F-theory, Black holes And Topological Strings

Cumrun Vafa

Based on joint work with

Haghighat, Murthy, Vandoren (2015)

And

Hayashi, Jefferson, H.-C. Kim, Ohmori (2018)

5 dimensional black holes that were studied in mid 90's, were among the first black holes studied in the post-duality era. These black holes can be constructed in various ways. For concreteness we will focus on:

$\underline{\text{II}}$B : $T^4 \times S^1$

$K3 \times S^1$

M-theory: T^6, $K3 \times T^2$

And, we consider the following duality equivalent brane configurations:

Type IIB: D3 branes wrapped on

$$\begin{array}{ccc} \Sigma & \subset & K3 \text{ or } T^4 \\ \times & & \times \\ S^1 & \subset & S^1 \end{array}$$

M-theory: M2 brane wrapped on a 2-cycle.

$$\Sigma \subset \begin{array}{c} T^6 \\ K3 \times T^2 \end{array}$$

These are dual to one another because:

Type IIB on Y x S^1 => M-theory on Y x T^2

Y=K3, or T^4

For concreteness let us focus on K3. Consider a genus g curve inside K3 and wrap a D3 brane on it. We get a sigma model on

$$\mathrm{Sym}^g(K3) \times R^4$$

These are dual to one another:

Type IIB on Y x S^1 => M-theory on Y x T^2

Y=K3, or T^4

For concreteness let us focus on K3.

$K3 \times S^1 \longrightarrow K3 \times T^2$

$\times n$

Holographic statement of this:

2d CFT

$$AdS^3 \times S^3 \times K3 \xleftrightarrow{\text{hol. dual}} Sym^g(K3)$$

The corresponding string enjoys (4,4) supersymmetry

Moreover, the $SO(4)=SU(2)_L \times SU(2)_R$ rotation symmetry is realized as current algebras on the left- and right-moving parts of the conformal theory, except that on the R^4 part of the sigma model it does not quite split to left/right moving action.

Central charge $c_L = 6g + 6$.

If we consider an extra circle and string wrapped around it and give it a KK momentum n, for large n we get the growth of entropy given by

$$S = 2\pi\sqrt{n c_L/6}$$

$$c_L = 6g+6$$

Which agrees with the prediction of Bekenstein-Hawking for large n and large g.

The 5d BPS black holes can be extended to spinning BPS black holes (BMPV) which further confirms this picture. The SU(2)_L captures the spinning BPS black hole.

F-theory and Spinning Black holes

In the context of M-theory we can reduce the amount of supersymmetry in the Bulk:

$$K3 \times T^2 \rightarrow CY^3$$

Again M2 branes wrapped on holomorphic curves give rise to spinning black hole. For general CY we do not know how to compute these—No strings!

The computation of partition function of the M2 branes on the CY3 fold is given by the topological string partition function (GV invariants).
This leads to the 5d spinning Black Hole partition function in terms of topological strings.
Can this be computed?
We need to know all loop topological string partition function—not known
If we can get a string as in K3 case we may have a better chance...

We want to get a string in 6 dimensions. Can this appear? Yes, if the CY is elliptic:

$$T^2 \propto B^4 = X$$

F-theory

$$B^4 \times S^1 =$$

M-theory

$$X$$

Moreover D3 branes wrapped on

$$\begin{array}{ccc} B^4 \times S^1 & \longrightarrow & T^2 \ltimes B^4 \\ \cup & & \cup \\ \Sigma \times \underset{\circlearrowleft n}{S^1} & & [nT^2] \cup \Sigma \end{array}$$

D3 branes wrapped around a curve in the base lead to strings with (0,4) supersymmetry.

It is not difficult to compute the central charge of this string and (ignoring CM) one finds that it is given by [V,1997]

$$(c_L, c_R) =$$

$$(6g + 12c_1\cdot\Sigma,\ 6g + 6c_1\cdot\Sigma)$$

Where c_1 is the first chern class of the base B.

This should lead to a holographic statement in the F-theory context [HMVV, 2015 (See also CLMSW,2017)]

$$AdS^3 \times S^3 \times B \longleftrightarrow \underset{(0,4)}{CFT}$$

$$Ell(CFT) =$$

$$\mathbf{Z}^{top}(\boldsymbol{X}) =$$

$$Z(BH)$$

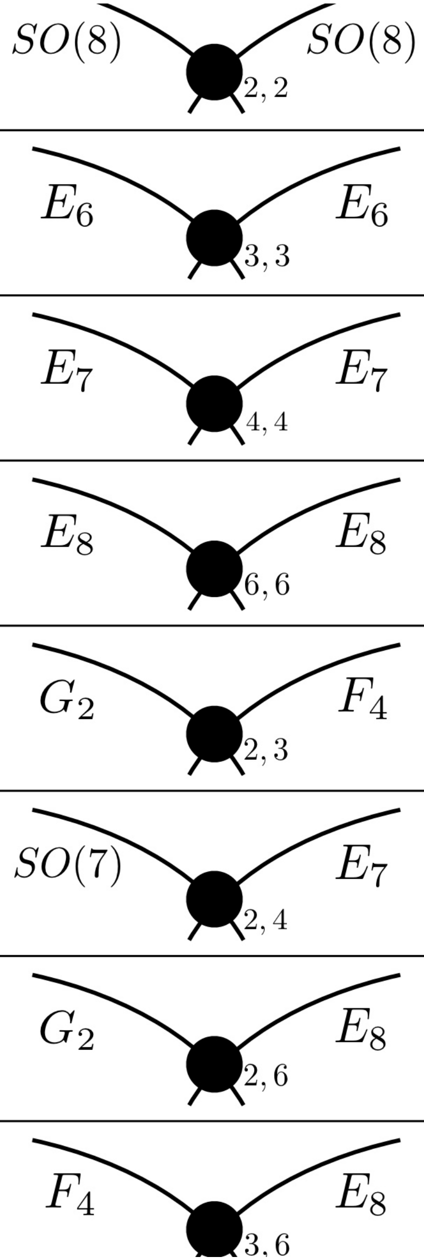
How can we come up with the (0,4) 2d CFT?

