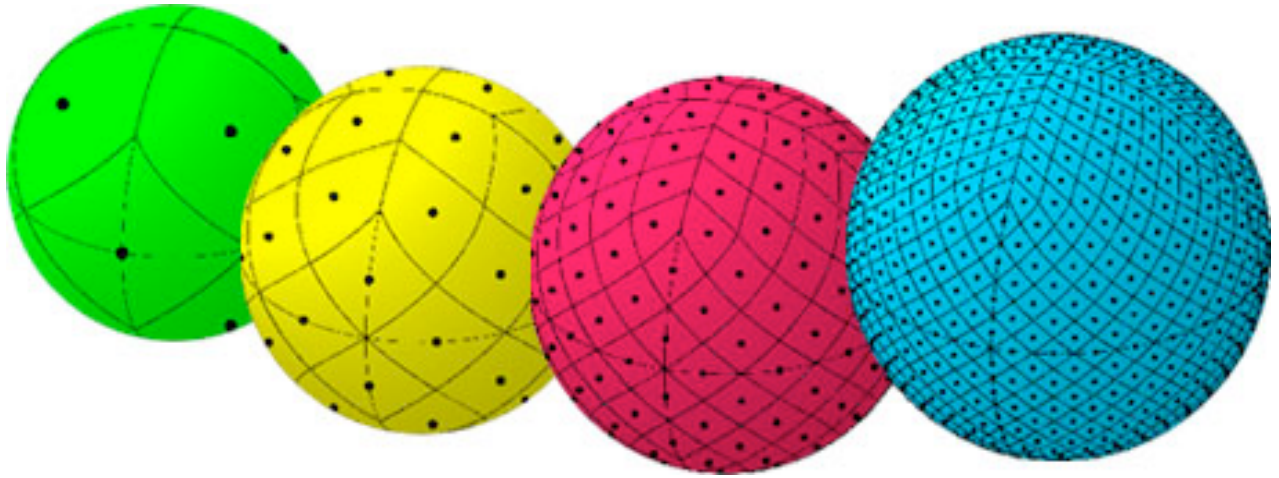


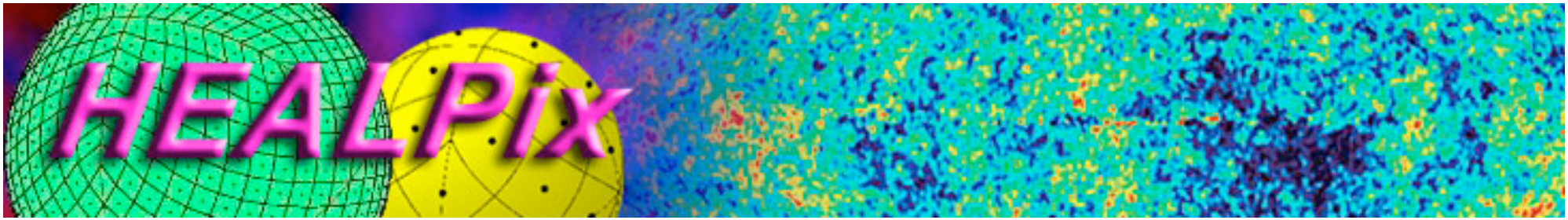
I. ***Introduction.***

II. ***Visualization routines overview: maps related tools.***

III. ***Subroutines overview: pixels related tools.***

IV. ***Spherical harmonics transforms. Power spectrum, a_{lm}, window functions.***





Useful links (maps and tools):

PLANCK: http://pla.esac.esa.int/

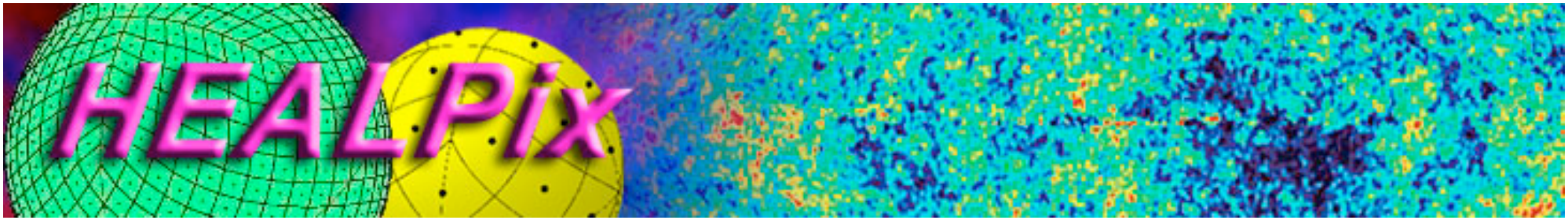
WMAP: http://lambda.gsfc.nasa.gov/product/map/current/

LAMBDA: http://lambda.gsfc.nasa.gov/
Foregrounds: http://lambda.gsfc.nasa.gov/product/foreground/

Planck Sky Model (PSM): http://www.apc.univ-paris7.fr/~delabrou/PSM/psm.html

CAMB web interface: https://lambda.gsfc.nasa.gov/toolbox/tb_camb_form.cfm

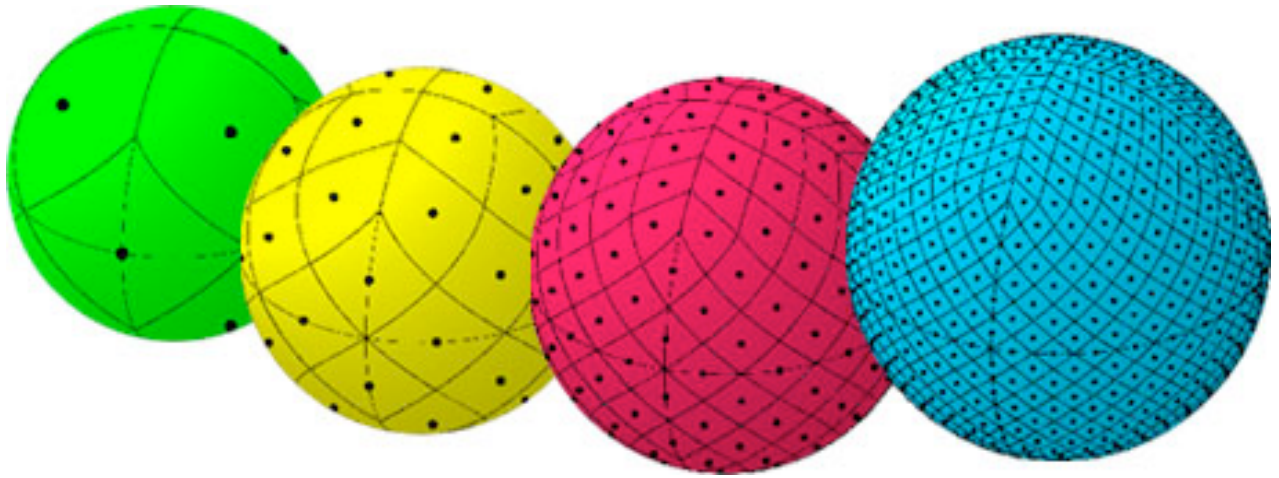
I. Introduction

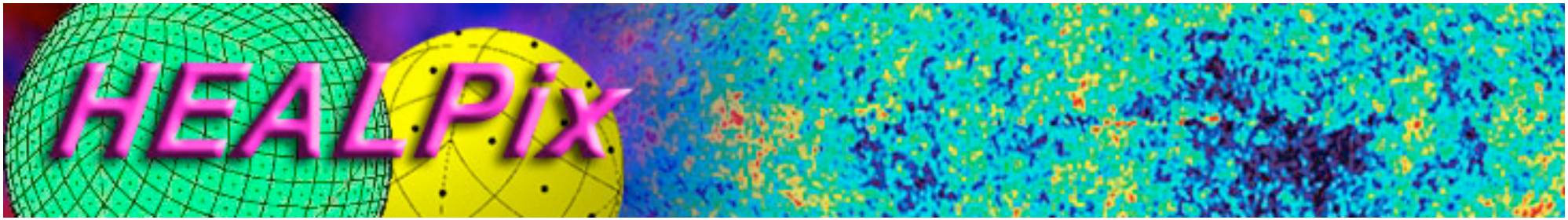


- **HEALPix** is an acronym for **H**ierarchical **E**qual **A**rea iso**L**atitude **Pix**elization of a sphere.
- Software for pixelization, hierarchical indexation, synthesis, analysis, and visualization of data on the sphere.
- Please acknowledge HEALPix by quoting the web page (http://healpix.sourceforge.net) and the publication: K.M. Gorski et al., 2005, ApJ., 622, 759.

- Routines written in C, C++, F90, IDL, Python and Java.
- Python version (based on the C++ library): **Healpy** (http://healpy.readthedocs.io/)

- Here I will focus on F90, IDL and Python routines mainly.



