# Toward Jet-Gap-Jet at Full NLO/NLL

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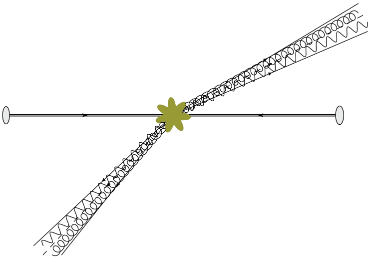

March 21, 2018

# Outline

- Review $\begin{cases} \text{Why Regge Theory?} \\ \text{BFKL Pomeron} \end{cases}$

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  - Why Regge Theory?
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- Mueller Tang Process
  - Previous Attempts
  - Improved Numerics for LL

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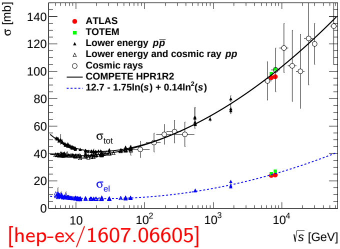
- Future
  - On Going Work
  - Future

# Regge theory and the Pomeron

Regge Theory grew out of pre-QCD S-matrix theory of the 50's and 60's. Amplitudes are seen as unitary, Lorentz invariant functions of analytic momenta. (doesn't assume an underlying theory) Amplitudes have poles representing particle exchange. Using a partial wave analysis, dominant contribution to simple amplitudes is the exchange of an *entire trajectory* of particles: Pomeron exchange: $\sigma_{tot} \sim s^{\alpha_0} - 1$




[hep-ex/1607.06605]

This soft Pomeron has been used to fit to p-p total cross sections since '70s.

| Authors | $\alpha_{\mathcal{P}}(0)$ |
|---|---|
| Donnachie-Landshoff (1992) | 1.0808 |
| Cudell, Kang and Kim (1997) | $1.096^{+0.012}_{-0.009}$ |
| Cudell *et al.* (2000) | $1.093 \pm 0.003$ |
| COMPETE Collaboration (2002) | $1.0959 \pm 0.0021$ |
| Luna and Menon (2003) | 1.085 - 1.104 |
| Menon and Silva (2013) | $1.0926 \pm 0.0016$ |

Throughout this talk I will mainly be focused on single Reggeon exchange although I might comment on the end about multi-Reggon exchange, Regge cuts, and saturation

# What is it good for?

- Regge theory predicts a wide rang of phenomena *connecting perturbative to non-perturbative regimes*. See work by TR & collaborators on holographic Pomeron/Odderon

- Regge limit of many amplitudes and cross sections simplifies

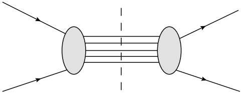


Figure: Optical theorem involving analytic properties of an amplitude.

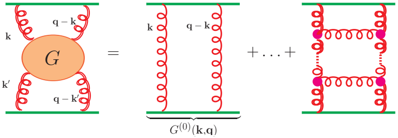
- Often gives model independent* results for BSM physics

- Weak coupling resums an infinite number of neglected Feynman diagrams.

- Assumption of simple physical arguments: Lorentz invariance, crossing symmetry, analyticity, etc. is similar to many other successful approaches: Forward physics, Amplitudes, Regge CFT, the Bootstrap, etc.

# BFKL

Balitsky, Fadin, Kuraev, Lipatov (BFKL): perturbative Pomeron. Large logs get in the way of usual perturabtion theory: resum $\alpha_s \, log(s)$ to all orders. Bfkl equation – integral equation for Green's function in Mellin space




$$G(\mathbf{k},\mathbf{k}',\mathbf{q},Y) = \int_{-i\infty}^{+i\infty} \frac{d\omega}{2\pi i} e^{Y\omega} f_\omega(\mathbf{k},\mathbf{k}',\mathbf{q}) \to \int_{-i\infty}^{+i\infty} \frac{d\omega}{2\pi i} e^{Y\omega} \sum_{n\in\mathcal{Z}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2\pi\, i} \frac{E_{\gamma,n}(k)E^*_{\gamma,n}(k')}{\omega - \bar{\alpha}_s\chi(\gamma,n)}$$

