Motivation

It's like the conifold!

- Geom. eng.: IIA on X_3 $\longrightarrow$ 4d, $\mathcal{N}=2$, SQED + 1 hyp.
  of charge 2.
- Is ell. fibered
- D3 in IIB: Non-Toric, but have quiver

Method

Non-Toric ... sorry Amihay & Co.

3 steps: 1) MF 2) NCCR 3) Quiver GIT

Outline

Will use conifold as ref.

I) Geom II) NCCR's III) QGIT

IV) Flop V) Divisors

"flops of length two" := admits a small (crepant) res

$\pi : \tilde{X}_3 \longrightarrow X_3$ s.t. $\pi^*(\mathcal{O}_1) = \mathcal{O}_{\mathbb{P}^1}^{\oplus \boxed{2}}$

Kollár

E.g.:

Laufer's examples: Built through: 1) patchwork

2) Def. of D_n [Morrison - Curto]

Conifold, Reid's Pagoda $\longleftarrow$ length one

appeared in [Cachazo, Katz, Vafa]

Lé g

I) Geometry: Hypersurface in $\mathbb{C}^4$!

$$y^2 + x^3 + w z^2 + w^{2n+1} y = 0 \qquad n \in \mathbb{N}_{>0}$$

$$\Delta = \underset{\substack{\uparrow \\ \mathrm{II}}}{w^2} \left(4 \underset{\substack{\uparrow \\ I_1}}{w^{6n+1}} + 27 z^4 \right) \qquad \mathrm{II} \cap I \longrightarrow I_0^* \; (D_4)$$

Has a $\mathbb{P}^1$ w/ Normal bdl $\quad N = \mathcal{O}(+1) \oplus \mathcal{O}(-3)$ higher obstructions

(Conif. $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$; Pagoda $\mathcal{O} \oplus \mathcal{O}(-2)$)

II) Can't use usual methods. Will now study NCCR's.

Primer: What is a singularity?

<u>Answers</u>:

1) Coord. ring $R = \mathbb{C}[x_1, \dots, x_n]/(W)$ is sing

$$\iff \mathbb{C}[x_1, \dots, x_n] / (\partial_i W, W) \neq \emptyset$$

2) Global dim $< \infty$

<u>E.g.</u>: Ideal $I = (x, y) \in R = \mathbb{C}[x, y]$

any $xP + yQ$, how to specify?

Duh, give me a pair (P, Q)

$$\begin{matrix} R \\ \oplus \\ R \end{matrix} \xrightarrow{(x,y)} I$$

$$(P, Q) \longmapsto (P, Q) \cdot (x, y)$$

Really? How about $(P, Q) = c(-y, x)$

$\implies$ kernel

mod out!

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} \begin{matrix} R \\ \oplus \\ R \end{matrix} \xrightarrow{(x,\, y)} I \longrightarrow 0$$

$$(P, Q) \longmapsto (P, Q) \cdot (x, y)$$

Done

Try $\quad R = \mathbb{C}[x, y, z, w] \big/ (xy - zw)$

$$R \xrightarrow{\begin{pmatrix} -z \\ x \end{pmatrix}} \begin{matrix} R \\ \oplus \\ R \end{matrix} \xrightarrow{(x,\, z)} I$$

Really? How 'bout $\quad (P, Q) = c(y, w)$?

$$\cdots\; R^{\oplus 2} \xrightarrow{\begin{pmatrix} x & -z \\ -w & y \end{pmatrix}} \begin{matrix} R \\ \oplus \\ R \end{matrix} \xrightarrow{\begin{pmatrix} y & z \\ w & x \end{pmatrix}} \begin{matrix} R \\ \oplus \\ R \end{matrix} \xrightarrow{(x,\, z)} I$$

$\infty \implies$ singular. $\qquad I$ has no finite projective res.

Idea of NCCR : Replace R w/ some NC ring A that :

1) "looks like" R, i.e. $Z(A) \cong R$ $\longleftrightarrow$ blows down to X

2) has gl dim $< \infty$ $\longleftrightarrow$ is non-sing

3) is MCM as an R-ring $\longleftrightarrow$ crepant [VdB]

Physics : Replace closed string geom. w/ open string noncom. coords [Berenstein, Leigh]

In practice : 1) Find MCM module $M_1 \in \text{mod}\, R$

2) Try $A_1 = \text{End}_R(R \oplus M_1)$

- If A_1 is NCCR $\longrightarrow$ done !
- else, try $A_2 = \text{End}(R \oplus M_1 \oplus M_2)$

$\vdots$

Here, will only have A_1

Iyama - Wemyss : $\text{Ext}^1_R(A, A) = 0 \iff A$ is CM

Thm. [VdB]: R admits NCCR $\Longleftrightarrow$ X admits small res. $\tilde{X}$

(for dim 3)