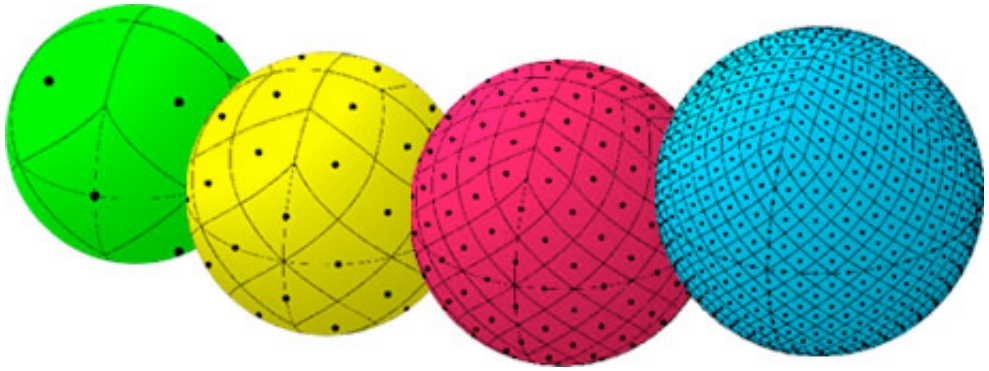


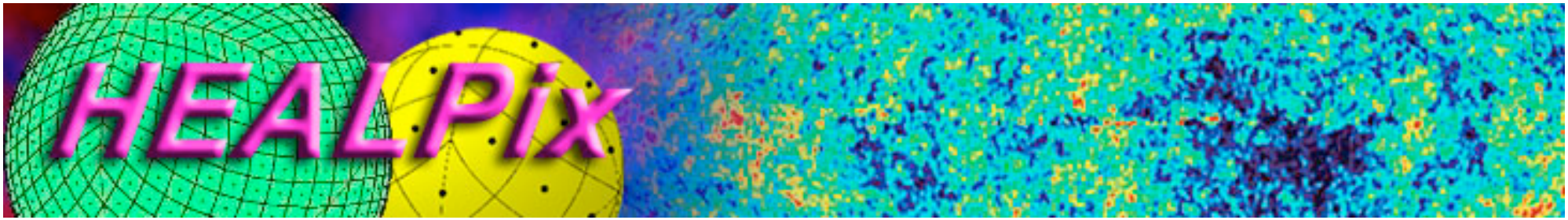
Basic properties:

1. The sphere is **hierarchically tessellated** into curvilinear quadrilaterals. The lowest resolution partition is comprised of 12 base pixels. Resolution of the tessellation increases by division of each pixel into four new ones.

2. Areas of all pixels at a given resolution **are identical**.

3. Pixels are distributed on lines of constant latitude. This property is essential for all harmonic analysis applications involving spherical harmonics. Due to the iso-latitude distribution of sampling points the speed of computation of integrals over individual spherical harmonics scales as $\sim N^{1/2}$ with the total number of pixels, as opposed to the $\sim N$ scaling for the non-iso-latitude sampling distributions.





Definitions.

1. Nside. The resolution of the grid is expressed by this parameter. Nside defines the number of divisions along the side of a base-resolution pixel that is needed to reach a desired high-resolution partition.

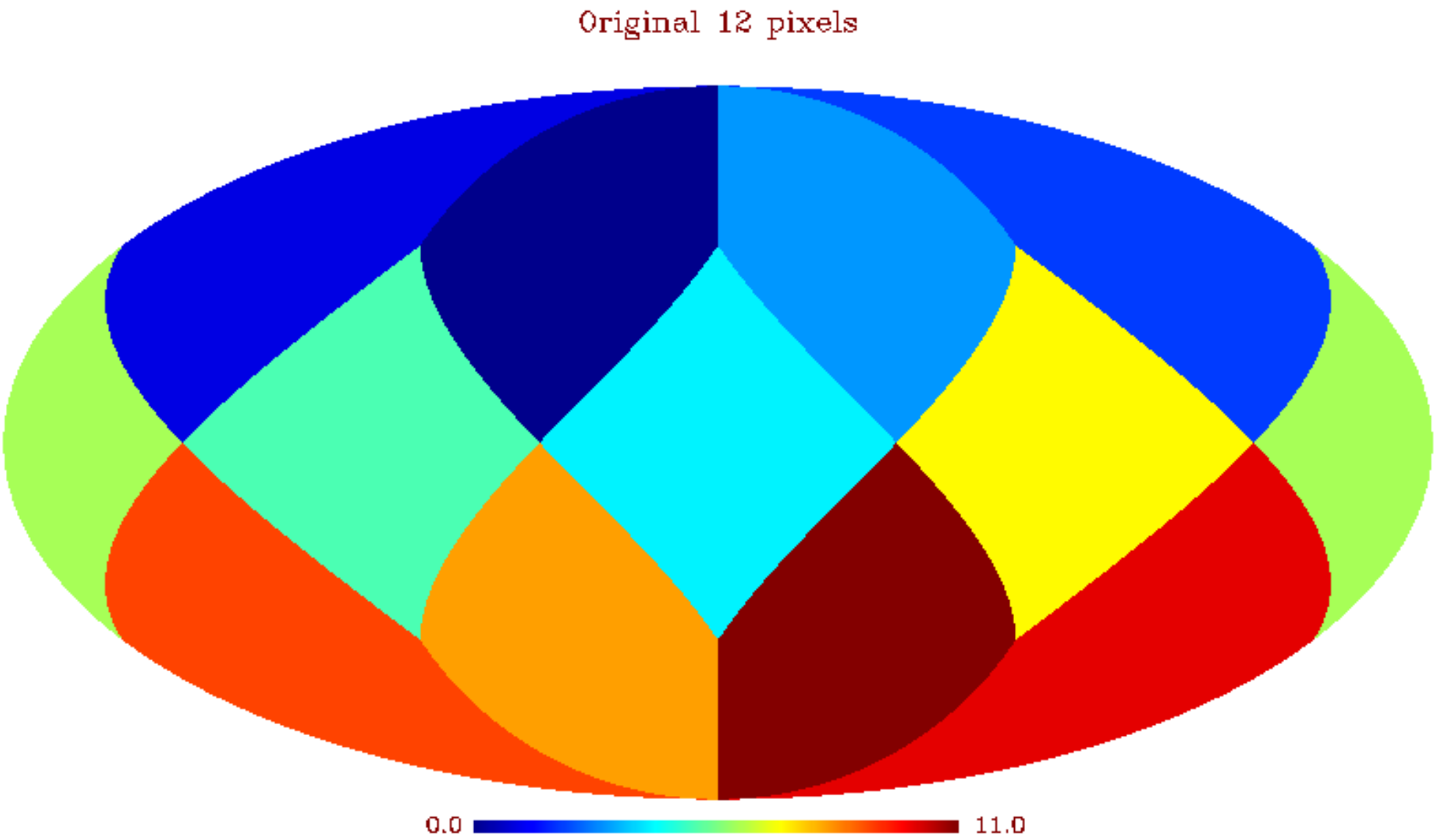
2. Nring. For a given value of Nside, all pixel centers are located on Nring = 4Nside-1 rings of constant latitude.

3. Npix. Denotes the total number of pixels. Npix=12*Nside^2.

4. Pixel numbering schemes (= "**ordering**").

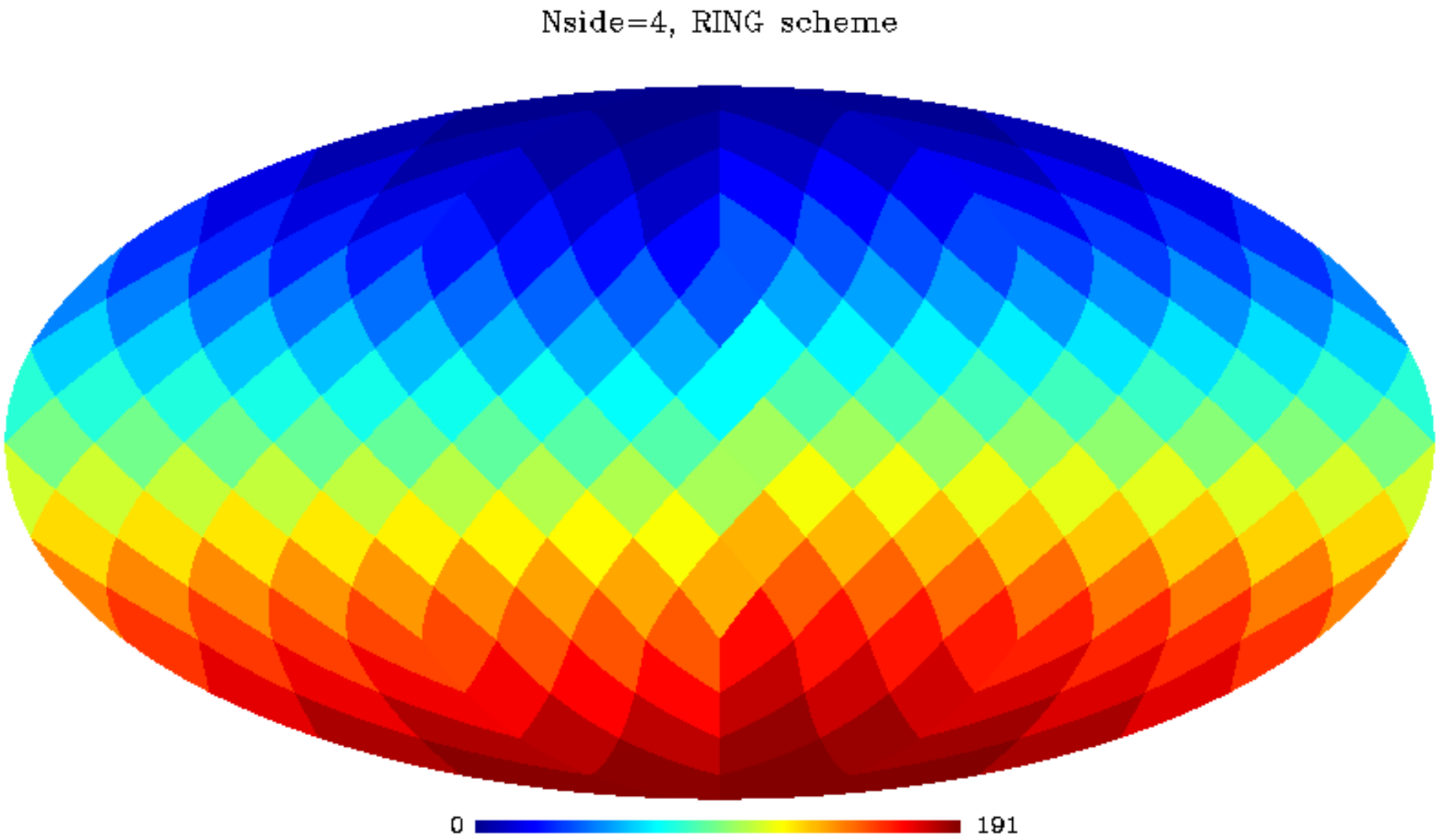
1. **RING scheme**: counting the pixels moving down from the north to the south pole along each iso-latitude ring. For this scheme, Fourier transforms with spherical harmonics are easy to implement.
2. **NESTED scheme**: arranging the pixel indices in twelve tree structures, corresponding to base-resolution pixels. Very useful for efficient applications involving nearest-neighbour searches.