Hint: For (1,0) 6d SCFT's the CFT for the the strings are known (0,4) quivers for some cases. Use these as local ingredient and `stitch them together' to get the CFT for the global model.

$$T^2 \times \mathbb{C} \times \mathbb{C} \,/\, \mathbb{Z}_n \times \mathbb{Z}_m$$

$$\mathbb{Z}_n : \quad \begin{pmatrix} \alpha^2 & & \\ & \alpha^{-1} & \\ & & \alpha^{-1} \end{pmatrix}$$

$$\mathbb{Z}_m : \quad \begin{pmatrix} \beta^2 & & \\ & \beta^{-1} & \\ & & \beta^{-1} \end{pmatrix}$$



$\mathbb{Z}_2 \times \mathbb{Z}_2$:	$SO(8)$	$SO(8)$	$\bullet_{2,2}$:	$[SO(8)]\ 1\ [SO(8)]$
$\mathbb{Z}_3 \times \mathbb{Z}_3$:	E_6	E_6	$\bullet_{3,3}$:	$[E_6]\ 1\ 3\ 1\ [E_6]$
$\mathbb{Z}_4 \times \mathbb{Z}_4$:	E_7	E_7	$\bullet_{4,4}$:	$[E_7]\ 1\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{so}_7}{3}\ \overset{\mathfrak{su}_2}{2}\ 1\ [E_7]$
$\mathbb{Z}_6 \times \mathbb{Z}_6$:	E_8	E_8	$\bullet_{6,6}$:	$[E_8]\ 1\ 2\ \overset{\mathfrak{sp}_1}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{f}_4}{5}\ 1\ \overset{\mathfrak{g}_2}{3}\ \overset{\mathfrak{sp}_1}{2}\ 2\ 1\ [E_8]$
$\mathbb{Z}_2 \times \mathbb{Z}_3$:	G_2	F_4	$\bullet_{2,3}$:	$[G_2]\ 1\ [F_4]$
$\mathbb{Z}_2 \times \mathbb{Z}_4$:	$SO(7)$	E_7	$\bullet_{2,4}$:	$[SO(7)]\ \overset{\mathfrak{su}_2}{2}\ 1\ [E_7]$
$\mathbb{Z}_2 \times \mathbb{Z}_6$:	G_2	E_8	$\bullet_{2,6}$:	$[G_2]\ 2\ 2\ 1\ [E_8]$
$\mathbb{Z}_3 \times \mathbb{Z}_6$:	F_4	E_8	$\bullet_{3,6}$:	$[F_4]\ 1\ \overset{\mathfrak{g}_2}{3}\ \overset{\mathfrak{sp}_1}{2}\ 2\ 1\ [E_8]$

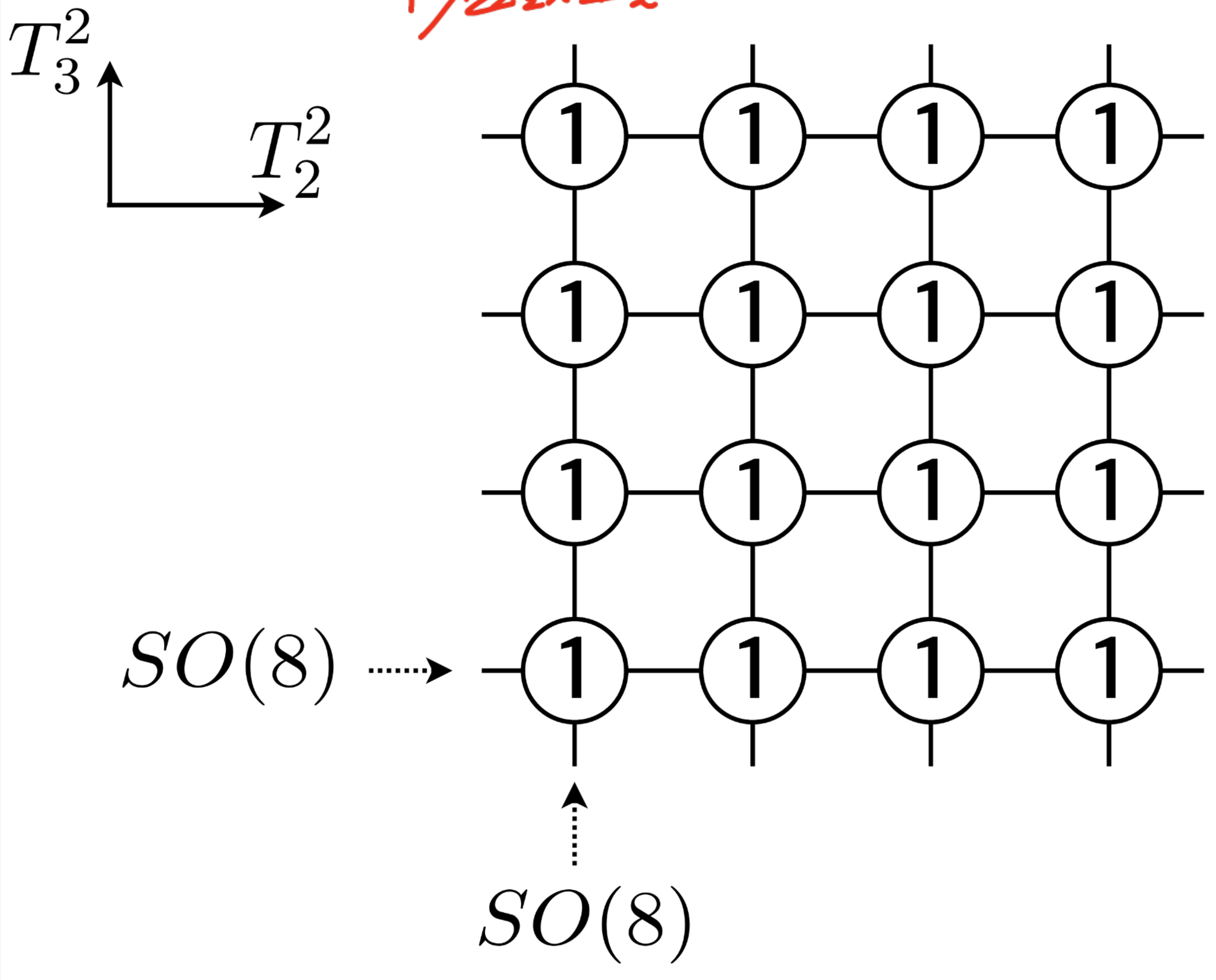
$$T^2 \times \mathbb{C} \times \mathbb{C} / \mathbb{Z}_n \times \mathbb{Z}_m$$

