Dark and shiny dresses around primordial black holes

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"la Caixa"

Dark matter candidates

10^{-22} eV — 1 keV — 1 GeV — 1 TeV — 10^{19} GeV (10^{-5} g) — 10^{57} GeV (10^{33} g)

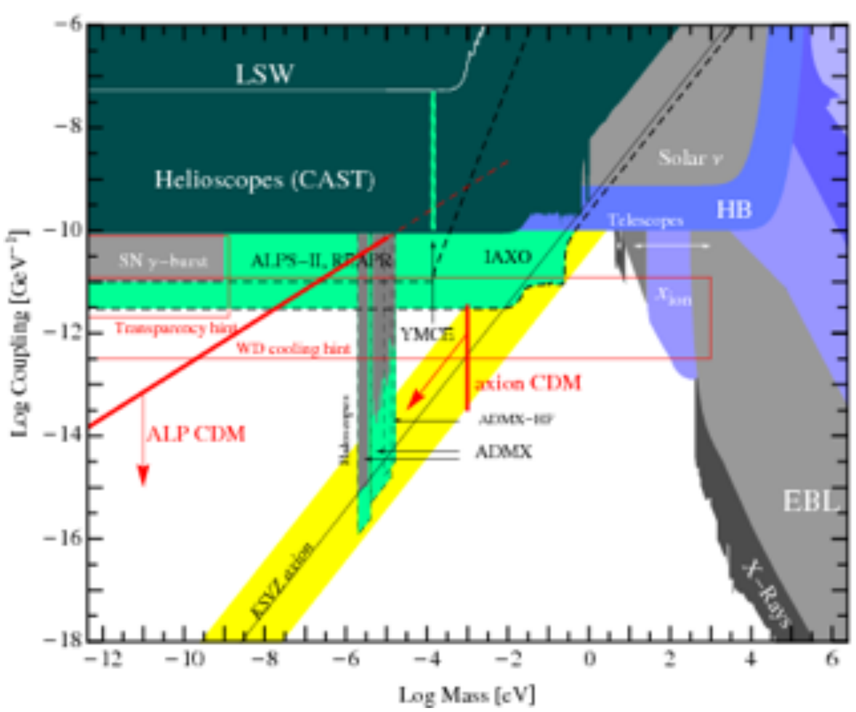
"Fuzzy dark matter"

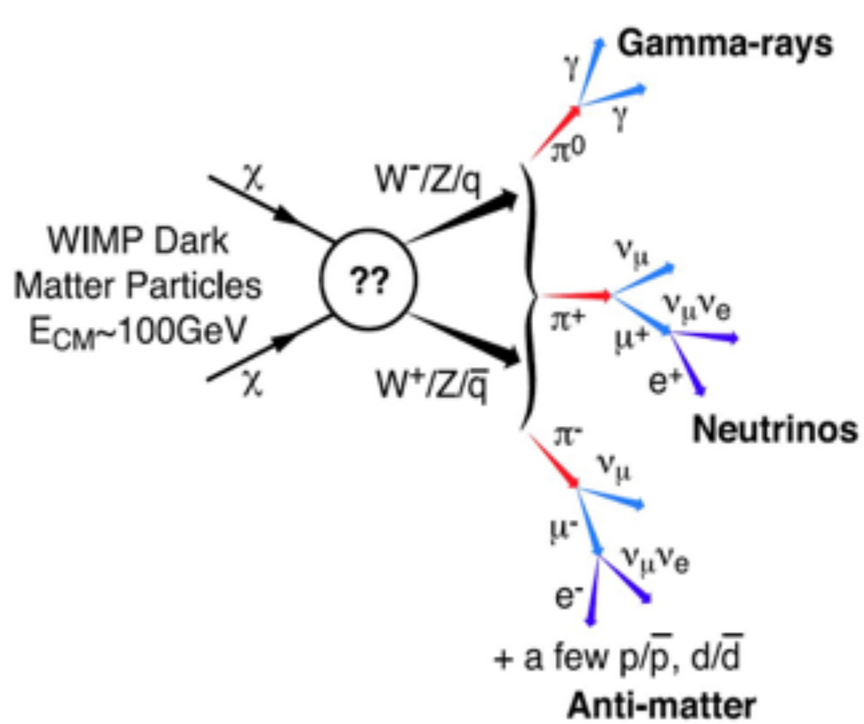
Axion-like particles

Sterile neutrinos

WIMPs

Primordial black holes (PBHs)