where in Leading Log (LL)

$$\chi(\gamma,n) = 2\psi(1) - \psi(\gamma + \frac{|n|}{2}) - \psi(1-\gamma+\frac{|n|}{2}) \quad and \quad \omega_0 = \frac{4\alpha_s N_c}{\pi} ln(2)$$

Surprising conformal symmetry greatly simplifies things in coordinate space

# First Complication

**Forward vs non-forward BFKL:** The forward solution is much simpler. The non-forward case is usually represented in coordinate space due to a surprising conformal symmetry found by Lipatov Zh.Eksp.Teor.Fiz. 90 (1986) 1536-1552.

$$\boldsymbol{t = 0}$$

$$\omega f = \delta^2(\vec{k}_1 - \vec{k}_2) + \mathcal{K}_0 f$$

Solutions to BFKL kernel, $\mathcal{K}_0$:

$$E^n_\nu = \frac{1}{\pi\sqrt{2}}(k^2)^{-\frac{1}{2}+i\nu} e^{i\,n\theta}$$

This simple form, coupled with the optical theorem, make tackling inclusive jet production an "easier" endeavor.

$$\boldsymbol{t \neq 0}$$

$$\omega \Delta_{b_1}\Delta_{b_1'} f = (2\pi)^4\delta^2(b_1 - b_1')\delta^2(b_2 - b_2') + \alpha_s\{\text{complicated}\}$$

$$f(\vec{b}) = FT\left\{\frac{f(\omega,\vec{k}_1,\vec{k}_2,\vec{q})}{k_2^2(\vec{k}_1-\vec{q})^2}\right\}_{\vec{k}_1,\vec{k}_2,\vec{q}-\vec{k}_1,\vec{q}-\vec{k}_2}$$

$$= \sum_n \int d\nu \int d^2\ell \left[\frac{16\nu^2+4n^2}{(4\nu^2+1-n^2)^2+16n^2\nu^2}\right] \times \frac{\tilde{E}^n_\nu(1)\tilde{E}^n_\nu(2)}{\omega-\bar{\alpha}_s\chi_n(\nu)}$$

$\tilde{E}^n_\nu$ involves ${}_2F_1$ (Will return!)

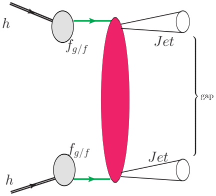
# Previous Applications

**In general:**
DIS
PP total/differential cross sections
Inclusive Jet Production
Higgs Production
.
.
.

**Here today:**
MN angular decorrelation (Colferai)
MN Multijet process (Gordo Gomez & Francesco Giovanni Celiberto)
Minijet Radiation (KU & UAM)
.
.
.

# Mueller Tang: Jet-Gap-Jet

Using rapidity gaps to investigate
BFKL effects dates back to the late 80's & early
'90s [Sov.J.Nucl.Phys. 46 (1987) 712-719]&[Phys. Rev. D 47 1 (1993)] Mueller and Tang: augmented BFKL hard Pomeron.