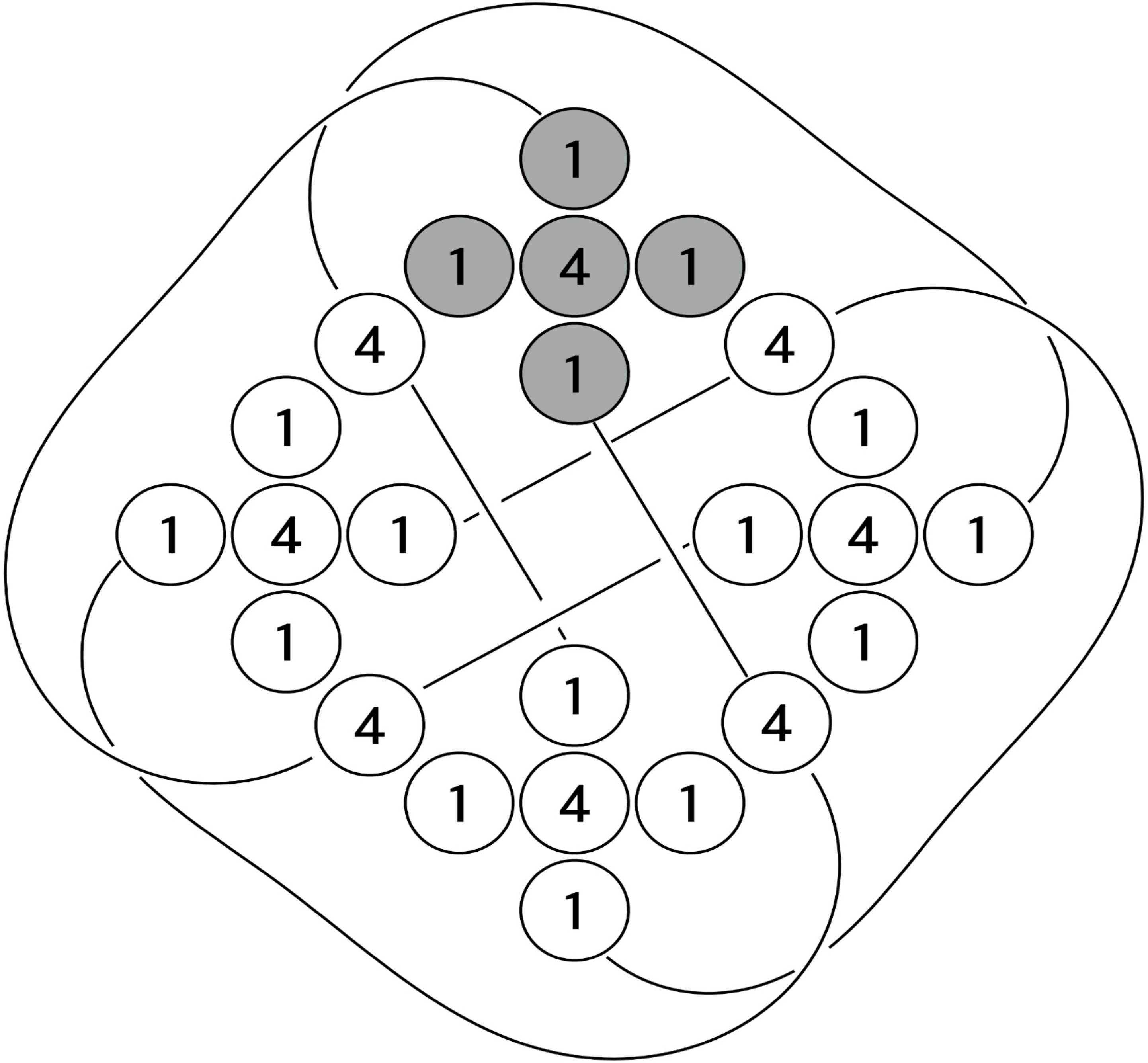
$$\downarrow$$

$$T^2 \times T^2 \times T^2 \Big/ \mathbb{Z}_n \times \mathbb{Z}_m$$

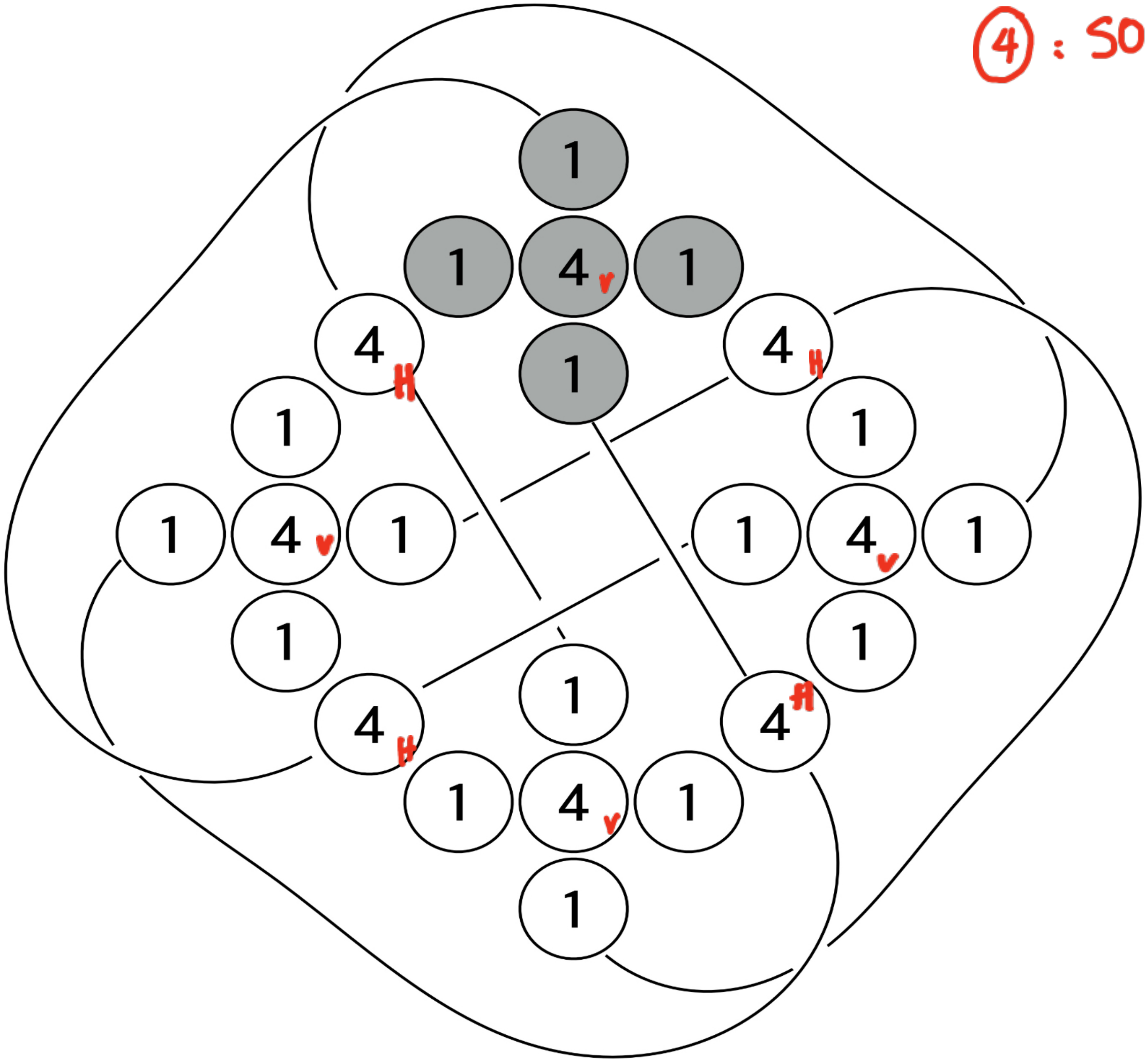
For each pair of fixed points we get a local picture as before, but now, the global symmetries get gauged.