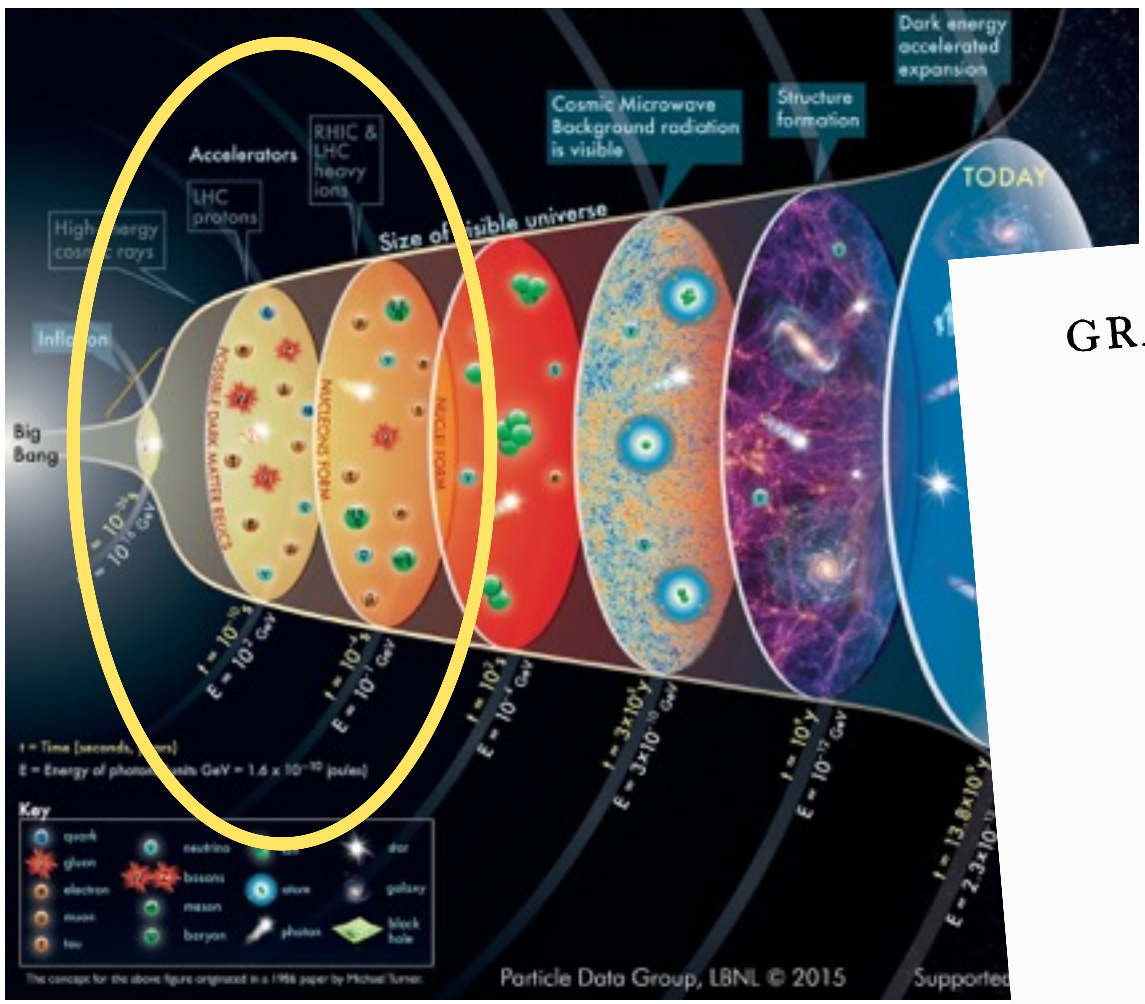
Proposed more than 40 years ago *[Zeld'ovich and Novikov 1966, Hawking 1971]*

Became less popular for a period after MACHO and EROS searches based on lensing *[Alcock 2001]*

Recently **reconsidered in the DM community**, as we will see

Primordial black holes

- Primordial black holes are **hypothetical black holes that could have formed in the early Universe**, deep into the radiation-dominated era, before BBN



GRAVITATIONALLY COLLAPSED OBJECTS OF VERY LOW MASS

Stephen Hawking

(Communicated by M. J. Rees)

(Received 1970 November 9)

SUMMARY

It is suggested that there may be a large number of gravitationally collapsed objects of mass 10^{-5} g upwards which were formed as a result of fluctuations in the early Universe. They could carry an electric charge of up to ± 30 electron units. Such objects would produce distinctive tracks in bubble chambers and could form atoms with orbiting electrons or protons. A mass of 10^{17} g of such objects could have accumulated at the centre of a star like the Sun. If such a star later became a neutron star there would be a steady accretion of matter by a central collapsed object which could eventually swallow up the whole star in about ten million years.

Primordial black hole formation

- **The early universe (t < 1 s) is extremely hot and dense: It is an ideal environment for black hole formation.**
  - Cosmological density in the early Universe, during *radiation domination*: $\rho_C = 10^6 \left(\frac{t}{\mathrm{s}}\right)^{-2} \frac{\mathrm{g}}{\mathrm{cm}^3}$
  - Density associated to the Schwarzschild length scale: $\rho_S = 10^{18} \left(\frac{M}{M_\odot}\right)^{-2} \frac{\mathrm{g}}{\mathrm{cm}^3}$
  - At very early times, the cosmological density is compatible with formation of black holes of mass
    - *$t \sim 10^{-45}$ s —> $M \sim M_{Planck}$*
    - *$t \sim 10^{-23}$ s —> $M \sim 10^{15}$ g*
    - *$t \sim 10^{-6}$ s —> $M \sim 10^{33}$ g $\sim M_{Sun}$*
    - *$t \sim 1$ s —> $M \sim 10^{6}$ M_{Sun}*
  - We can convince ourselves that, for each time, there is a mass scale for which