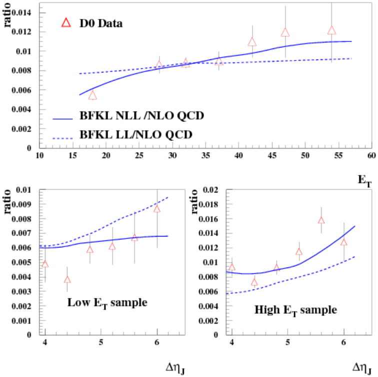


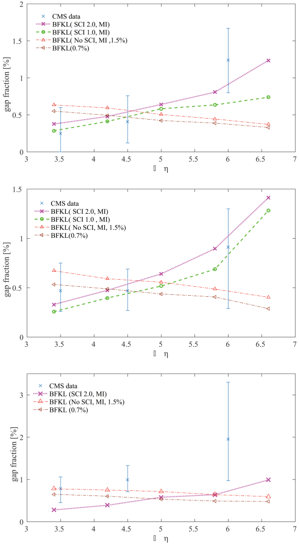

$$E_{n\nu}(\rho_1, \rho_2) = \underbrace{\left(\frac{\rho_1 - \rho_2}{\rho_1 \rho_2}\right)^h \left(\frac{\rho_1^* - \rho_2^*}{\rho_1^* \rho_2^*}\right)^{\bar{h}}}_{\text{Lipatov term}} \underbrace{- \left(\frac{1}{\rho_2}\right)^h \left(\frac{1}{\rho_2^*}\right)^{\bar{h}} - \left(\frac{-1}{\rho_1}\right)^h \left(\frac{-1}{\rho_1^*}\right)^{\bar{h}}}_{\text{Mueller-Tang correction}}$$

At finite momentum transferred can be investigated at hadron colliders looking for highly exclusive processes where two jets far apart in rapidity represent the sole observed radiation. The absence of any additional emission over a large rapidity region suggests that the color-singlet exchange contributes substantially to the jet-gap-jet cross section.

DGLAP suppressed at large $\Delta y$ $\rightarrow$ Good window into BFKL effects.

# previous fits and analysis



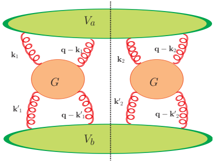



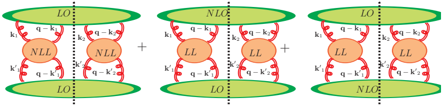

Left: LL & NLL BFKL at Tevatron [hep-ph/1012.3849].

Right LL+SCI at CMS [hep-ph/1703.10919].

# incorporating NLO impact factor

A full NLL/O calculation is within reach. NLO MT impact factors recently calculated [1406.5625,1409.6704]. Very complicated! (not in a factorizable form!) But...only certain combinations of jet vertex and Green's function approximation orders contribute effectively to the NL order of the cross section. The most complicated combinations can be discarded because they are subleading.




- GGF NLL + LO vertices. For this special case the general formula for the cross section can be expressed in a much simpler form because LL vertices are idependent from the reggeon momenta.
- GGF LL + LO vertex + NLO vertex. The non trivial dependence of the NLO jet vertex from the reggeon momenta introduces an important complication.
- GGF LL + both NLO vertices. Discarded because subleading.