healpix_intro.py; healpix_intro.pro



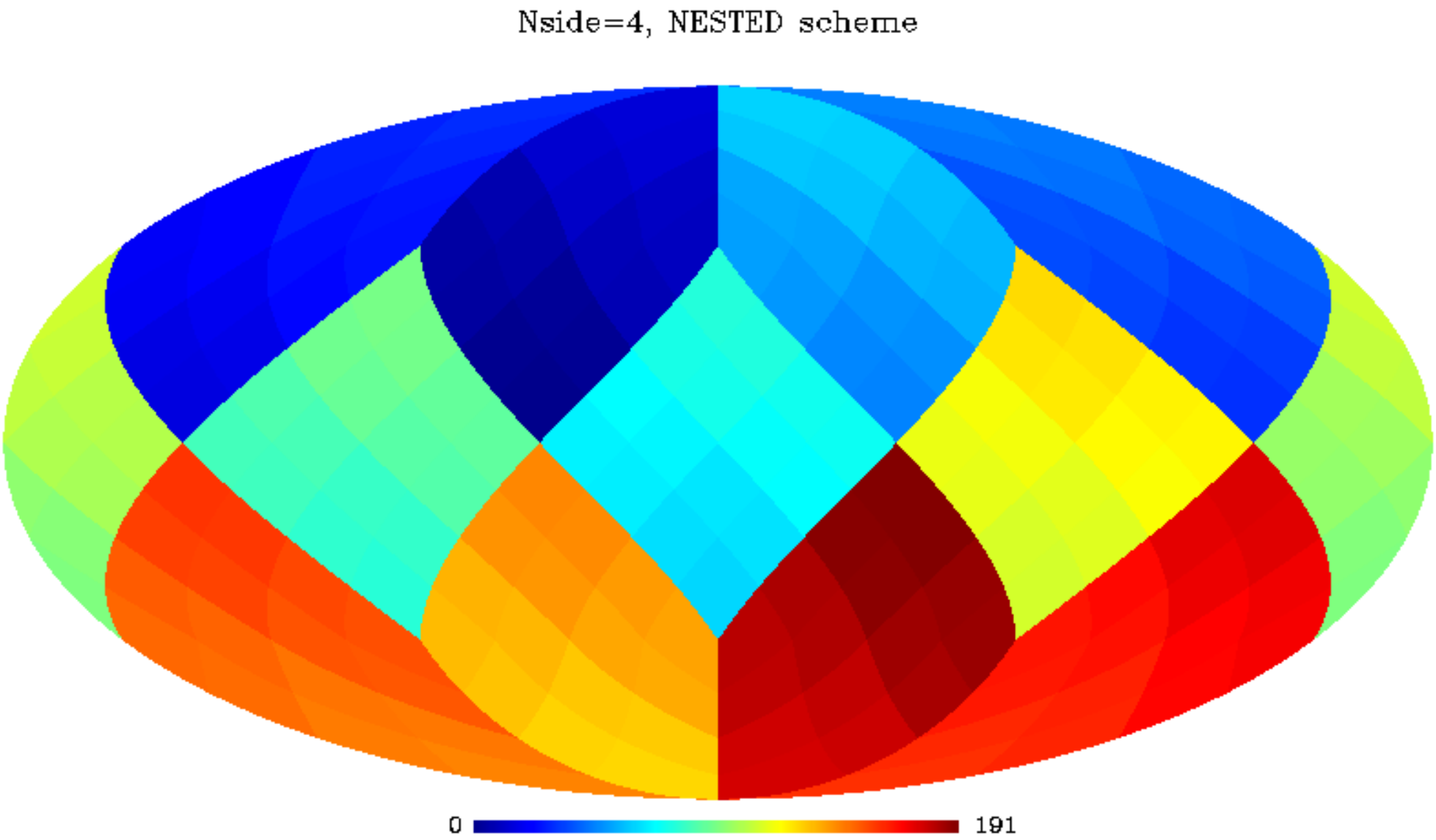
Base12 pixels in mollview projection.

healpix_intro.py; healpix_intro.pro

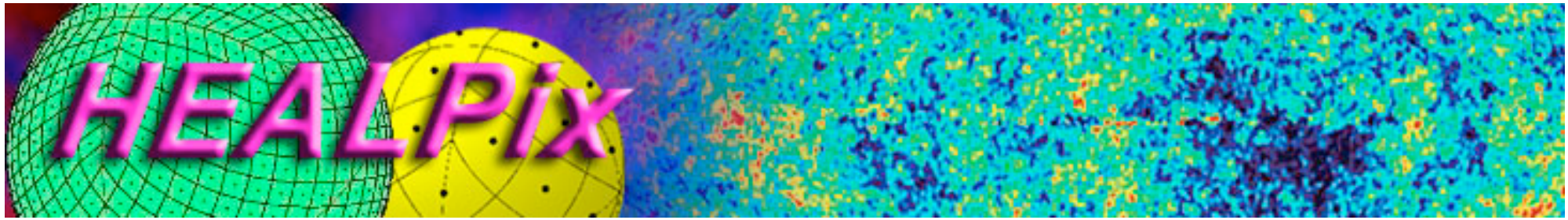


Nside=4, Npix=192

healpix_intro.py; healpix_intro.pro



Nside=4, Npix=192



Conventions

1.Expansion in spherical harmonics.

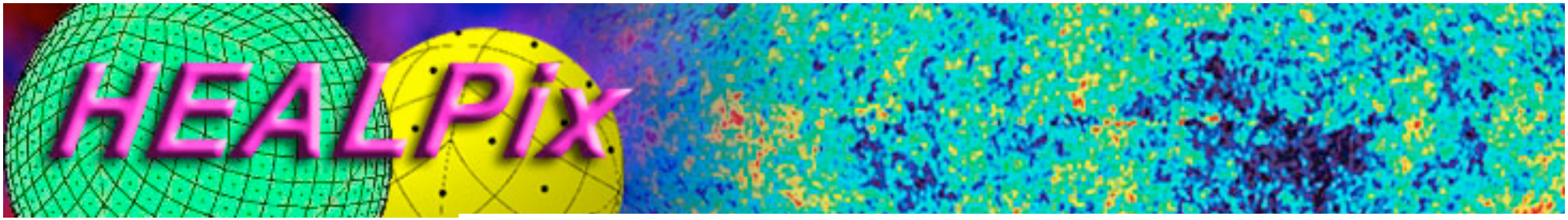
$$f(\gamma) = \sum_{\ell=0}^{l_{max}} \sum_{m} a_{\ell m} Y_{\ell m}(\gamma) \qquad \hat{a}_{\ell m} = \frac{4\pi}{N_{\rm pix}} \sum_{p=0}^{N_{\rm pix}-1} Y^*_{\ell m}(\gamma_p) f(\gamma_p)$$

γ denotes a unit vector pointing at polar angle $\theta \in [0, \pi]$ and azimuth $\phi \in [0, 2\pi)$.