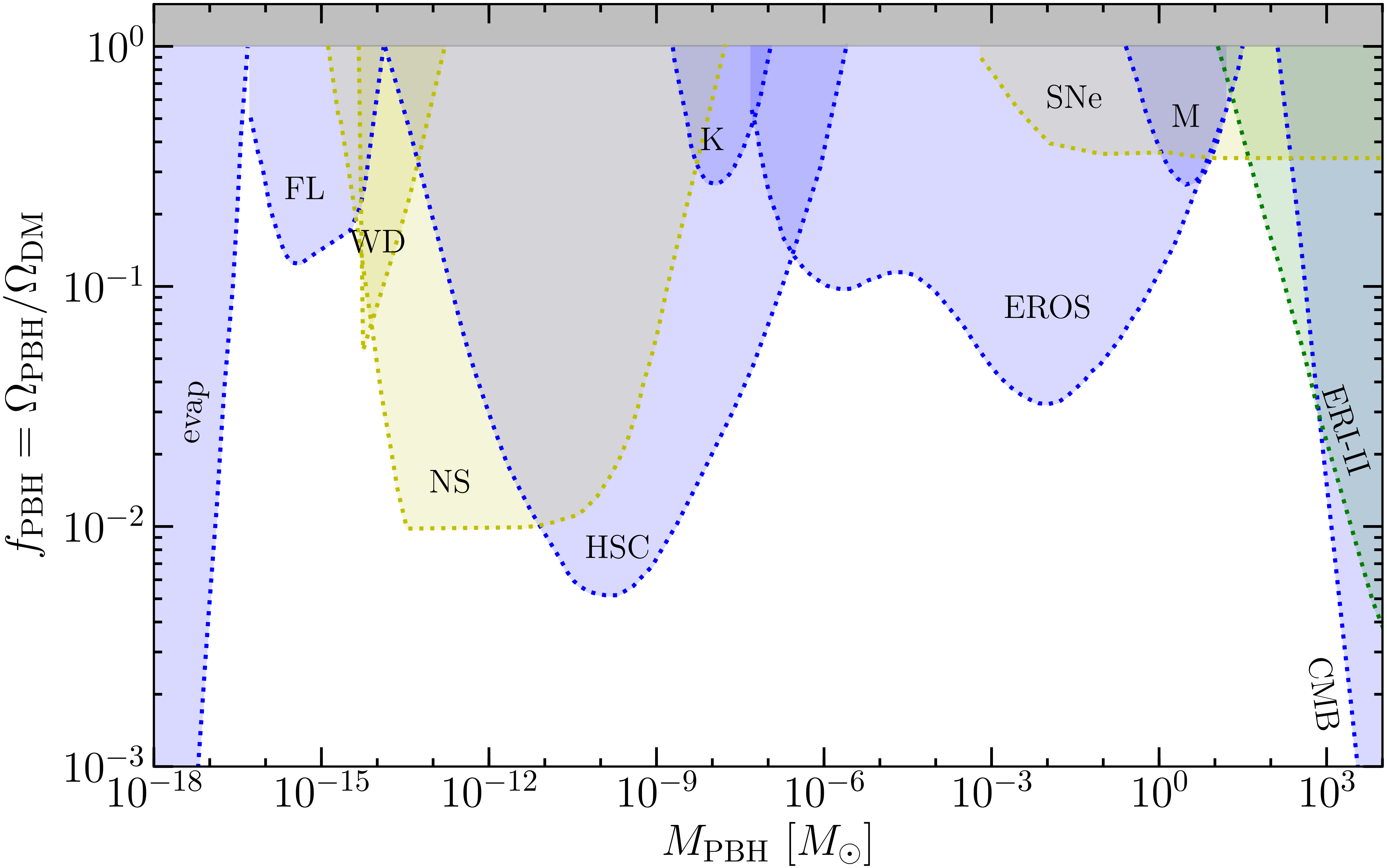
Jeans scale ~ Schwarzschild length scale

$$\lambda_J = \frac{c_s}{\sqrt{G\rho}} \simeq (10^{19}\,\mathrm{km}) \left(\frac{\rho}{3.9 \cdot 10^{-27} \mathrm{g/cm^3}}\right)^{-(1/2)}$$

$$\lambda_J \sim (6.7 \cdot 10^5\,\mathrm{km}) \left(\frac{t}{1\,\mathrm{s}}\right) \sim (2.95\,\mathrm{km}) \frac{M}{M_\odot} \;\rightarrow\; M(t) \sim 2.2 \cdot 10^5 M_\odot \left(\frac{t}{1\,\mathrm{s}}\right)$$