# Second Complication

## details of NLO impact factor

$$
\begin{aligned}
&\frac{d\hat{V}^{(1)}(x, \mathbf{k}, \mathbf{l}_1, \mathbf{l}_2; x_J, \mathbf{k}_J; M_{X,\max}, s_0)}{dJ} = \\
&= v_q^{(0)} \frac{\alpha_s}{2\pi} \Bigg[ S_J^{(2)}(\mathbf{k}, x) \cdot \Bigg[ -\frac{\beta_0}{4} \Bigg[ \Bigg\{ \ln\left(\frac{\mathbf{l}_1^2}{\mu^2}\right) + \ln\left(\frac{(\mathbf{l}_1 - \mathbf{k})^2}{\mu^2}\right) + \{1 \leftrightarrow 2\} \Bigg\} - \frac{20}{3} \Bigg] - 8C_f \\
&+ \frac{C_a}{2} \Bigg[ \Bigg\{ \frac{3}{2\mathbf{k}^2} \Bigg\{ \mathbf{l}_1^2 \ln\left(\frac{(\mathbf{l}_1 - \mathbf{k})^2}{\mathbf{l}_1^2}\right) + (\mathbf{l}_1 - \mathbf{k})^2 \ln\left(\frac{\mathbf{l}_1^2}{(\mathbf{l}_1 - \mathbf{k})^2}\right) - 4|\mathbf{l}_1||\mathbf{l}_1 - \mathbf{k}|\phi_1 \sin\phi_1 \Bigg\} \\
&\quad - \frac{3}{2}\left[ \ln\left(\frac{\mathbf{l}_1^2}{\mathbf{k}^2}\right) + \ln\left(\frac{(\mathbf{l}_1 - \mathbf{k})^2}{\mathbf{k}^2}\right) \right] - \ln\left(\frac{\mathbf{l}_1^2}{\mathbf{k}^2}\right) \ln\left(\frac{(\mathbf{l}_1 - \mathbf{k})^2}{s_0}\right) - \ln\left(\frac{(\mathbf{l}_1 - \mathbf{k})^2}{\mathbf{k}^2}\right) \ln\left(\frac{\mathbf{l}_1^2}{s_0}\right) - 2\phi_1^2 + \{1 \leftrightarrow 2\} \Bigg\} + 2\pi^2 + \frac{14}{3} \Bigg] \\
&+ \int_{z_0}^{1} dz \Bigg\{ \ln\frac{\lambda^2}{\mu_F^2} S_J^{(2)}(\mathbf{k}, zx) \left[ P_{qq}(z) + \frac{C_a^2}{C_f^2} P_{gq}(z) \right] + \left[ (1-z)\left[1 - \frac{2}{z}\frac{C_a^2}{C_f^2}\right] + 2(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ \right] S_J^{(2)}(\mathbf{k}, zx) + 4S_J^{(2)}(\mathbf{k}, x) \Bigg\} \\
&+ \int_0^1 dz \int \frac{d^2\mathbf{q}}{\pi} \Bigg[ P_{qq}(z)\Theta\left(\hat{M}^2_{X,\max} - \frac{(\mathbf{p} - z\mathbf{k})^2}{z(1-z)}\right) \Theta\left(\frac{|\mathbf{q}|}{1-z} - \lambda\right) \\
&\qquad \times \frac{\mathbf{k}^2}{\mathbf{q}^2(\mathbf{p} - z\mathbf{k})^2} S_J^{(3)}(\mathbf{p}, \mathbf{q}, (1-z)x, x) + \Theta\left(\hat{M}^2_{X,\max} - \frac{\mathbf{\Delta}^2}{z(1-z)}\right) S_J^{(3)}(\mathbf{p}, \mathbf{q}, zx, x) P_{gq}(z) \\
&\qquad \times \Bigg\{ \frac{C_a}{C_f}[J_1(\mathbf{q}, \mathbf{k}, \mathbf{l}_1, z) + J_1(\mathbf{q}, \mathbf{k}, \mathbf{l}_2, z)] + \frac{C_a^2}{C_f^2} J_2(\mathbf{q}, \mathbf{k}, \mathbf{l}_1, \mathbf{l}_2)\Theta(\mathbf{p}^2 - \lambda^2) \Bigg\} \Bigg] \Bigg]
\end{aligned}
$$

# Second Complication Cont'd

BFKL Eigenfunctions: A nice derivation can be found in hep-ph/0102221

$$\tilde{E}_{n\nu}(k_1, k_2) = N(n,\nu)\left[ {k_1^*}^{\bar{h}-2} {k_2}^{h-2} \; {}_2F_1(1-h, 2-h; 2, -\frac{k_1}{k_2}) \; {}_2F_1(1-\bar{h}, 2-\bar{h}; 2, -\frac{k_2^*}{k_1}^{*}) + \{1 \to 2\} \right]$$

where $h = \left(\frac{1+n}{2} + i\nu\right)$, $\bar{h} = \left(\frac{1-n}{2} + i\nu\right)$
Still to compute physical observables this must be convoluted with the vertex functions and parton distribution functions:

$$R \sim \frac{d\sigma}{dx_1 dx_2 dY dp_T} \sim \Big[ f_{q/g} \times v^{(1)} \otimes G^{(LL)} \otimes G^{(LL)} \otimes v^{(0)} \\ + \times v^{(0)} \otimes G^{(NLL)} \otimes G^{(NLL)} \otimes v^{(0)} \times f_{q/g} \Big]$$