2.Spherical harmonics: $Y_{\ell m}(\theta, \phi) = \lambda_{\ell m}(\cos\theta) \mathrm{e}^{im\phi}$

$$\begin{aligned} \lambda_{\ell m}(x) &= \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_{\ell m}(x), \quad \text{for } m \geq 0 \\ \lambda_{\ell m} &= (-1)^m \lambda_{\ell |m|}, \quad \text{for } m < 0, \\ \lambda_{\ell m} &= 0, \quad \text{for } |m| > \ell. \end{aligned}$$

3.Angular Power Spectrum. $\hat{C}_\ell = \frac{1}{2l+1} \sum_m |\hat{a}_{\ell m}|^2$

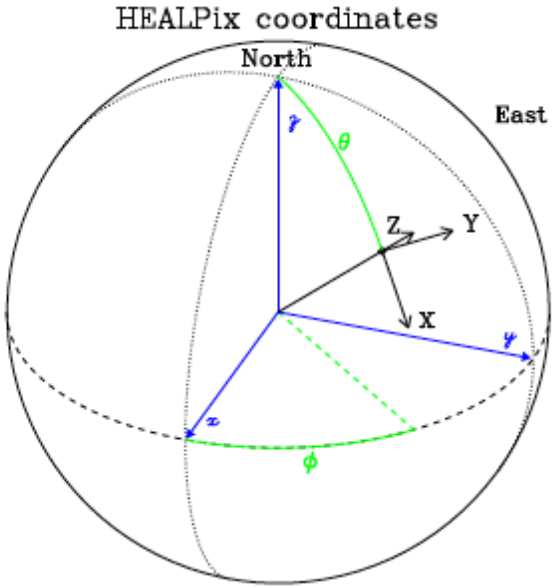


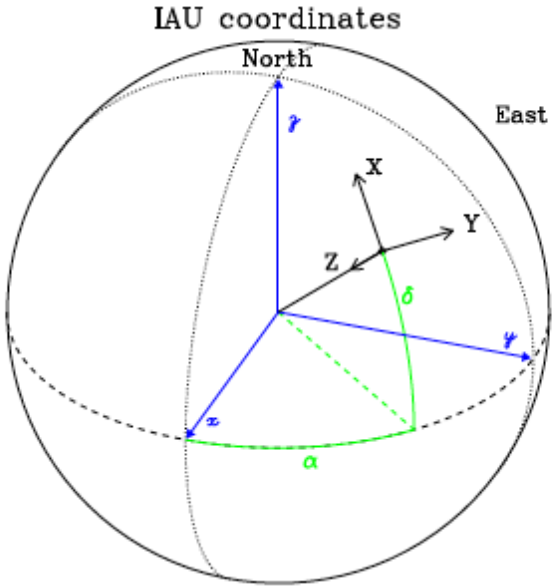
Conventions

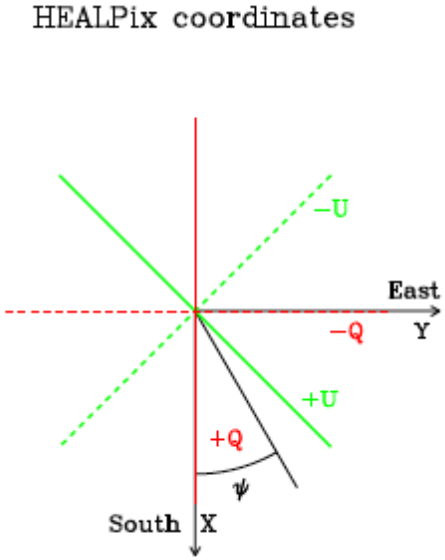
4.Polarization.

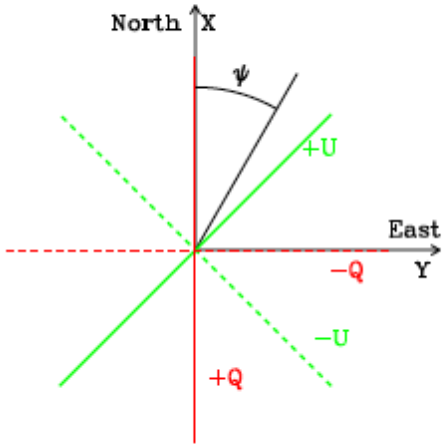
a) Q > 0; U = 0. It will be along the North-South axis.
b) Q = 0; U > 0. It will be along a North-West to South-East axis.

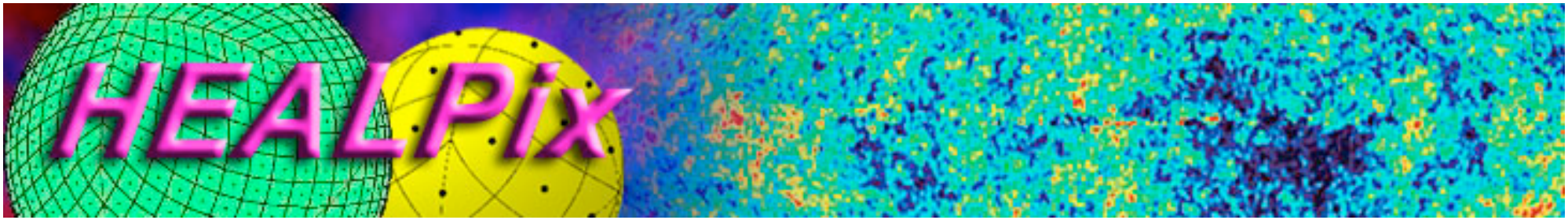
Note the change of sign U→ -U.











Maps, powers spectra and alms are stored in FITS format.

- FITS support office https://fits.gsfc.nasa.gov/
- MAPS. Specific keywords (NSIDE, ORDERING, COORD in ext header).

I/O routines (HIDL/HGDL):

- Read:
  - read_fits_cut4, read_fits_map, read_fits_s
  - read_tqu
- Write:
  - write_fits_cut4, write_fits_map, write_fits_sb
  - write_tqu
- Others:
  - getsize_fits
  - change_polcconv

- **IDL**: healpix_io.pro
- **Python**: healpix_io.py

I/O routines (Python):

- read_map, read_alm, read_cl, mrdfits
- write_map, write_alm, write_cl, mwrfits

II. Visualization routines. Maps related tools

Visualization (HIDL/HGDL):

- Gnomic, Mollweide, Cartesian, orthographic and azimuthal equatorial projections:
  - mollview
  - gnomview
  - cartview
  - orthview
  - azeqview
- Interactive cursor for all of them:
  - cartcursor
  - mollcursor
  - gnomcursor
  - orthcursor
- HEALPix maps in Google Sky and Dome Master format: hpx2gs, hpx2dm.
- Planck colors. (HIDL> planck_colors,1, /show)

Visualization (pre-compiled F90):

- map2gif (http://healpix.sourceforge.net/html/facilitiesnode8.htm)

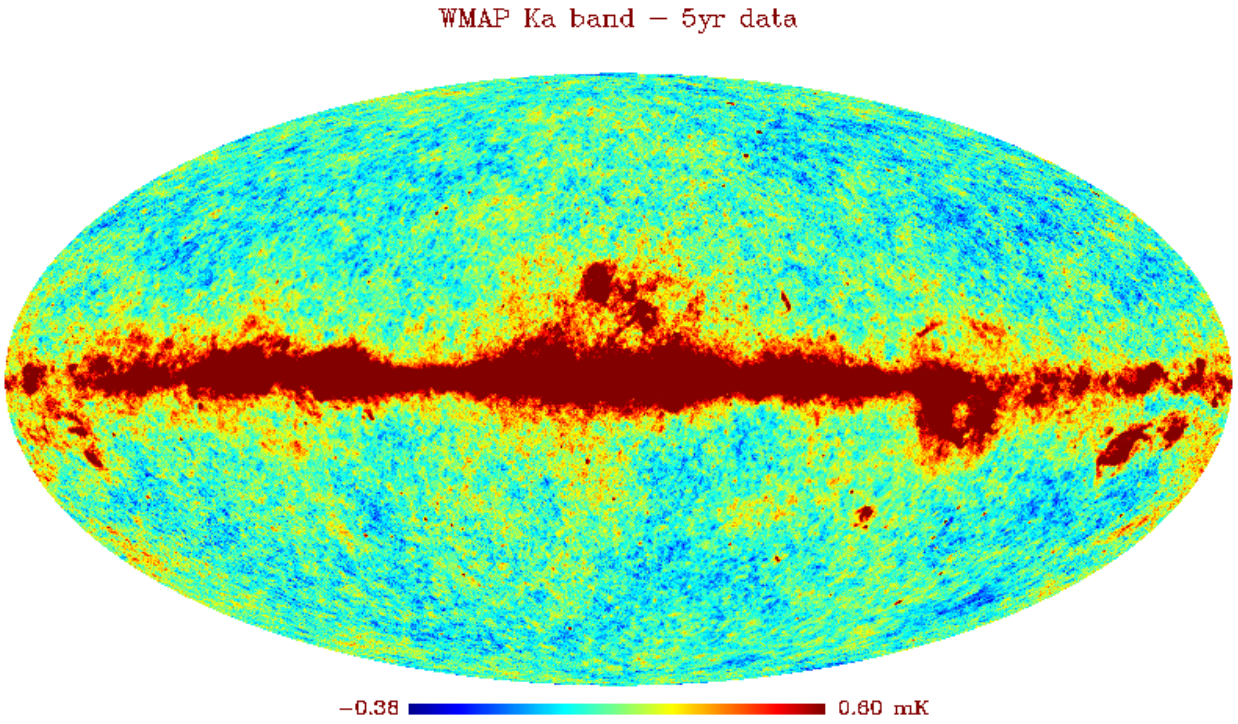
Visualization (Python): http://healpy.readthedocs.io/en/latest/healpy_visu.html