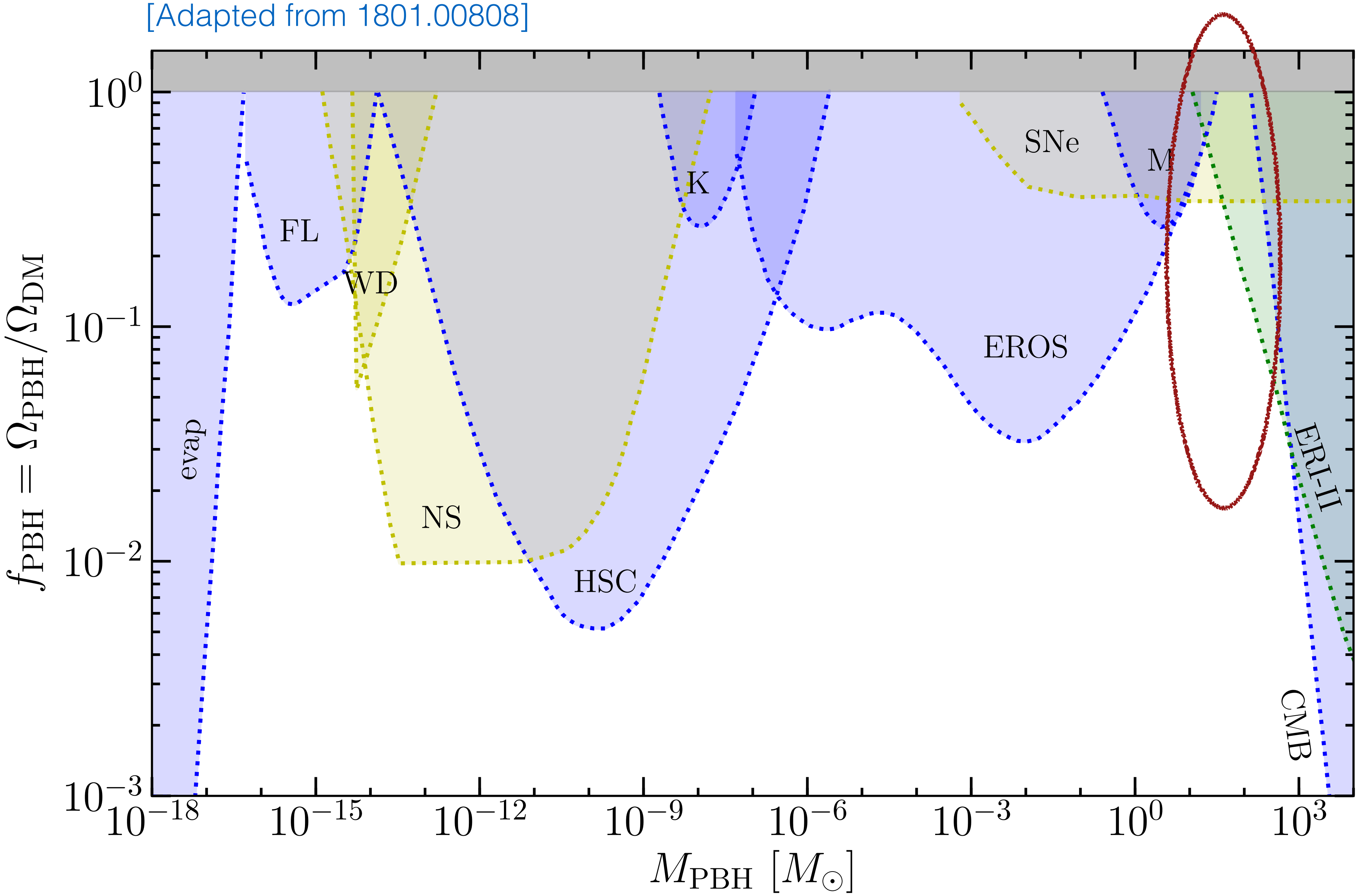
PBH phenomenology

[Adapted from 1801.00808]



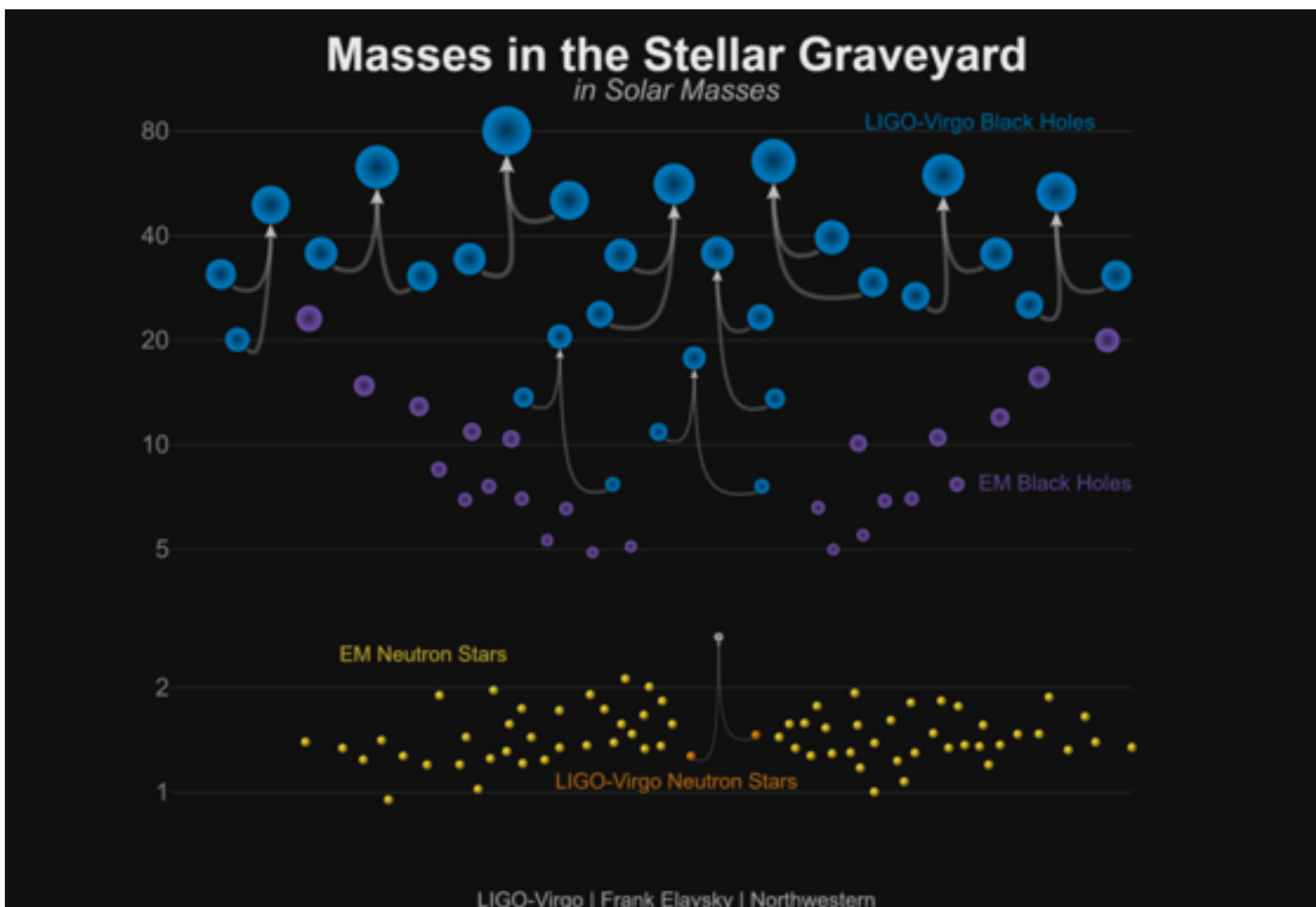
- **The [1 — 100] M_{Sun} mass window is particularly interesting**
- The most studied constraints are very weak in this window