$T^6/\mathbb{Z}_2\times\mathbb{Z}_2$




1
1
4
1
4
4
1
1
1
1
4
1
1
4
1
1
1
1
4
4
1
4
1
1

4 : SO(8)



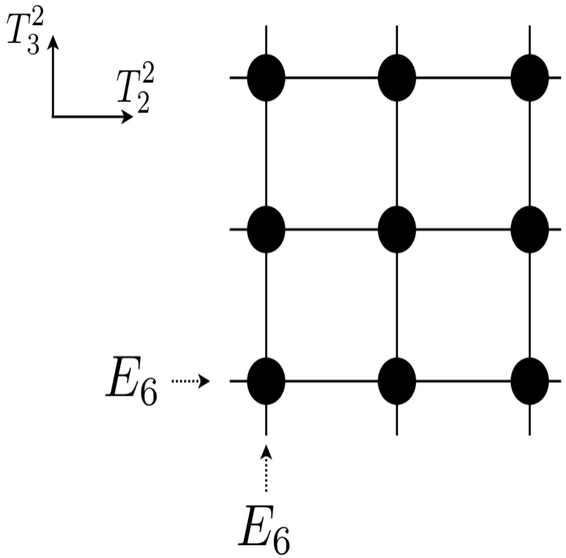
Gravitational anomaly cancels:

$$H - V + 29T - 273 = 4 - 8 \cdot 28 + 29 \cdot 17 - 273 = 0$$

Similarly one checks that mixed gravitational/gauge anomalies also cancel.
One can also check that the 2-cycle lattice (modulo identifications) is self-dual, as it should.

We seem to have a perfectly complete model.

Similarly for T^6/Z3xZ3 we get:



SU(3)

● : (E_6, E_6) conformal matter 1 3 1

| : $\mathcal{O}(-6)$ minimal CFT 6

E_6

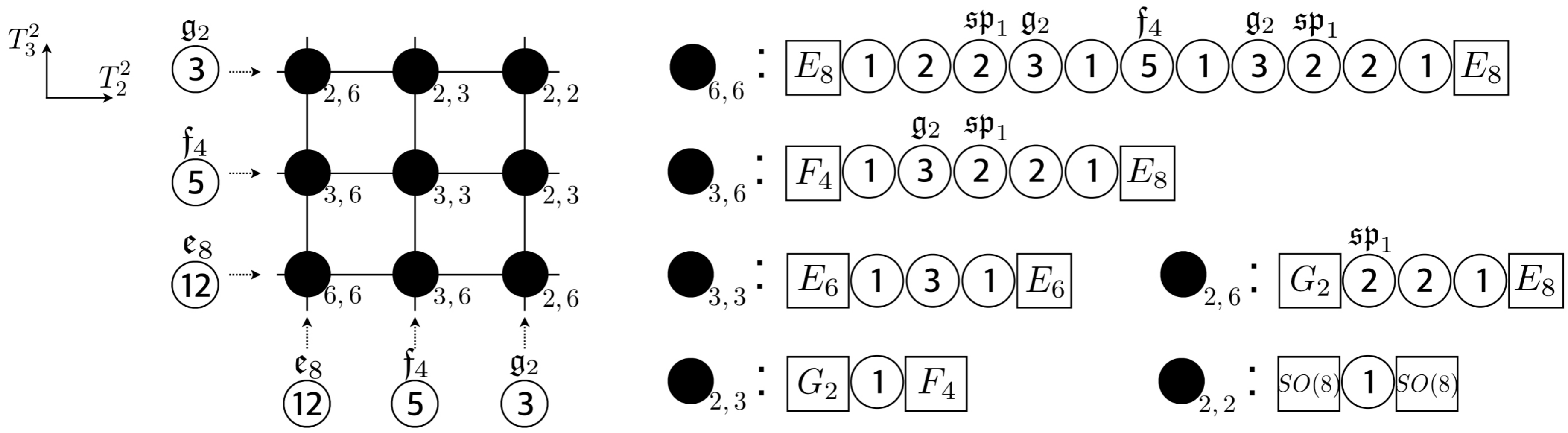
Gravitational anomalies cancel:

$3\times 9+2-1$

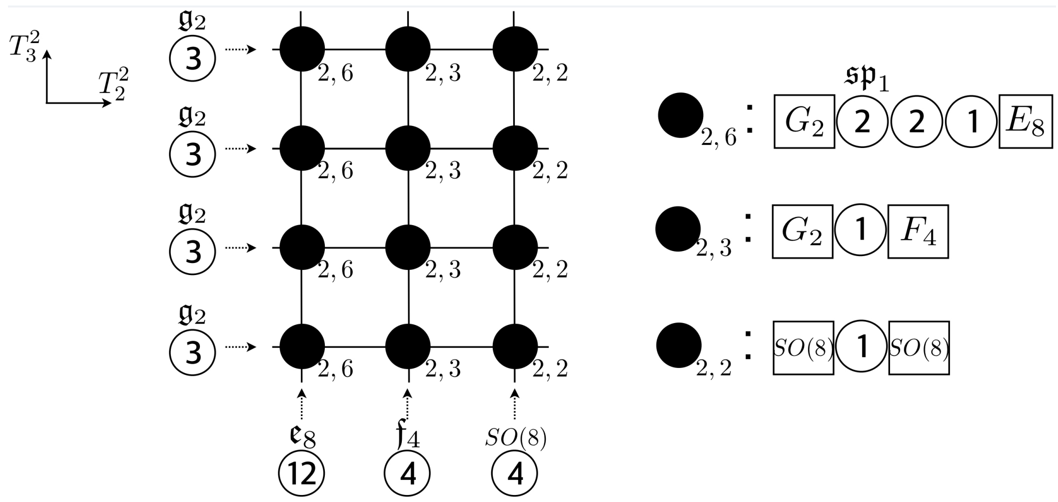
$$H - V + 29T - 273 = 1 - 540 + 29 \times 28 - 273 = 0$$