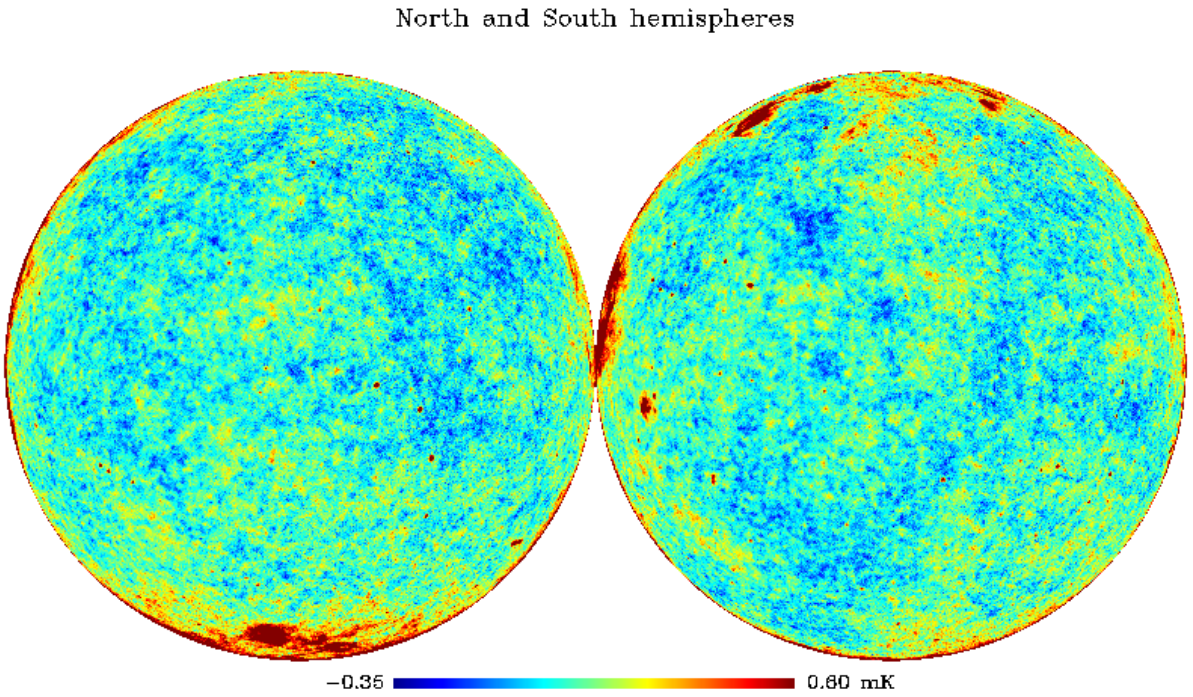
- Gnomic, Mollweide, Cartesian, orthographic projections:
  - mollview
  - gnomview
  - cartview
  - orthview

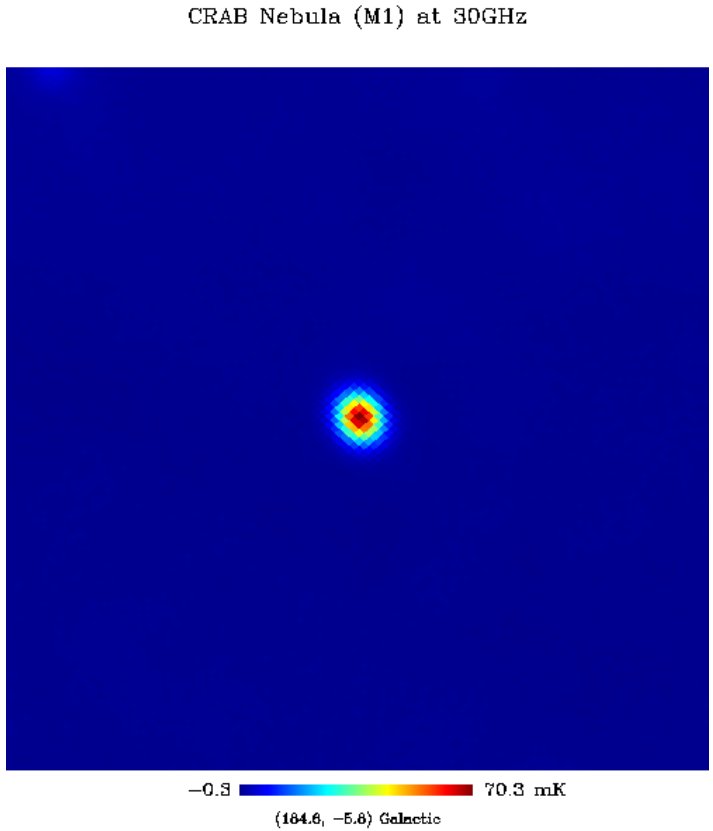
- Graticule:
  - graticule
  - delgraticule

- Interactive:
  - mollzoom

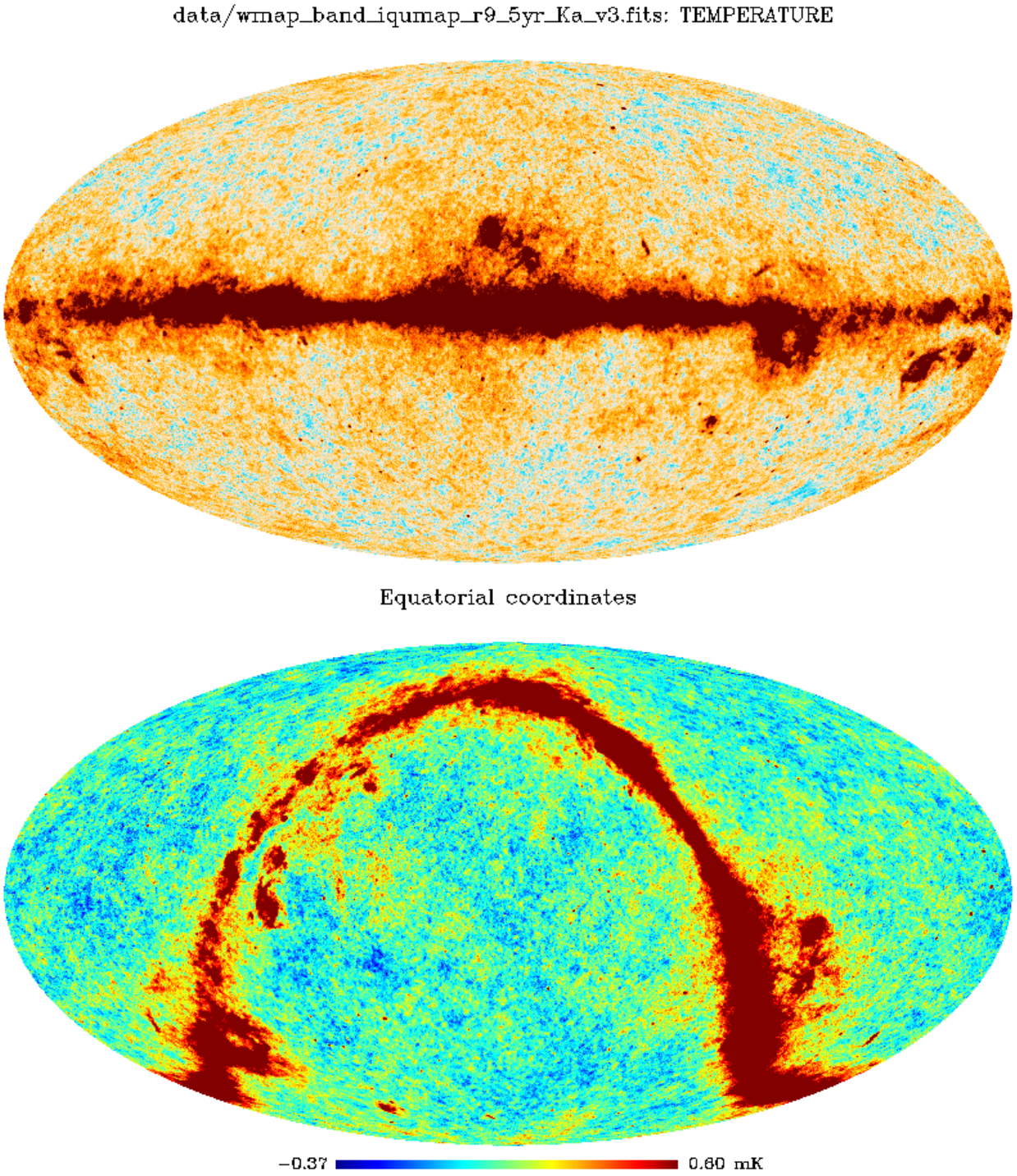
- Masked maps:
  - UNSEEN
  - ma(). Numpy MaskedArray class

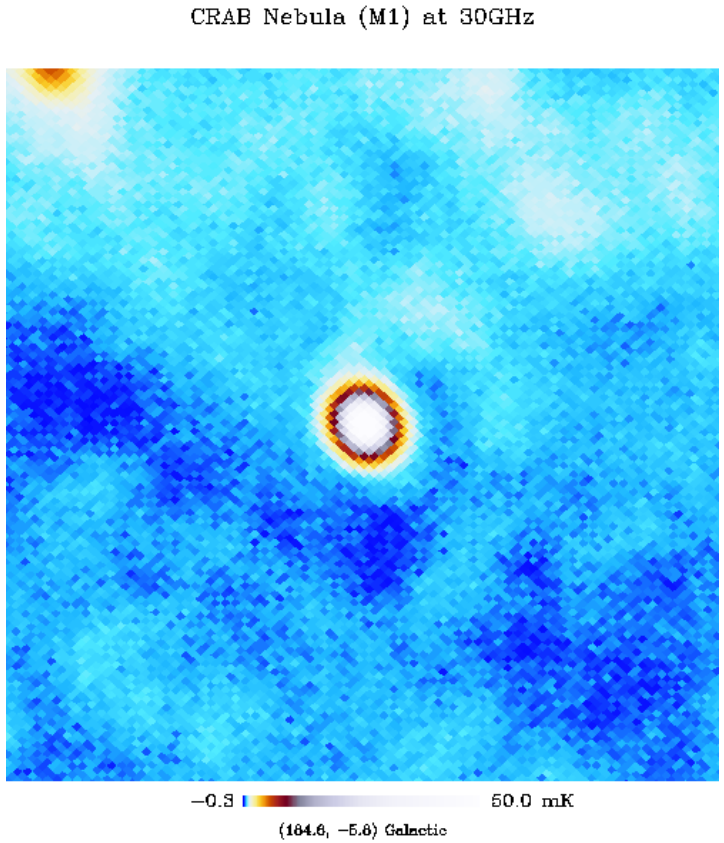






- **IDL**: healpix_visu.pro
- **F90**: healpix_visu.exe
- **Python**: healpix_visu.py