What is MCM? : Module that can be written as

$$\cdots \xrightarrow{\phi} R^{\oplus 2\ell} \xrightarrow{\varphi} R^{\oplus 2\ell} \xrightarrow{\phi} R^{\oplus 2\ell} \longrightarrow M \longrightarrow 0$$

w/ $\varphi \cdot \phi = \phi \cdot \varphi = \mathbb{1}_{2\ell}\, W$ matrix factorization

E.g. : $R = \mathbb{C}[x, y, z, w] / (xy - zw)$

$$- - - . \quad R^{\oplus 2} \xrightarrow{\begin{pmatrix} x & -z \\ -w & y \end{pmatrix}} \begin{matrix} R \\ \oplus \\ R \end{matrix} \xrightarrow{\begin{pmatrix} y & z \\ w & x \end{pmatrix}} \begin{matrix} R \\ \oplus \\ R \end{matrix} \xrightarrow{(x, z)} M \longrightarrow 0$$

$M = \mathrm{coker} \underbrace{\begin{pmatrix} y & z \\ w & x \end{pmatrix}}_{\varphi}$ $\qquad \varphi = \begin{pmatrix} y & z \\ w & x \end{pmatrix}$; $\psi = \begin{pmatrix} x & -z \\ -w & y \end{pmatrix}$

$A = \mathrm{End}_R(R \oplus M)$

$$\mathrm{Hom}_R(R, M) := \left\{ \begin{tikzcd} & & R \arrow[dl, dashed, "\delta"'] \arrow[d, "\alpha"] \\ R^{\oplus 2} \arrow[r, "\psi"] & R^{\oplus 2} \arrow[r, "\varphi"] & R^{\oplus 2} \end{tikzcd} \; ; \quad \alpha \sim \alpha + \varphi \circ \delta \right\}$$

$$\left\langle \alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \; \alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\mathrm{Hom}_R(M, R) := \left\{ \begin{tikzcd} R^{\oplus 2} \arrow[r, "\psi"] & R^{\oplus 2} \arrow[r, "\varphi"] & R^{\oplus 2} \arrow[d, "\beta"] \\ & & R \end{tikzcd} \;\middle|\; \beta \circ \varphi = 0 \right\}$$

$$\left\langle \beta_1 = (1 \; 0) \cdot \psi \; ; \; \beta_2 = (0 \; 1) \cdot \psi \right\rangle$$

$$\begin{tikzcd} R \arrow[r, bend left, "\alpha_i"] & M \arrow[l, bend left, "\beta_i"] \end{tikzcd}$$

$$\varphi = \begin{pmatrix} y & z \\ w & x \end{pmatrix} ; \quad \psi = \begin{pmatrix} x & -z \\ -w & y \end{pmatrix}$$

<u>Relations</u> :

$$\beta_1 \alpha_1 \beta_2 = \underbrace{(1,0) \cdot \psi \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{x} \cdot \underbrace{(0,1) \cdot \psi}_{(-w, \, y)} = (-xw, \; xy)$$

$$\beta_2 \alpha_1 \beta_1 = -w(x, -z) = (-xw, \; \underset{xy}{\underset{\shortparallel}{wz}})$$

conifold

$$\longrightarrow \boxed{W_{KW} = \beta_1 \alpha_1 \beta_2 \alpha_2 - \beta_2 \alpha_1 \beta_1 \alpha_1}$$

Back to Laufer

$$y^2 + x^3 + w z^2 + w^{2n+1} y = 0$$

$$\Phi_{\mathrm{L}} := \begin{bmatrix} x & -y & -z & -w^n \\ y^2 & x & w^n y & -z \\ wz & -w^{n+1} & x & y \\ w^{n+1}y & wz & -y^2 & x \end{bmatrix}, \quad \Psi_{\mathrm{L}} := \begin{bmatrix} x & y & z & w^n \\ -y^2 & x & -w^n y & z \\ -wz & w^{n+1} & x & -y \\ -w^{n+1}y & -wz & y^2 & x \end{bmatrix}$$

$$x \longleftrightarrow y$$

$$\begin{tikzcd} R \arrow[r, bend left, "d"] & M \arrow[l, bend left, "c"] \arrow[loop above, "a"] \arrow[loop below, "b"] \end{tikzcd}$$

$$\mathcal{W}_{\mathrm{L}} = dcb^2 + \tfrac{1}{2}cdcd + a^2 b + \tfrac{1}{4}b^4$$

Great! Now what?

Quiver GIT

1) Replace each node w/ $\mathbb{C}^{d_i}$
2) Each arrow becomes appropriate $d_j \times d_i$ matrix
3) Impose D-flatness or King stability

3) Pick $\vec{\theta} = (\theta_L, \theta_R)$ w/ $\vec{\theta} \cdot \vec{d} = 0$

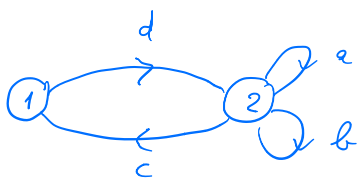
Impose $$\theta_i \mathbb{1}_{d_i} = \sum_{a:\text{ arrows to } i} \phi_a \phi_a^\dagger - \sum_{b:\text{ arrows from } i} \phi_b^\dagger \phi_b$$

$\longrightarrow$ comm. crepant res.