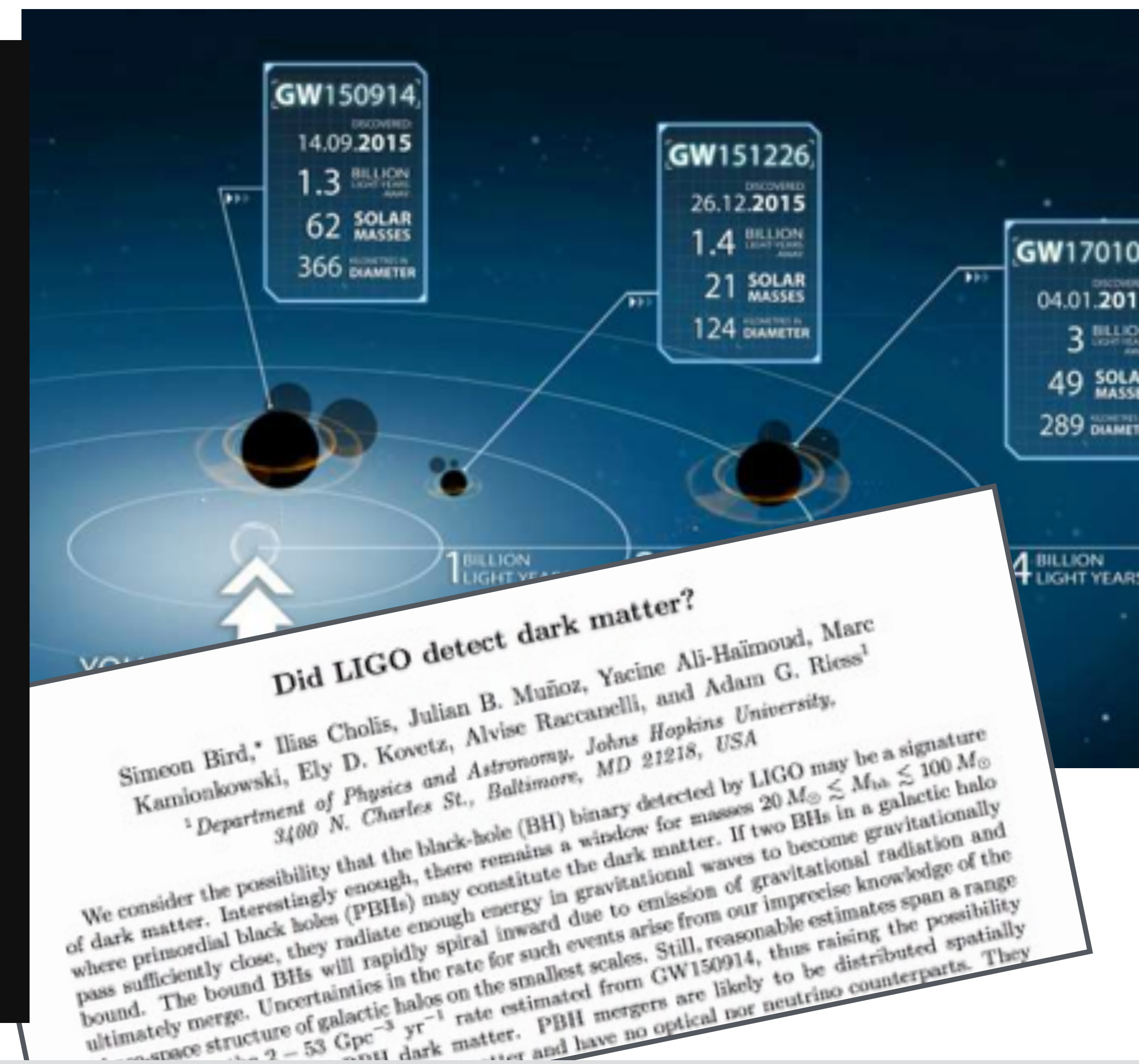
[Adapted from 1801.00808]