# NLL BFKL with LO vertices

The normalization of the Gluon Green function fixes the jet vertex leading order.

$$\lim_{Y\to 0} G(\mathbf{k},\mathbf{k}',\mathbf{q},Y) = G(\mathbf{k},\mathbf{k}',\mathbf{q},0) = \frac{\delta^2(\mathbf{k}-\mathbf{k}')}{\mathbf{k}^2(\mathbf{q}-\mathbf{k})^2}.$$

At this order, apart for the jet distribution function $S$ that fixes the jet momentum, the jet vertex is a simple color factors ($c$-number)

$$V_a(x,\mathbf{q},x_J,\mathbf{k}_J) = S_J^0(x,\mathbf{q};x_J,\mathbf{k}_J)h_a^0,$$

$$h_a^0 = C_{q/g}^2\frac{\alpha_s^2}{N_c^2-1}, \qquad S_J^{(0)} = x\delta^2(\mathbf{k}_J-\mathbf{q})\delta(x_J-x).$$

The independence of the LO vertices from the reggeon momenta allow for considerable simplification.

# Hard Hypergeometrics

(1) How does one compute Gauss Hypergeometric functions: ${}_2F_1(a, b, c, z)$? For some special values of the arguments they simplify, but the Mueller-Tang case involves:

$$ {}_2F_1\left(\frac{1}{2} - \frac{n}{2} \mp i\,\nu, \frac{3}{2} - \frac{n}{2} \mp i\,\nu, 2, \frac{k_1}{k_2}\right) $$

Must perform a Mellin transform in $\nu$ and sum over n, clearly won't always be in "nice" regions.

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Must perform a Mellin transform in $\nu$ and sum over n, clearly won't always be in "nice" regions.

(2) What about the rest of the complex plane outside $|z|<1$? For example, for $d=b-a$:

$$ {}_2F_1(a,b,c,z)=\Gamma\left(\begin{array}{c} c,-d \\ a,c-b \end{array}\right)(-z)^{-b}\,{}_2F_1(b,1-c+b,1+d,\frac{1}{z})+(a\leftrightarrow b) $$

But each term naively diverges when $d\in\mathbb{Z}$!

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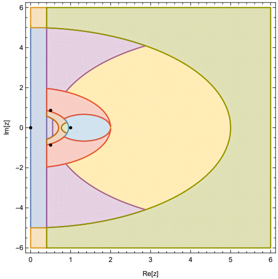
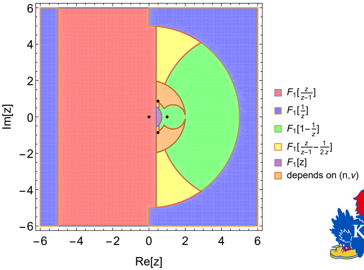
But each term naively diverges when $d \in \mathbb{Z}$!

But we know the Gauss hypergeometric function converges for a wide range of these values, this must just be a computational approach. (Physicists are cavalier about canceling infinities!)

# Recent Numerical Methods

The solution is to write $d \to m + \epsilon$, expand everything in terms of $\epsilon$, and try to cancel the resulting divergences. Bühring, SIAM J. Math. Anal. 18 (1987), no. 3 For our case, $d = b - a = 1$, this can be done explicitly.