Which rep?

Wemyss: Take $\vec{d} = (\mathrm{rk}_R R, \ \mathrm{rk}_R M)$

$\longrightarrow$ will get crepant geometric resolution as moduli space.



d is 1×2

c is 2×1

a, b are 2×2

$$\hat{X}_3 = \left(\mathcal{M}\big((1,2)\big) - D\text{-terms}\right) \Big/ \underset{U(1) \times U(2)}{\left(\mathbb{C}^* \times GL_2 \mathbb{C}\right)}$$

$$a, b \in \Gamma\left(N = \mathcal{O}(1) \oplus \mathcal{O}(-3)\right)$$

D-term: $\quad cc^\dagger - d^\dagger d = -2\xi$

$$dd^\dagger - c^\dagger c + [a, a^\dagger] + [b, b^\dagger] = \mathbb{1}_2 \xi$$

The flop at last!

How to find exceptional $\mathbb{P}^1$? $\qquad \mathbb{P}^1 = \pi^{-1}(0)$

Reineke: Set all $\underbrace{\text{gauge invariants}}_{\text{all loops \& their powers}}$ to zero

Ansatz: $\quad a = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$; $b = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$; $c = (\sqrt{2}\gamma, 0)$; $d = \begin{pmatrix} 0 \\ \sqrt{2}\delta \end{pmatrix}$;

rel: $\quad \delta\gamma = 0$

$$|\delta|^2 - |\gamma|^2 = \xi \quad ; \quad |\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2$$

After some gauge fixing s.t. $\mathbb{C}^* \times GL_2\mathbb{C} \longrightarrow \mathbb{C}^*$

$\mathbb{C}^*$ - action : $\quad (\alpha, \beta) \longmapsto (\lambda\alpha, \lambda\beta)$

flop $\quad \xi > 0 \longrightarrow \xi = 0 \longrightarrow \xi < 0$

Light states : $\quad (0,1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1)[1] \dashleftarrow \overline{D2} + F_{DBI}$

$\qquad (1,0) \longrightarrow$ something on $\mathbb{P}^1$

but $\quad$ D0-brane $= (1,2)$-rep $= (1,0) + 2\times(0,1)$

$\Rightarrow$ $(1,0)$ = bound state (T-brane) of two D2's.

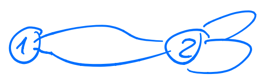
depending on B-field : $\begin{cases} (0,1) \text{ massless} \longrightarrow \text{charge-1 hyp.} \\ (1,0) \text{ massless} \longrightarrow \text{charge-2 hyp.} \end{cases}$

$Z(B+iJ) \sim \int (F-B+iJ)$ no corrections

$\Rightarrow$ both states will admit zeros

Divisors

Expectation: Small res $\longrightarrow$ Weil div on X_3 $\longrightarrow$ normalizable 2-form $\downarrow$ $U(1)$

Construction: 

has taut bdle V_2 w/ $S(\mathbb{C}^* \times GL_2\mathbb{C})$ str. gp.

$\Gamma(V_2)$ = paths from ① $\longrightarrow$ ②

$\Rightarrow$ det V_2 is line bdle s.t. div cuts $\mathbb{P}^1$ @ pt.

$(G - S, V)$

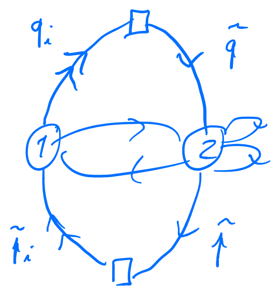
$\Gamma(\det V_2) = \text{path}_{1\to 2} \wedge \text{path}_{1\to 2}$

$\underline{\xi > 0}$: $\Gamma = \{ad \wedge d, \; bd\wedge d, \; abd \wedge d\}$ 3- sections

$\Gamma|_{\mathbb{P}^1} = \{\alpha, \beta, 0\} = \Gamma(\mathcal{O}(1)_{\mathbb{P}^1})$ ✓

Can get this in singular description ...

Flavored quiver



$$W = M_m^{\ \alpha}\, p_\alpha\, \tilde{p}_m + \tilde{M}_{m\alpha}\, \tilde{q}^\alpha\, q_m$$

$$M_m^{\ \alpha} = C_m^{\ i}\, s_i^{\ \alpha} \qquad \tilde{M}_{m\alpha} = \tilde{C}_m^{\ i}\, \tilde{s}_{i\alpha}$$

$$\det M = 0 \longrightarrow \mathrm{rk} < 2 \implies \text{D3 on top of D7}$$

$$\uparrow$$

$$c_1^{\ i} s_i^{\ \alpha} \parallel c_2^{\ i} s_i^{\ \alpha} \qquad \checkmark$$