The LIGO/Virgo window: The merger rate

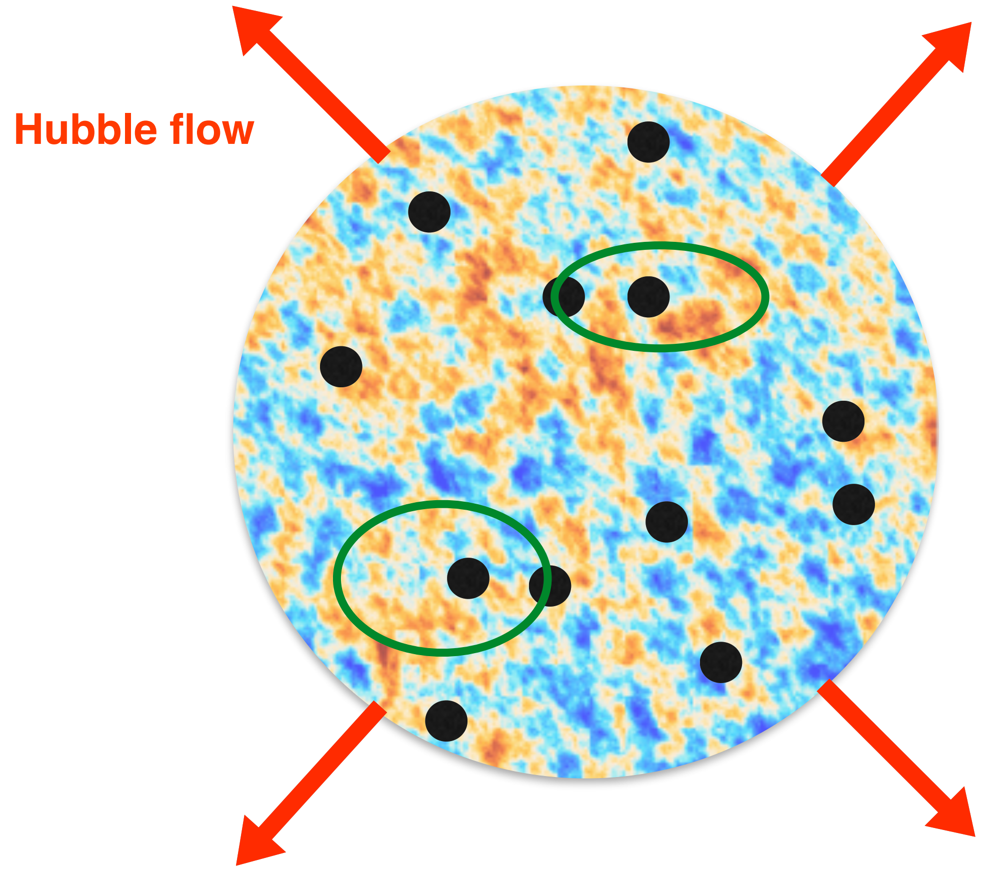
- **Virgo and LIGO** are detecting mergers of binary-black-hole systems in that mass range
- **10 events** associated to BBH mergers. Large masses, low spins. Cosmological distances.
- More massive than BHs detected as part of X-ray binary systems, which are mostly Galactic and cover the ~5 to ~15 M_{Sun} range (GRS 1915+105, M = 14±4 Msun, arXiv:0111540)
- Are the BBH systems detected by LIGO of primordial origin? [Bird+ PRL 2017, Clesse&García-Bellido PDU 2016]





The merger rate of a PBH population

A) Binaries formed in the early Universe



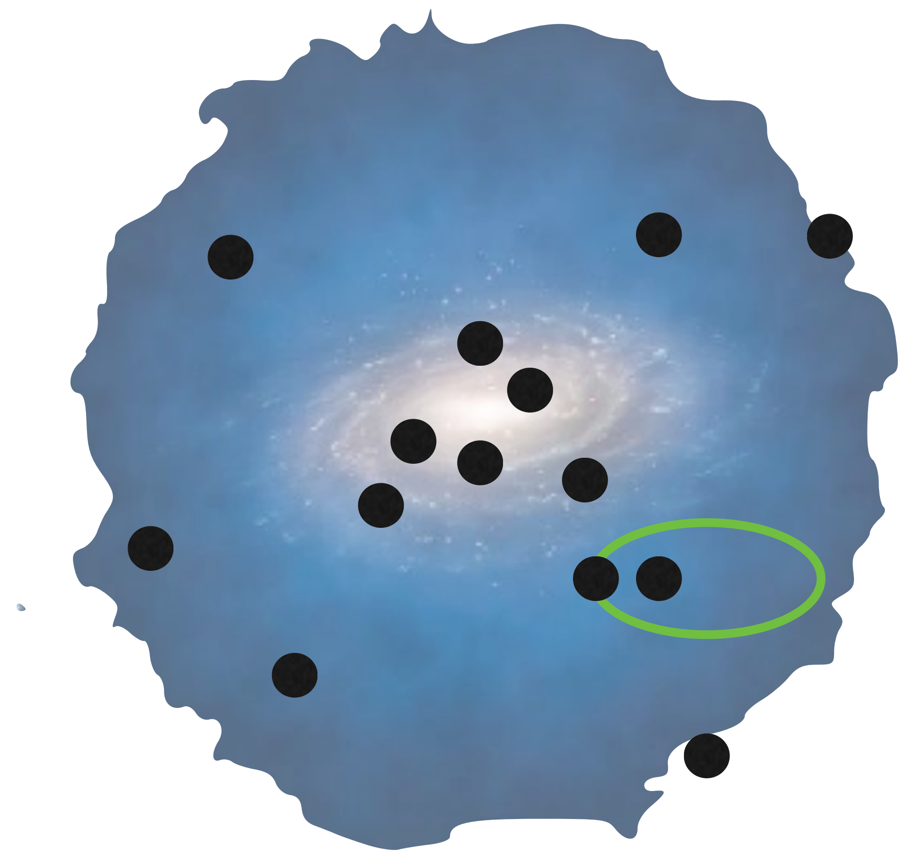
Require:

$$M_{\rm BH} R^{-3} > \rho(z) \text{ before } z_{\rm eq}$$

- If most of the DM is made of PBHs, **most pairs decouple from the Hubble flow** and form a binary deep in the radiation era.
- If f < 0.01, only **rare pairs with small separation form binary systems**.

Sasaki+ PRL 2017
Ali-Haimoud+ 2017

B) Binaries formed after close encounters within a DM halo



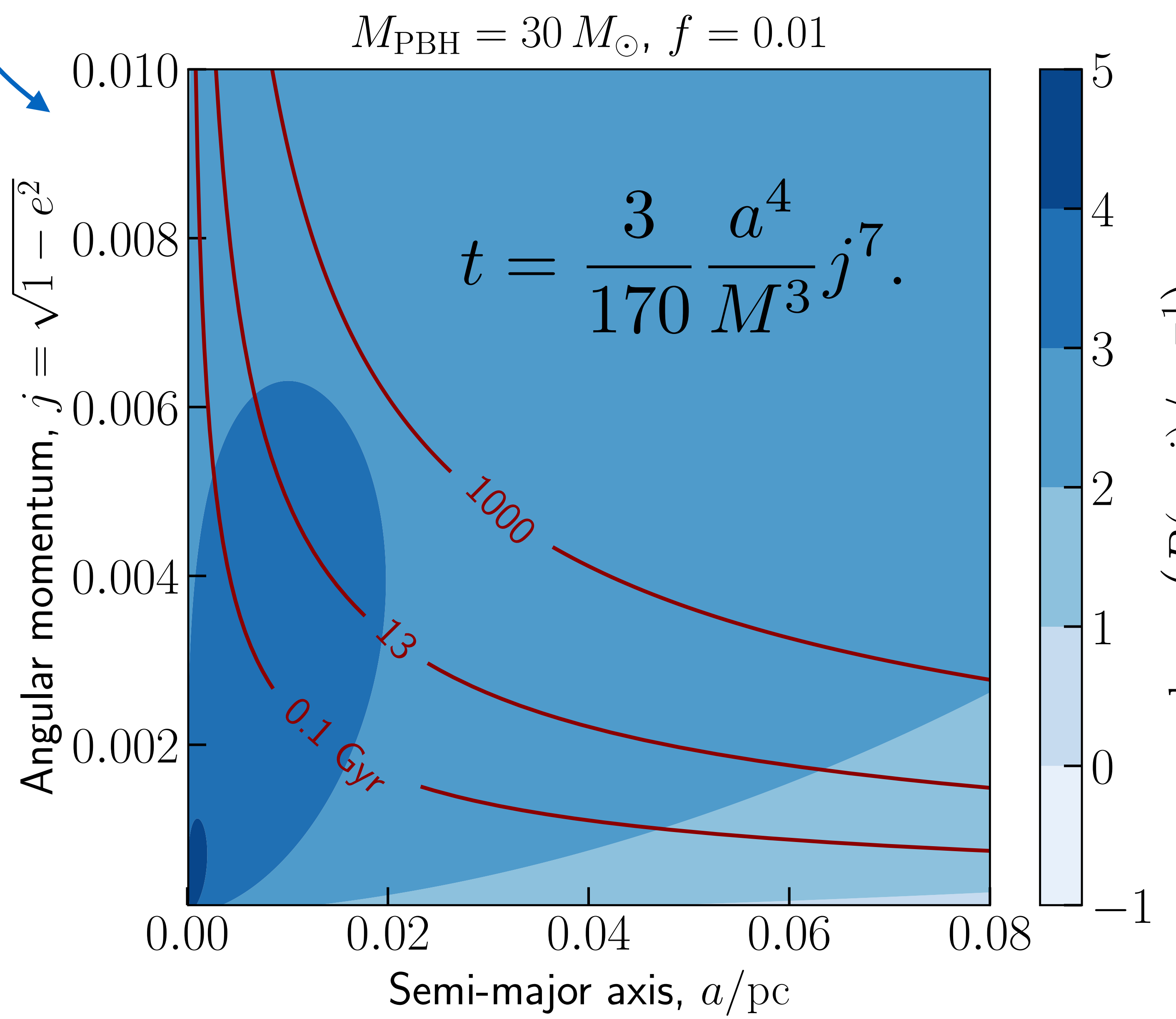
$$\sigma = \pi \left(\frac{85\,\pi}{3}\right)^{2/7} R_s^2 \left(\frac{v_{\rm pbh}}{c}\right)^{-18/7}$$
$$= 1.37 \times 10^{-14}\, M_{30}^2\, v_{\rm pbh-200}^{-18/7}\, {\rm pc}^2$$