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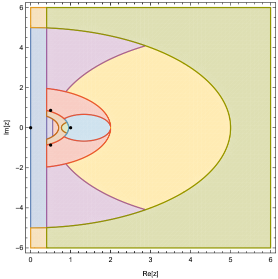
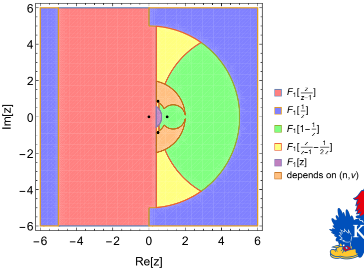
**But!** Not all expansions are created equal. There are a variety of maps that can cover the complex plane: $z, 1/z, 1/(z - z_0), (z-1)/z$. Want to select a covering of the complex plane. Special care must be taken near $z = \{0, 1, \infty, exp(\pm i\pi/3)\}$ Michel and Stoitsov, arXiv:0708.0116 [math-ph] and Doornik, Math. Comp. 84 (2015), 1813-1833.



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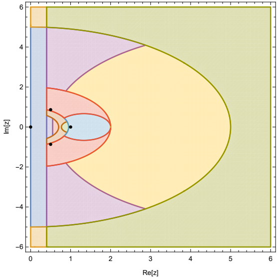
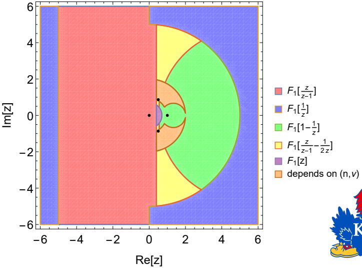
In all cases you end up with a numerically manageable convergent power series.



# Numerical Methods Cont'd

Numerical accuracy can be evaluated by testing how well solutions satisfy Hypergeometric equation.

$$T = \frac{|z(1-z)f_2 + [c-(a+b+1)z]f_1 - F(z)|}{|F(z)| + |f_1| + |f_2|}$$

Various algorithms have been proposed Michel, Stoitsov, Doornik and implemented in CPP (including our own makeshift, but Doornik's is most efficient) All give accuracy $-log_{10}(T) \rightarrow$ single to double precision. All algorithms are roughly the same "speed" $\sim$ 1000s of evaluations per second on personal computer.

The LO/LL calculation is currently being used for calibration/debugging and various series convergence acceleration (example Willis, arXiv:1102.3003 [math.NA]) are being considered.

# Upgraded Numerics

- Simpler calculations have taken several weeks to run on pc.
- HPC ordered at KU: 40-cores at 2.4 GHz with 192 GB 2666 MHz DDR4 memory.
- want to focus time on (1) maximizing efficiency for LO/LL calculation (i.e. Gauss hypergeometrics) and (2) identify convenient parameterizations/order of integrals for NLO impact factor
- Want to identify physical regions where NLO impact factor is “more” factorizable

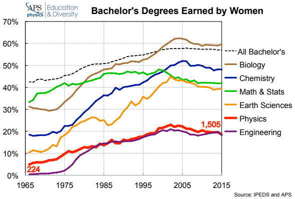
# Conclusions

**Conclusions**

- Single Pomeron exchange is still an exciting exploration ground.
- JGJ at LHC seems like fertile ground for identifying BFKL signals
- Already seen hints that full NLO description will fit data.

**Special thanks for useful discussions:** Dmitri Colferai, Evan Weinberg, Grigorios Chachamis, Agustin Sabio Vera, David Gordo Gomez.