- **IDL**: healpix_visu.pro
- **F90**: healpix_visu.exe
- **Python**: healpix_visu.py

III. Subroutines overview. Pixels related tools

Pixel related tools in IDL (similar routines in F90, C and Python)

(http://healpix.sourceforge.net/html/idl.htm)

- init_healpix

- Coordinate tools: ang2vec, euler_matrix_new, rotate_coord, vec2ang

- Coordinates to pixel transforms, and back: nside2npix, npix2nside, ang2pix_*, pix2ang_*, pix2vec_*, vec2pix_*

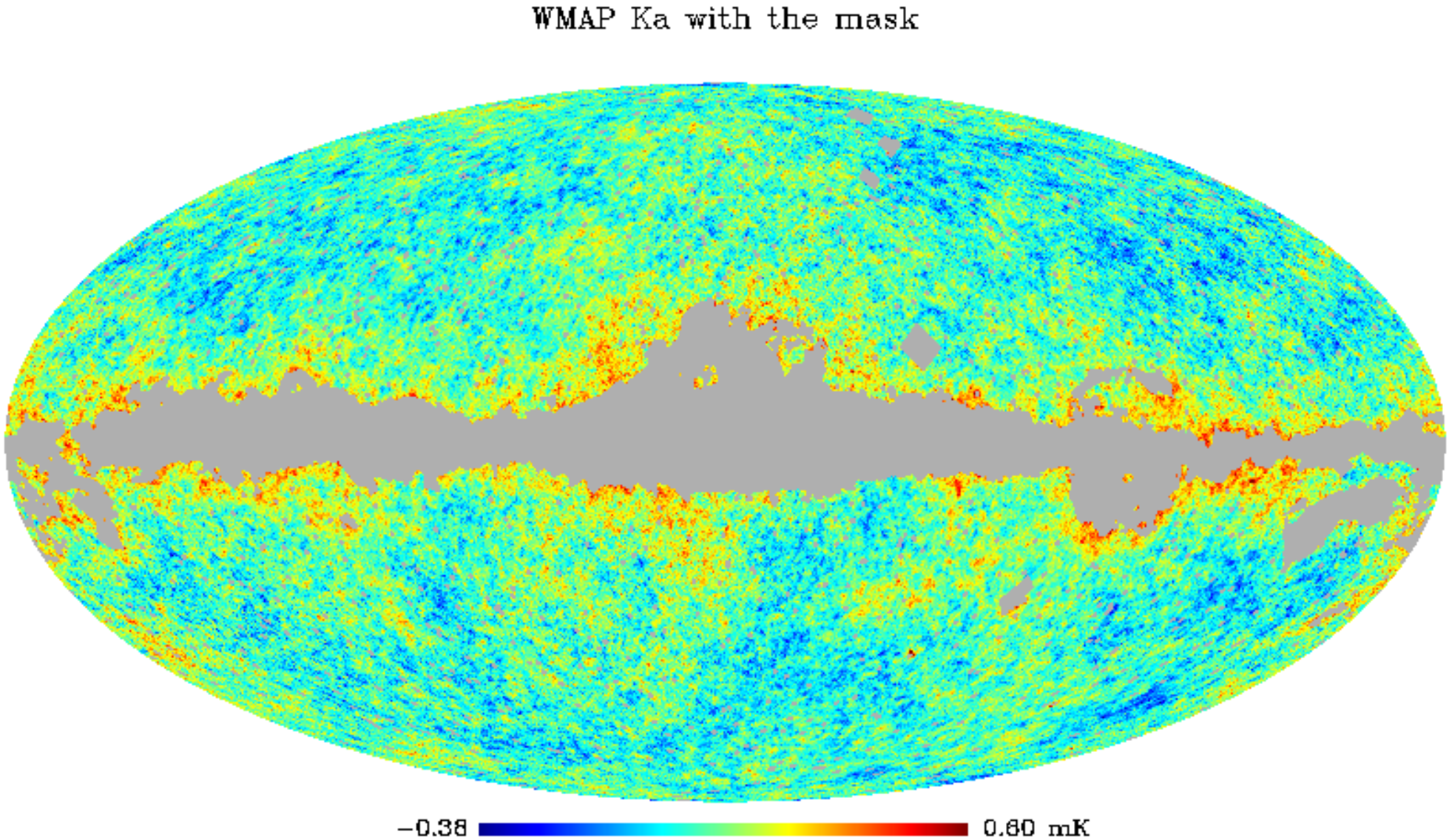
- RING/NESTED transforms: nest2ring, ring2nest

- Neighbouring pixels: neighbours_nest, neighbours_ring

- Pixel query within a disc, polygon, strip or triangle: query_disc, query_polygon, query_strip, query_triangle.

- Template pixels: nside2templates, same_shape_pixels_ring, same_shape_pixels_nest, template_pixel_ring, template_pixel_nest

- Pixel pro/down-grading and NESTED/RING pixel reordering: ud_grade, reorder



IDL: healpix_pixel.pro
Python: healpix_pixel.py

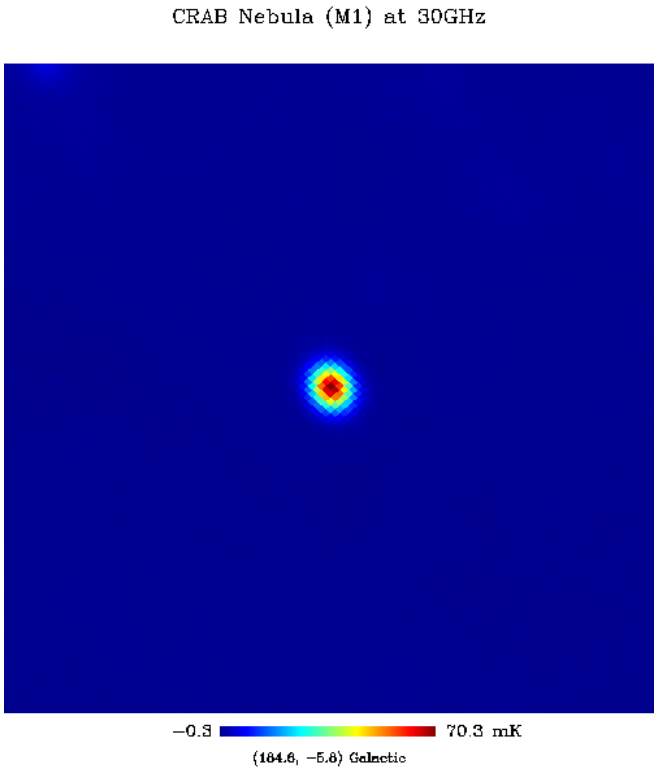
Exercise: Aperture photometry of a radio-source in the WMAP maps. Compute the flux density associated to the CRAB nebula at 30GHz, by using an aperture of 1º radius.

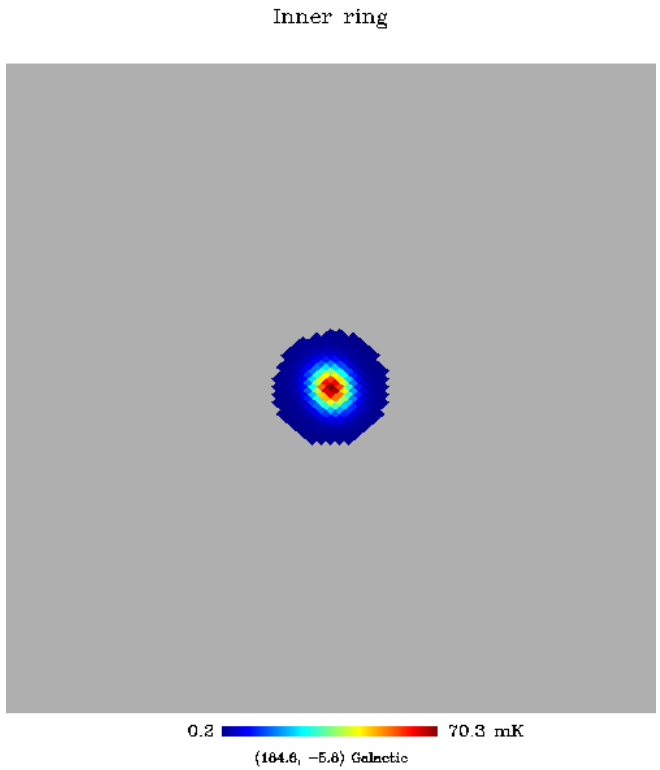
Solution: crab_flux.pro. The derived value is 325.6 Jy, fully consistent with the result of Page et al. (2003), based on WMAP3, of 322 ± 6 Jy.