$78\times 6 + 8\times 9$

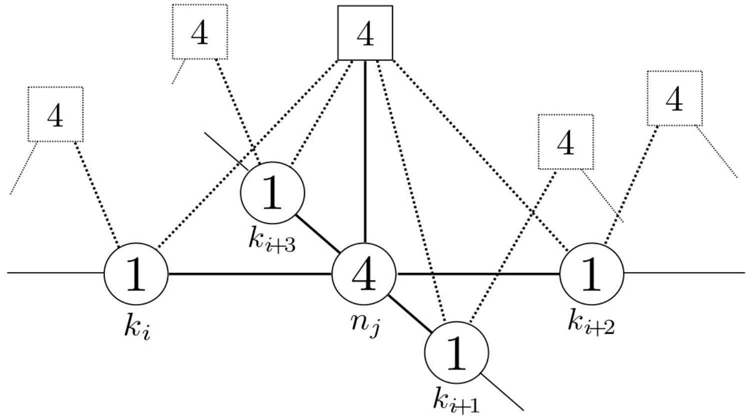
$\mathbb{Z}_6 \times \mathbb{Z}_6$



$\mathbb{Z}_2 \times \mathbb{Z}_6$



For the Z_2 x Z_2 model we have a concrete quiver description of the strings (using various string dualities from previous works):



Quiver diagram near a -4 curve which intersects with four -1 curves. This is a subset of the full quiver diagram for 6d strings. Self-dual strings over -4 curve (denoted by a circle with 4) are described $Sp(n)$ gauge theory and strings over -1 curves (denoted by circles with 1) have $O(k)$ gauge theory description. A $Sp(n)$ gauge node includes a real antisymmetric hypermultiplet and 4 fundamental hypermultiplets denoted by a vertical solid line. A $O(k)$ gauge node includes a real symmetric hypermultiplet. A solid line between two gauge node denotes a 1/2 twisted hyper- and a fermi- multiplet in bifundamental representation. The dotted lines stand for fermi multiplets. When all other $\mathcal{O}(-4)$ tensor multiplets are decoupled (when the external lines in this diagram are removed), this diagram describes the little string theory of $T^4 \times \mathbb{C}/\mathbb{Z}_2 \times \mathbb{Z}_2$.

We get a holographic statement:

$$AdS^3 \times \frac{S^3 \times T^4}{Z_2 \times Z_2} \longleftrightarrow \overbrace{\prod_{ij} O(k_i) \times Sp(n_j)}^{\text{Quiver}}$$

Check: We can compute the anomalies for this quiver theory using:

$$c_R = 3\mathrm{Tr}(\gamma^3 R_{\mathrm{cft}}^2)\ , \quad c_R - c_L = \mathrm{Tr}(\gamma^3)$$

where R_{cft} is the right-moving R-charge in the IR superconformal algebra and γ^3 is the 2d chirality operator acting on a chiral fermion $\psi_\pm$ as $\gamma^3\psi_\pm = \mp\psi_\pm$.

There are two choices for R-symmetry. For one of them, the BH branch we get the expected answer:

$$c_R = 3C\cdot C + 3c_1(B)\cdot C\ , \quad c_L = 3C\cdot C + 9c_1(B)\cdot C + 2$$

$$Z^{top}\left(\frac{T^6}{Z_2\times Z_2}\right) = Z^{BH}$$

$$= Ell\,(\,\text{Quiver})$$

It would be interesting to check this. The first time there is a concrete proposal for topological strings for compact CY.

$$
\begin{aligned}
Z_{\vec{k},\vec{n}^H,\vec{n}^V} = \oint \prod_{i,j=1}^{4} Z_{k_{ij}}^{\mathcal{O}(-1)}(\varphi_{ij}, m_i^H, m_j^V) \times \prod_{i=1}^{4} Z_{n_i^H}^{\mathcal{O}(-4)}(\tilde{\varphi}_i^H, m_i^H) \times \prod_{j=1}^{4} Z_{n_j^V}^{\mathcal{O}(-4)}(\tilde{\varphi}_i^V, m_i^V) \\
\times \prod_{i,j=1}^{4} Z_{k_{ij},n_i^H}^{\mathcal{O}(-1)\times\mathcal{O}(-4)}(\varphi_{ij}, \tilde{\varphi}_i^H) \times Z_{k_{ij},n_j^V}^{\mathcal{O}(-1)\times\mathcal{O}(-4)}(\varphi_{ij}, \tilde{\varphi}_j^V)
\end{aligned}
$$

$$
\begin{aligned}
Z_k^{\mathcal{O}(-1)}(\varphi, m) = \frac{1}{|W_k|} \prod_{I=1}^{r} \left(\frac{d\varphi_I}{2\pi i} \cdot \frac{\theta_1(2\epsilon_+)}{i\eta} \right) \prod_{e\in\mathrm{root}} \frac{\theta_1(e(\varphi))\theta_1(2\epsilon_+ + e(\varphi))}{i^2\eta^2} \\
\times \prod_{\rho\in\mathrm{sym}} \frac{i^2\eta^2}{\theta_1(\epsilon_{1,2} + \rho(\varphi))} \prod_{\rho\in\mathrm{fund}} \prod_{a=1}^{8} \frac{\theta_1(m_a + \rho(\varphi))}{i\eta} \ ,
\end{aligned}
$$

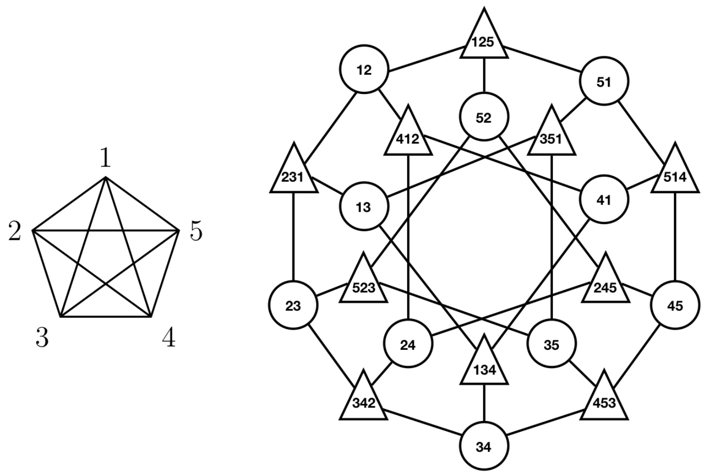
$$
\begin{aligned}
Z_n^{\mathcal{O}(-4)}(\tilde{\varphi}, m) = \frac{1}{|W_n|} \prod_{I=1}^{n} \left(\frac{d\tilde{\varphi}_I}{2\pi i} \cdot \frac{\theta_1(2\epsilon_+)}{i\eta} \right) \prod_{e\in\mathrm{root}} \frac{\theta_1(e(\tilde{\varphi}))\theta_1(2\epsilon_+ + e(\tilde{\varphi}))}{i^2\eta^2} \\
\times \prod_{\rho\in\mathrm{anti}} \frac{i^2\eta^2}{\theta_1(\epsilon_{1,2} + \rho(\tilde{\varphi}))} \prod_{\rho\in\mathrm{fund}} \prod_{p=1}^{4} \frac{i\eta}{\theta_1(\epsilon_+ + \rho(\tilde{\varphi}) \pm m_p)}
\end{aligned}
$$