**Outreach**

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# NLO impact factors

In general the cross section for these processes is given as a multiple convolution between the the jet vertices and the GGFs.

$$\frac{d\hat{\sigma}}{d\mathcal{J}_1 d\mathcal{J}_2 d^2\mathbf{q}} = \int d^2\mathbf{k}_1 d^2\mathbf{k'}_1 d^2\mathbf{k}_2 d^2\mathbf{k'}_2 V_a(\mathbf{k}_1, \mathbf{k}_2, \mathcal{J}_1, \mathbf{q}) \times$$
$$G(\mathbf{k}_1, \mathbf{k'}_1, \mathbf{q}, Y) G(\mathbf{k}_2, \mathbf{k'}_2, \mathbf{q}, Y) V_b(\mathbf{k'}_1, \mathbf{k'}_2, \mathcal{J}_2, \mathbf{q}), \qquad \mathcal{J} = \{\mathbf{k}_\mathcal{J}, x_\mathcal{J}\}.$$

Jet Functions for NLO impact factor

$$J_1(\boldsymbol{q}, \boldsymbol{k}, \boldsymbol{l}, z) = \frac{1}{2}\frac{\boldsymbol{k}^2}{(\boldsymbol{q}-\boldsymbol{k})^2}\left(\frac{(1-z)^2}{(\boldsymbol{q}-z\boldsymbol{k})^2} - \frac{1}{\boldsymbol{q}^2}\right) - \frac{1}{4}\frac{1}{(\boldsymbol{q}-\boldsymbol{l})^2}\left(\frac{(\boldsymbol{l}-z\cdot\boldsymbol{k})^2}{(\boldsymbol{q}-z\boldsymbol{k})^2} - \frac{\boldsymbol{l}^2}{\boldsymbol{q}^2}\right)$$
$$- \frac{1}{4}\frac{1}{(\boldsymbol{q}-\boldsymbol{k}+\boldsymbol{l})^2}\left(\frac{(\boldsymbol{l}-(1-z)\boldsymbol{k})^2}{(\boldsymbol{q}-z\boldsymbol{k})^2} - \frac{(\boldsymbol{l}-\boldsymbol{k})^2}{\boldsymbol{q}^2}\right);$$

$$J_2(\boldsymbol{q}, \boldsymbol{k}, \boldsymbol{l}_1, \boldsymbol{l}_2) = \frac{1}{4}\left[\frac{\boldsymbol{l}_1^2}{(\boldsymbol{q}-\boldsymbol{k})^2(\boldsymbol{q}-\boldsymbol{k}+\boldsymbol{l}_1)^2} + \frac{(\boldsymbol{k}-\boldsymbol{l}_1)^2}{(\boldsymbol{q}-\boldsymbol{k})^2(\boldsymbol{q}-\boldsymbol{l}_1)^2}\right.$$
$$+ \frac{\boldsymbol{l}_2^2}{(\boldsymbol{q}-\boldsymbol{k})^2(\boldsymbol{q}-\boldsymbol{k}+\boldsymbol{l}_2)^2} + \frac{(\boldsymbol{k}-\boldsymbol{l}_2)^2}{(\boldsymbol{q}-\boldsymbol{k})^2(\boldsymbol{q}-\boldsymbol{l}_2)^2} - \frac{1}{2}\left(\frac{(\boldsymbol{l}_1-\boldsymbol{l}_2)^2}{(\boldsymbol{q}-\boldsymbol{l}_1)^2(\boldsymbol{q}-\boldsymbol{l}_2)^2}\right.$$
$$\left.\left. + \frac{(\boldsymbol{k}-\boldsymbol{l}_1-\boldsymbol{l}_2)^2}{(\boldsymbol{q}-\boldsymbol{k}+\boldsymbol{l}_1)^2(\boldsymbol{q}-\boldsymbol{l}_2)^2} + \frac{(\boldsymbol{k}-\boldsymbol{l}_1-\boldsymbol{l}_2)^2}{(\boldsymbol{q}-\boldsymbol{k}+\boldsymbol{l}_2)^2(\boldsymbol{q}-\boldsymbol{l}_1)^2} + \frac{(\boldsymbol{l}_1-\boldsymbol{l}_2)^2}{(\boldsymbol{q}-\boldsymbol{k}+\boldsymbol{l}_1)^2(\boldsymbol{q}-\boldsymbol{k}+\boldsymbol{l}_2)^2}\right)\right].$$