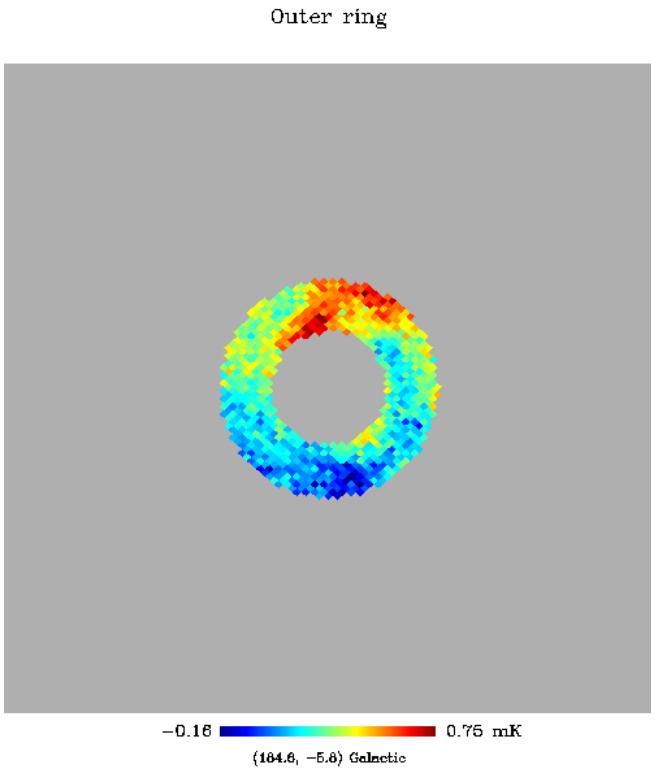
IDL: crab_flux.pro
Python: crab_flux.py

Definitions: 1Jy = 10^{-26} W/m^2/Hz.

$$S_\nu = \frac{2k_B}{\lambda^2}\int T_A(\theta,\phi)d\Omega$$







IV. Spherical harmonics transforms: power spectra, a_{lm} and window functions

Power spectra, a_{lm} and window functions

Fortran stand-alone facilities. http://healpix.sourceforge.net/html/facilities.htm

- anafast
- synfast
- smoothing
- hotspot
- ud_grade

IDL routines. http://healpix.sourceforge.net/html/idl.htm

- Spherical Harmonics analysis and synthesis: ianafast, isynfast
- Smoothing and filtering: ismoothing, median_filter, remove_dipole

- Window function generation: gaussbeam, beam2bl, bl2beam, healpixwindow
- C(l) binning: bin_llcl
- alm tools: alm_i2t, alm_t2i, index2lm, lm2index,
- C(l), B(l) and a_{lm} I/O: fits2cl/cl2fits, bl2fits, fits2alm/alm2fits

Power spectra, a_{lm} and window functions

Python routines. http://healpy.readthedocs.io/en/latest/healpy_spht.html

- Spherical Harmonics analysis and synthesis: anafast, synfast
- Smoothing and filtering: smoothing

- Window function generation: gaussbeam, pixwin.
- alm tools: alm2map, map2alm
- Power spectra: alm2cl, almxfl

SPHERICAL HARMONICS, POWER SPECTRUM, COSMIC VARIANCE

$$\frac{T(\mathbf{r}_0,\hat{\mathbf{n}})-T_0(\mathbf{r}_0)}{T_0(\mathbf{r}_0)}=\sum_{l>0}\sum_{m=-l}^{+l}a_{lm}(\mathbf{r}_0)Y_l^m(\hat{\mathbf{n}})$$

$$\left\langle |a_{lm}(\mathbf{r})|^2\right\rangle=\left\langle a_{lm}(\mathbf{r})a^*_{l'm'}(\mathbf{r})\right\rangle\equiv\delta_{ll'}\delta_{mm'}C_l$$

$$\begin{aligned}
C_l&=\left\langle |a_{lm}(\mathbf{r})|^2\right\rangle= && \text{isotropy}\\
&=\left\langle\frac{\sum_m|a_{lm}(\mathbf{r})|^2}{2l+1}\right\rangle= && \text{ergodicity}\\
&=\left\langle\frac{\sum_m|a_{lm}(\mathbf{r})|^2}{2l+1}\right\rangle_{\text{space}}\simeq\\
&\simeq\frac{\sum_m|a_{lm}(\mathbf{r}_0)|^2}{2l+1}.
\end{aligned}$$

WINDOW FUNCTION

The instrumental response of an imaging device can be written as the convolution in real space of the actual (underlying) temperature of the sky emission with the instrumental response (beam).

$$\Delta T^{obs}(\hat{n}) = \int \Delta T(\hat{n}) B(|\hat{n} - \hat{n}'|) d\Omega_{\hat{n}'}$$

In Fourier space, the equation implies that

$$C_\ell^{obs} = B_\ell^2 C_\ell$$

Where B_l is the harmonic transform of the beam.

There is an equivalent effect due to the finite pixelization of the map. If the pixel size is small compared to the signal correlation length (determined by the beam size), then the pixel effect can be written as a window function with no m-dependence. This function, for Healpix, is provide by the routine healpixwindow.pro.

Cosmic variance (e.g. Knox 1995)

$$Var(C_\ell) = \frac{2}{2\ell+1} C_\ell^2$$

For the case of beam and instrumental noise, the expression changes to:

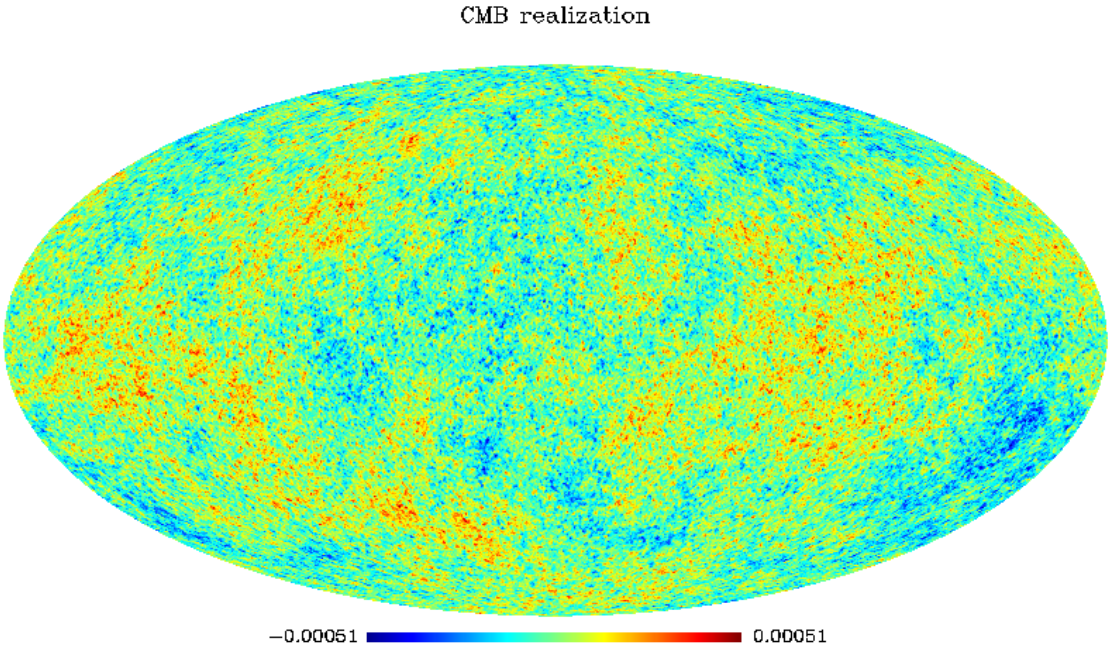
$$Var(C_\ell) = \frac{2}{2\ell+1} \left(C_\ell + \frac{1}{w B_\ell^2} \right)^2$$

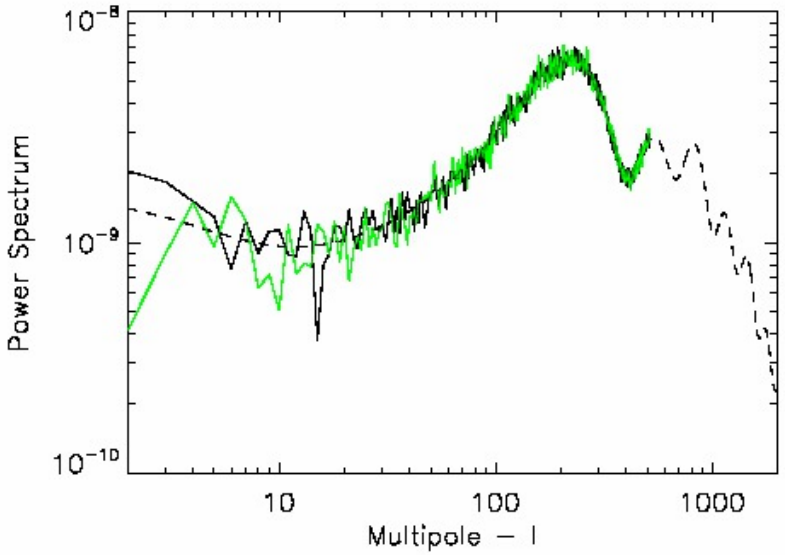
$$w^{-1} = \sigma_{noise}^2 \Omega_{pix}$$

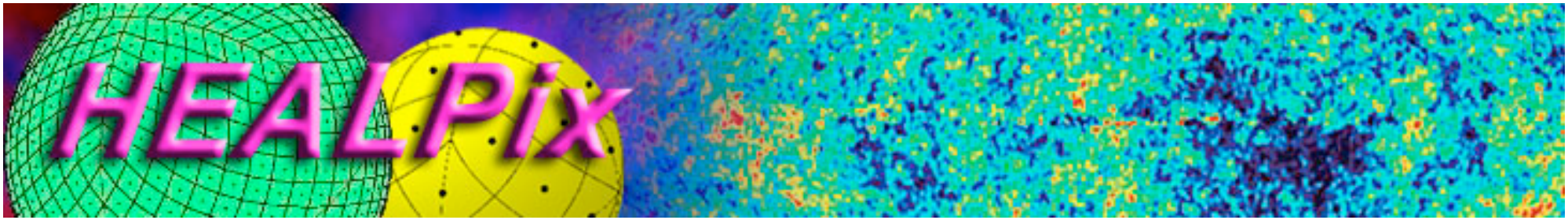
Exercise.
1. Starting from a given cosmological model, make a realization of a CMB map.
2. Repeat the exercise for several realizations. Concept of cosmic variance.
3. Repeat the exercise including the effect of the instrumental beam.