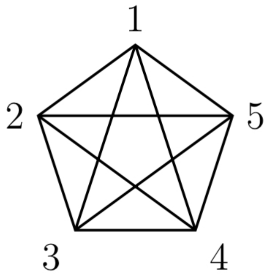
Extension to Non-elliptic Calabi-Yau threefold

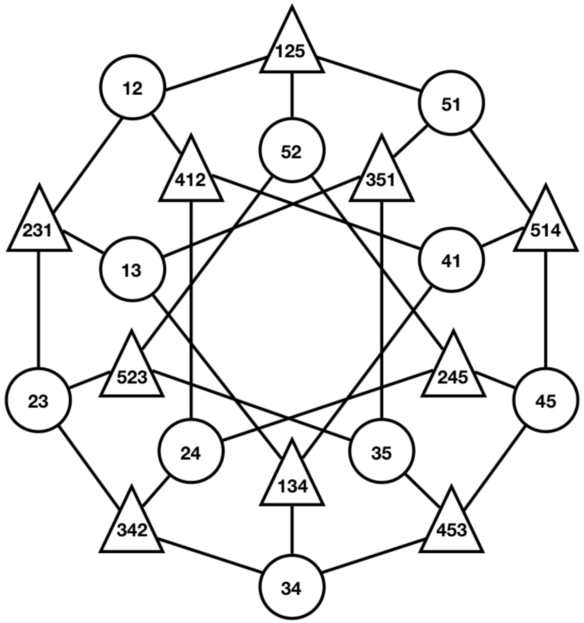
The idea is to use 5d SCFTs. Consider the simple example of the mirror quintic:

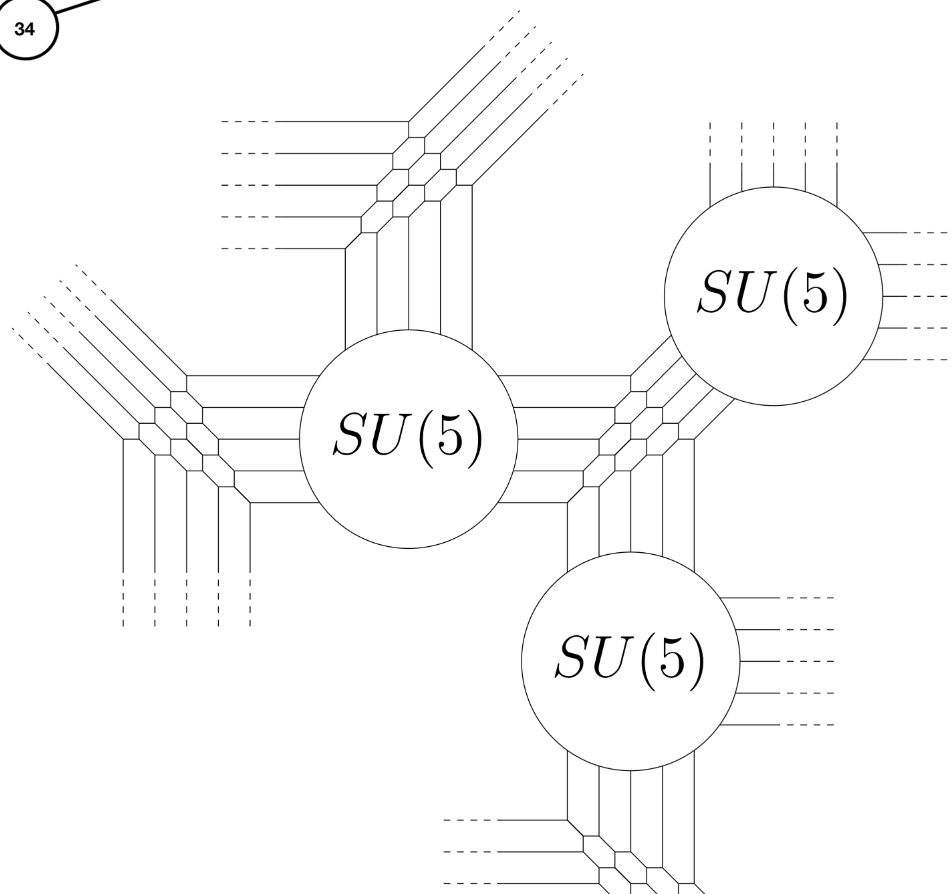
(Quintic threefold/Z5xZ5xZ5)

There are point (zi=zj=zk=0) and curve singularities where (zi=zj=0) with an A4 singularity: Geometry is:



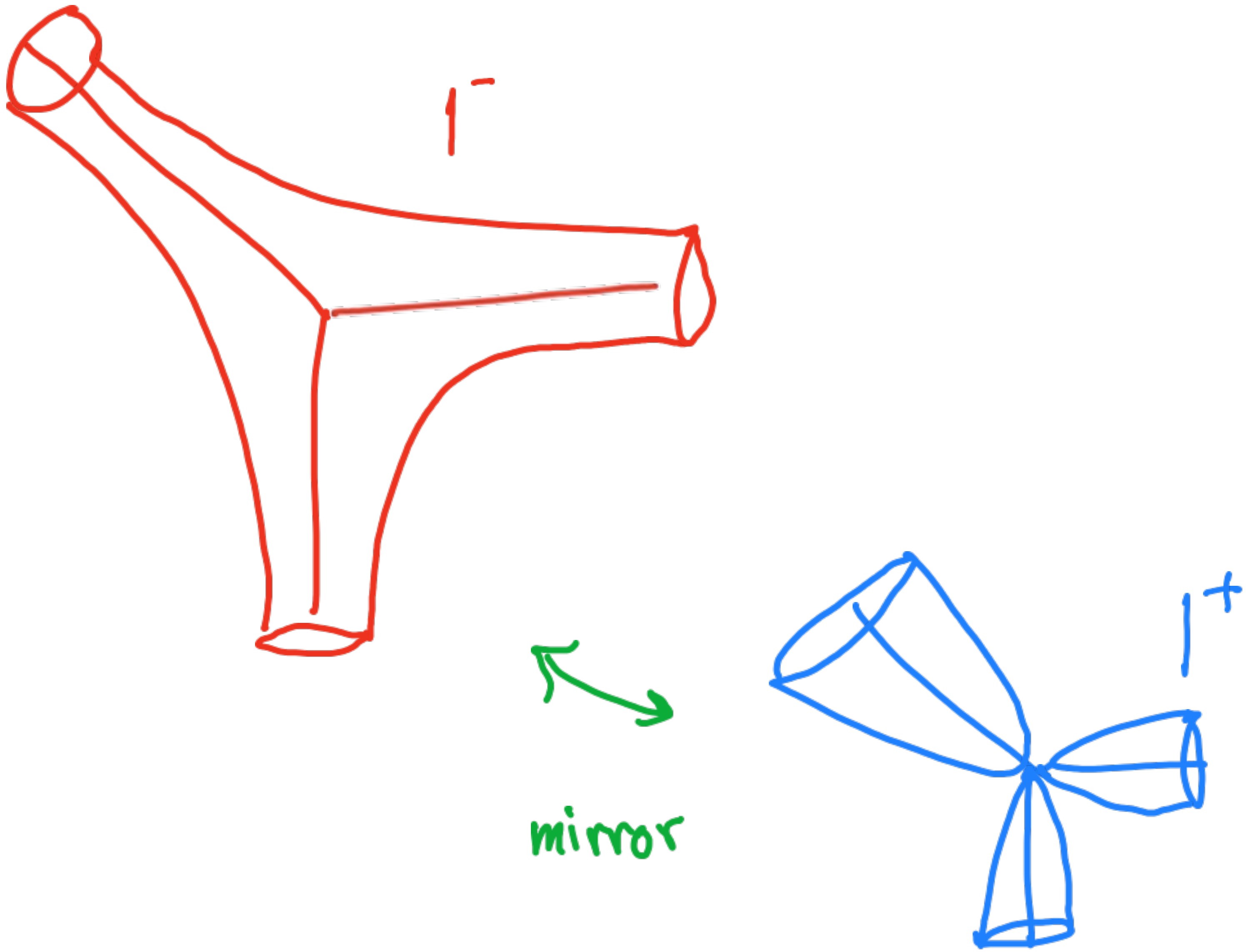




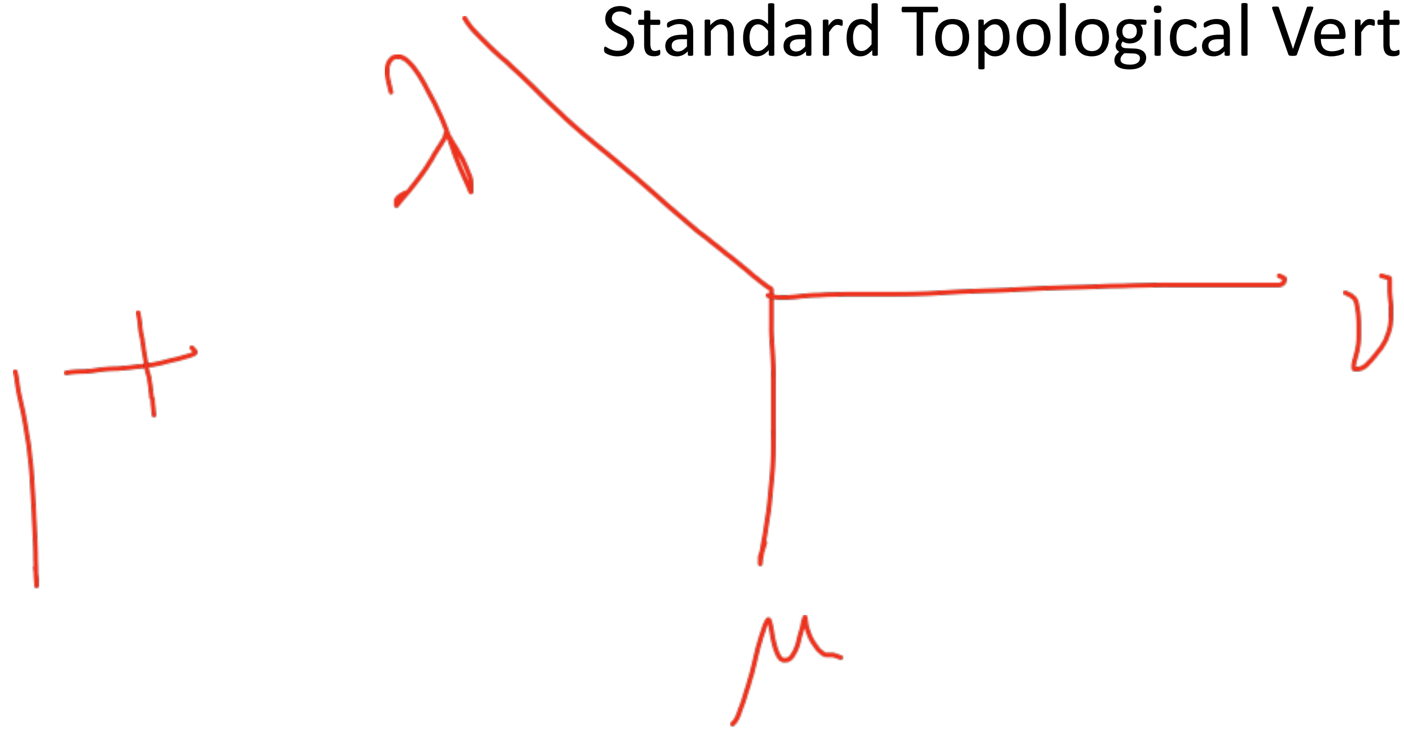


$h^{1,1} =$

$6 \times 10 + 4 \times 10 + 1$
$= 101$


1^-
mirror
1^+

Standard Topological Vertex



$$C_{\lambda\mu\nu}(q) = q^{-\frac{||\mu^t||^2}{2}+\frac{||\mu||^2}{2}+\frac{||\nu||^2}{2}}\tilde{Z}_\nu(q)\sum s_{\lambda^t/\eta}(q^{-\rho-\nu})s_{\mu/\eta}(q^{-\rho-\nu^t})$$

$s_{\mu/\nu}(x)$ is the skew Schur function

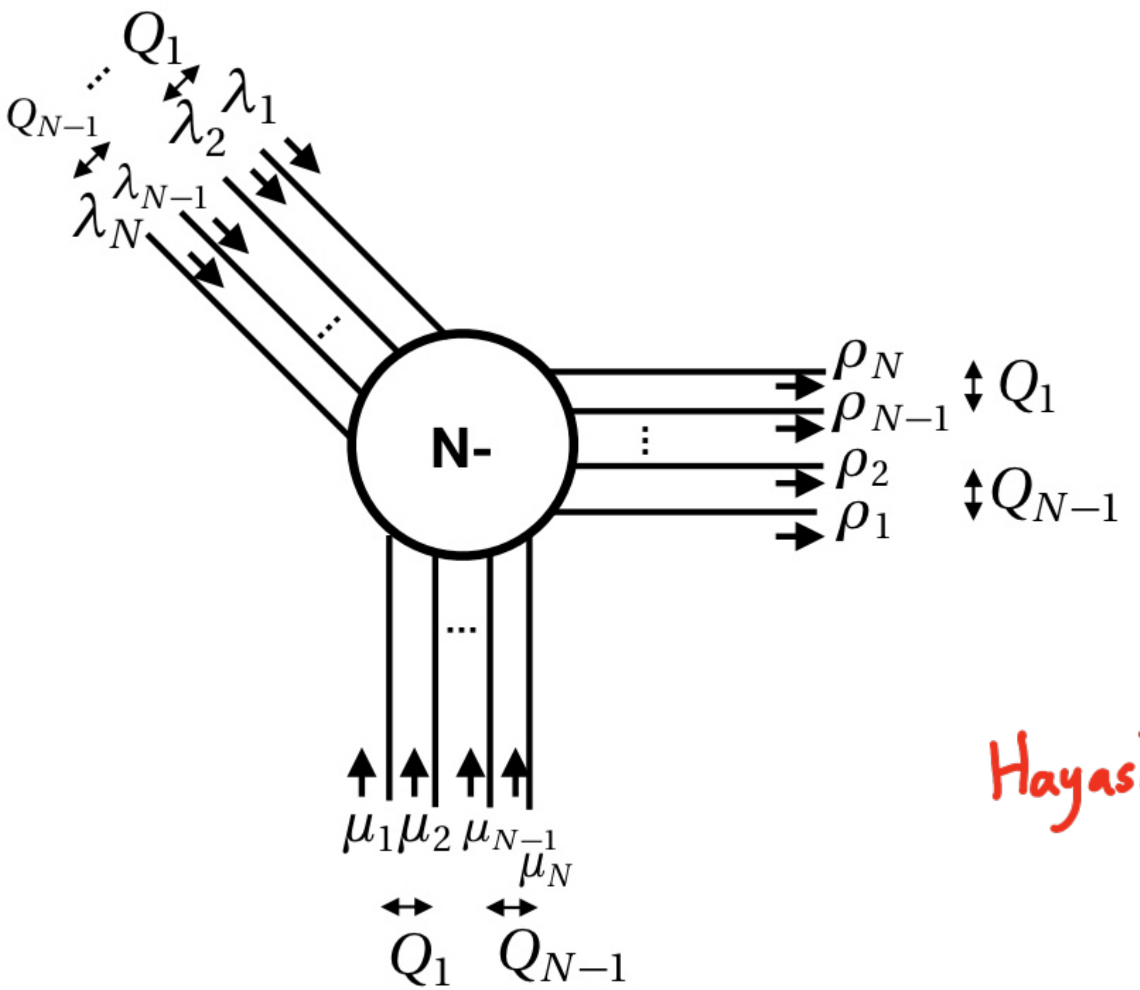
Bryan, Pandharipande

Aganagic, Ooguri,
Saulina, V.

$$C^{(1^-_k)}_{\lambda,\mu,\rho}(q) = \frac{f_\lambda(q)^{-k}\delta_{\lambda,\mu,\rho}}{q^{\frac{1}{2}||\lambda||^2}\tilde{Z}_\lambda(q)}.$$

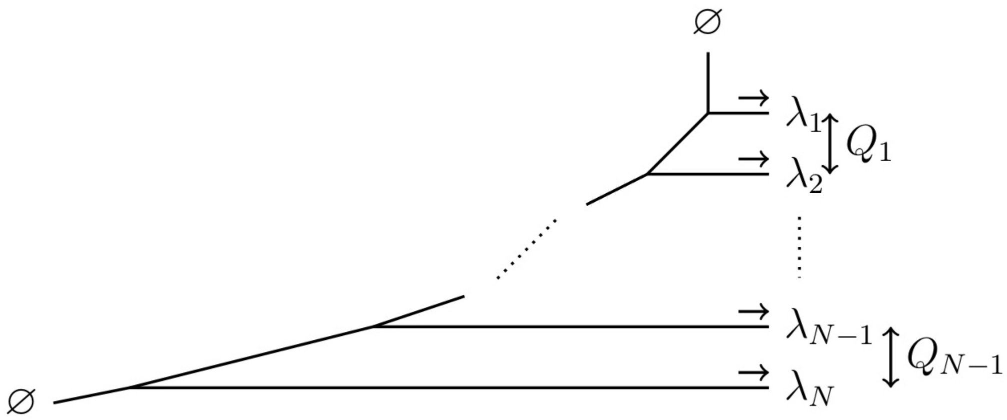
$$\tilde{Z}_\nu(q) = \prod_{(i,j)\in\nu}\left(1 - q^{\nu_i - j + \nu^t_j - i + 1}\right)^{-1}$$

$$f_\lambda(q) = (-1)^{|\lambda|} q^{\frac{||\lambda^t||^2 - ||\lambda||^2}{2}}$$



Hayashi, Ohmori

$$C^{(N_k^-)}_{\vec{\lambda},\vec{\mu},\vec{\rho}}(\vec{Q};q) = \frac{\prod_{a=1}^{N} f_{\lambda_a}(q)^{N-k+1-2a}\delta_{\lambda_a,\mu_a,\rho_a}}{Z^{\text{half vector}}_{SU(N),\vec{\lambda}}(\vec{Q};q)}$$



The web diagram that defines $Z^{\text{half vector}}_{SU(N),\vec{\lambda}}(\vec{Q};q)$. Each vertex represents the usual unrefined topological vertex, which is also called "+" vertex here.

$$
\begin{aligned}
Z^{\text{half vector}}_{SU(N),\vec{\lambda}}(\vec{Q};q) = q^{\frac{1}{2}\sum_{a=1}^{N}||\lambda_a||^2} \prod_{a=1}^{N} \tilde{Z}_{\lambda_a}(q) \\
\prod_{1\le a<b\le N-1} \Big[\prod_{i,j=1}^{\infty} \left(1 - Q_a Q_{a+1}\cdots Q_b q^{i+j-1}\right)^{-1} \\
\prod_{(i,j)\in\lambda_a} \left(1 - Q_a Q_{a+1}\cdots Q_b q^{-\lambda_{a,i}+j-\lambda^t_{b+1,j}+i-1}\right)^{-1} \\
\prod_{(i,j)\in\lambda_{b+1}} \left(1 - Q_a Q_{a+1}\cdots Q_b q^{\lambda_{b+1,i}-j+\lambda^t_{a,j}-i+1}\right)^{-1} \Big]
\end{aligned}
$$

Conclusion

We have seen that we can stitch local CFT's in 6 and 5 dimensions to have a complete description of non-gravitational sector of some compact CY 3-folds.

We can use these to propose:

Concrete holographic duality AdS3xS3XB with (0,4) CFT
Propose all genus answer for compact Calabi-Yau 3-fold.

Checks: The leading growth agrees with BH expectations
The local contributions are known to be correct

Further checks are necessary to be sure.