Solutions: cosmic_variance.pro ; cosmic_variance.py

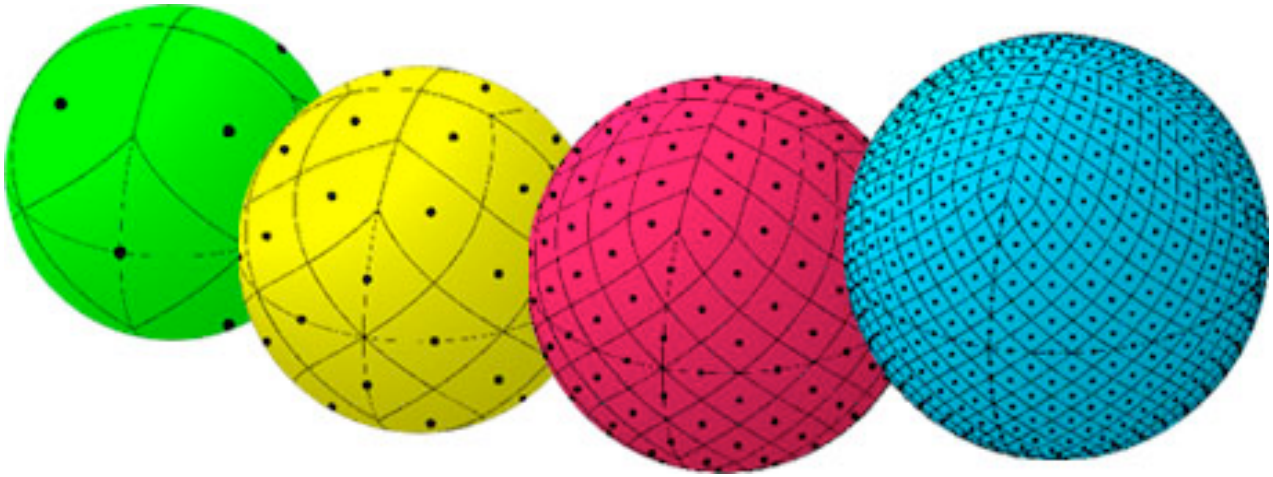
Tools: CAMB web interface: https://lambda.gsfc.nasa.gov/toolbox/tb_camb_form.cfm

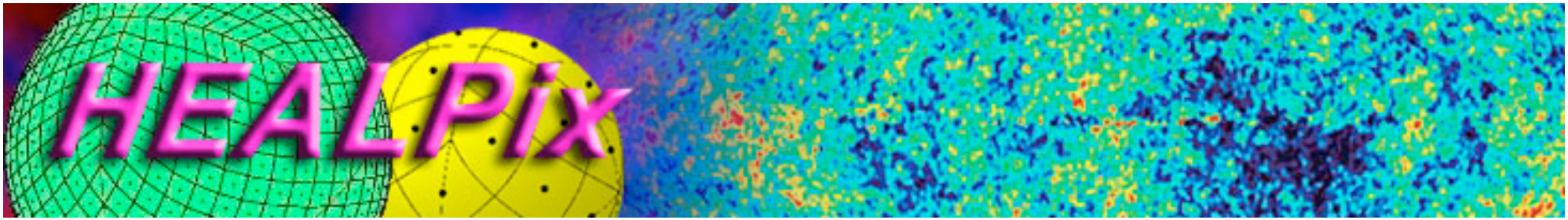






- Complete documentation of the package can be found at http://healpix.sourceforge.net and http://healpy.readthedocs.io/





Using Healpix GDL in HYDRA.

module load gsl
module load fftw
source .profile

Using HEALPY in HYDRA.

module load scipy
module load numpy
module load healpy

Solutions to exercises in HYDRA:

/home/prof7/HEALPIX/

The IFT School on Cosmology tools. Madrid, 13-17 March 2017.
José Alberto Rubiño Martín

HEALPIX EXERCISES

1. **Aperture photometry from a Healpix map.** Write a code to extract the flux of a compact (point-like) source from a Healpix map. Apply the code to compute the flux of the CRAB nebulae at 30GHz using the WMAP maps. Compare the result with that obtained with the Planck 30GHz map.

2. **Two point correlation function.**
   a. Starting from the harmonic decomposition of the temperature map ($\Delta T(n)=\Sigma\, a_{lm}\, Y_{lm}(n)$), derive the analytic equation for the two-point correlation function, $C(\theta)$, defined as $C(\theta) = < \Delta T(n_1)\, \Delta T(n_2) >$, where θ is the angle defined by the two directions n_1 and n_2. Hint: you will need to use the Spherical Harmonic Addition Theorem. The final solution should be:

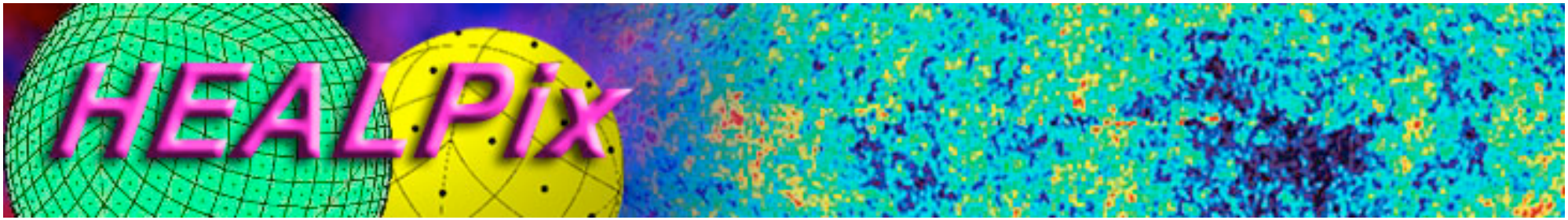
$$C(\theta) = \sum_{\ell} \frac{2\ell+1}{4\pi} C_\ell P_\ell(cos\theta)$$

   b. Plot this function $C(\theta)$ for two different cosmological models generated with CAMB. Model1 and model2 should have identical cosmological parameters except for the baryon density: Ω_b(model 1)=0.04 and Ω_b(model 2)=0.08. What angular scales are different between those two plots?

3. **Cosmic Variance.**
   a. Use Monte Carlo simulations (nsim=1000) to demonstrate that the cosmic variance is given by:

$$Var(C_\ell) = \frac{2}{2\ell+1} C_\ell^2$$

   b. Impact of white (uncorrelated) noise. Add white noise to the simulations and estimate the contribution to the variance. Demonstrate analytically that the power spectrum of a map of white noise is given by $C_l = 1/w$, with $w^{-1}=\sigma^2\Omega_{pix}$.



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4. **Pseudo power spectra and the effect of a mask.** Incomplete sky coverage affects our estimation of the angular power spectrum of a given map. In the incomplete sky, the spherical harmonics are no longer a orthonormal base function, and the inversion of the temperature map into a_{lm} produces an artificial coupling of modes.
   a. Demonstrate the effect of a mask on the power spectrum estimation by using a CMB simulation and a galactic mask excluding the region |b|<10º.
   b. Demonstrate that, at first order, the power spectrum can be corrected by the approximate factor 1/fsky, where fsky is the fraction of the total sky observed. Show that this approximation fails for more restrictive masks.
   c. The effect of the mask can be written in terms of a mixing matrix, as shown below. Use simulations to compute this mixing matrix numerically.

$$\tilde{a}_{lm} = \int W(\hat{n}) x(\hat{n}) Y^*_{lm}(\hat{n}) d\Omega_n \qquad \tilde{C}_l = \frac{1}{2l+1} \sum_{m=-l}^{l} |\tilde{a}_{lm}|^2$$

$$\langle \tilde{C}_l \rangle = \sum_{l'} M_{ll'} C_{l'}$$

   d. MASTER code. Hivon et al. (2002) provide analytical expressions for those mixing matrices. Compare the analytical result with the simulations.

$$M_{l_1 l_2} = \frac{2l_2+1}{4\pi} \sum_{l_3} (2l_3+1) \mathcal{W}_{l_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^2$$

5. **Aberration.** Simulate the effect of the aberration due to the movement of the observer in a CMB map with Healpix. Hint: the basic equations can be found in Chluba (2011), MNRAS, 415, 3227 (see eq.3 in the paper).