Bird+ PRL 2017
Clesse, García-Bellido, PDU 2016

The merger rate of primordial BBHs

- If PBHs make up all of the DM, most of the pairs decouple from the Hubble flow before matter-radiation equality and form bound systems.
- Probability distribution of PBH binaries that form deep in the radiation era from a randomly (unclustered) distribution of compact objects.
- The angular momentum stems from the torque exerted by all the other PBHs

$e > 0.9999$

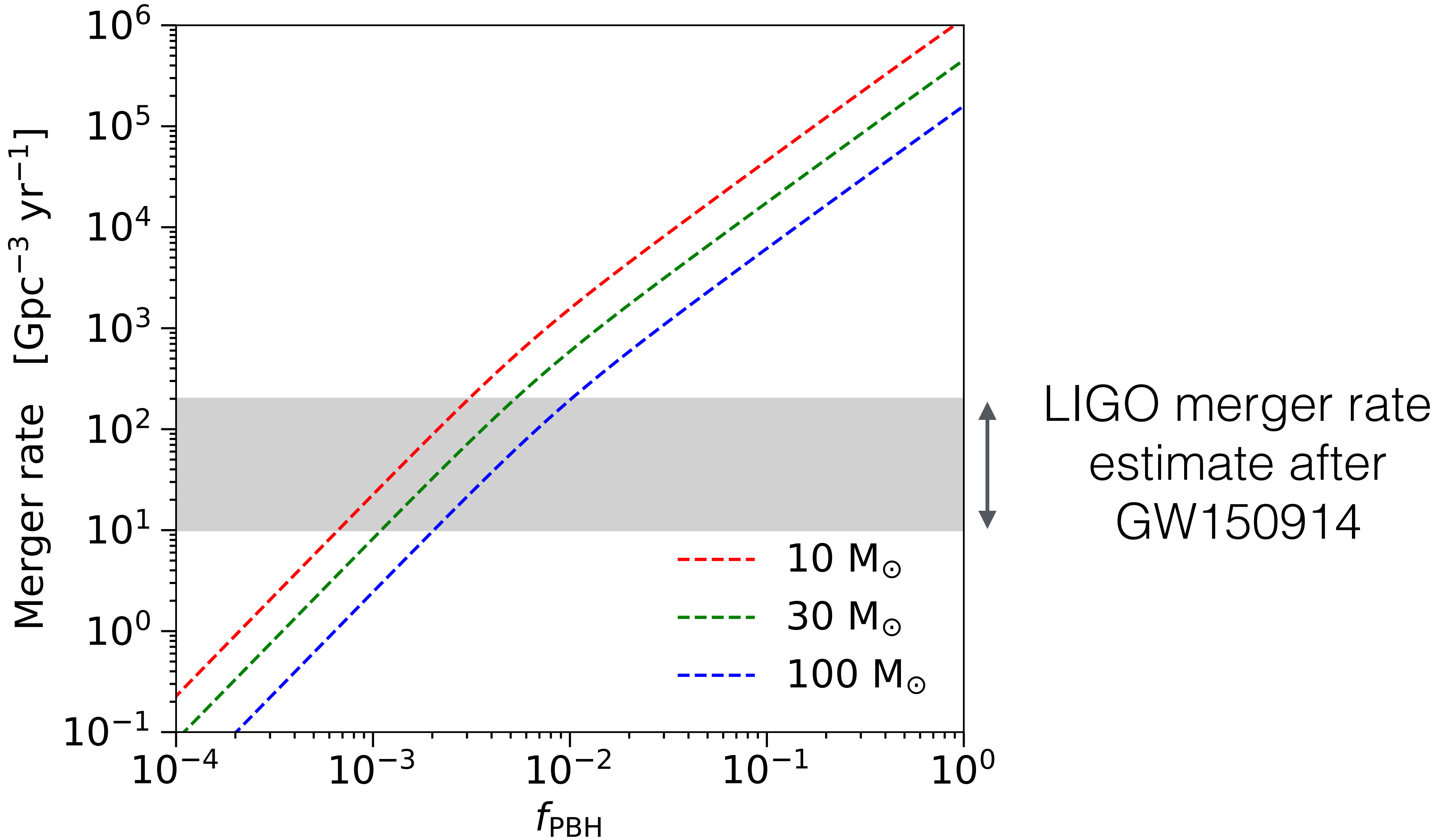


Sasaki+ PRL 2017
Ali-Haimoud+ 2017

The merger rate of primordial BBHs

- The **merger rate** associated to the BBH systems that formed in the early universe if $f_{PBH} \ll 1$ is way **larger than the one inferred by the LIGO/Virgo collaborations**

- **Caveats:**
  - Survival over cosmological timescales
  - Clustering (may be potentially relevant for broad mass functions)
  - "***Dark Dress***" if $f_{PBH} \ll 1$: *Multi-dark components!*

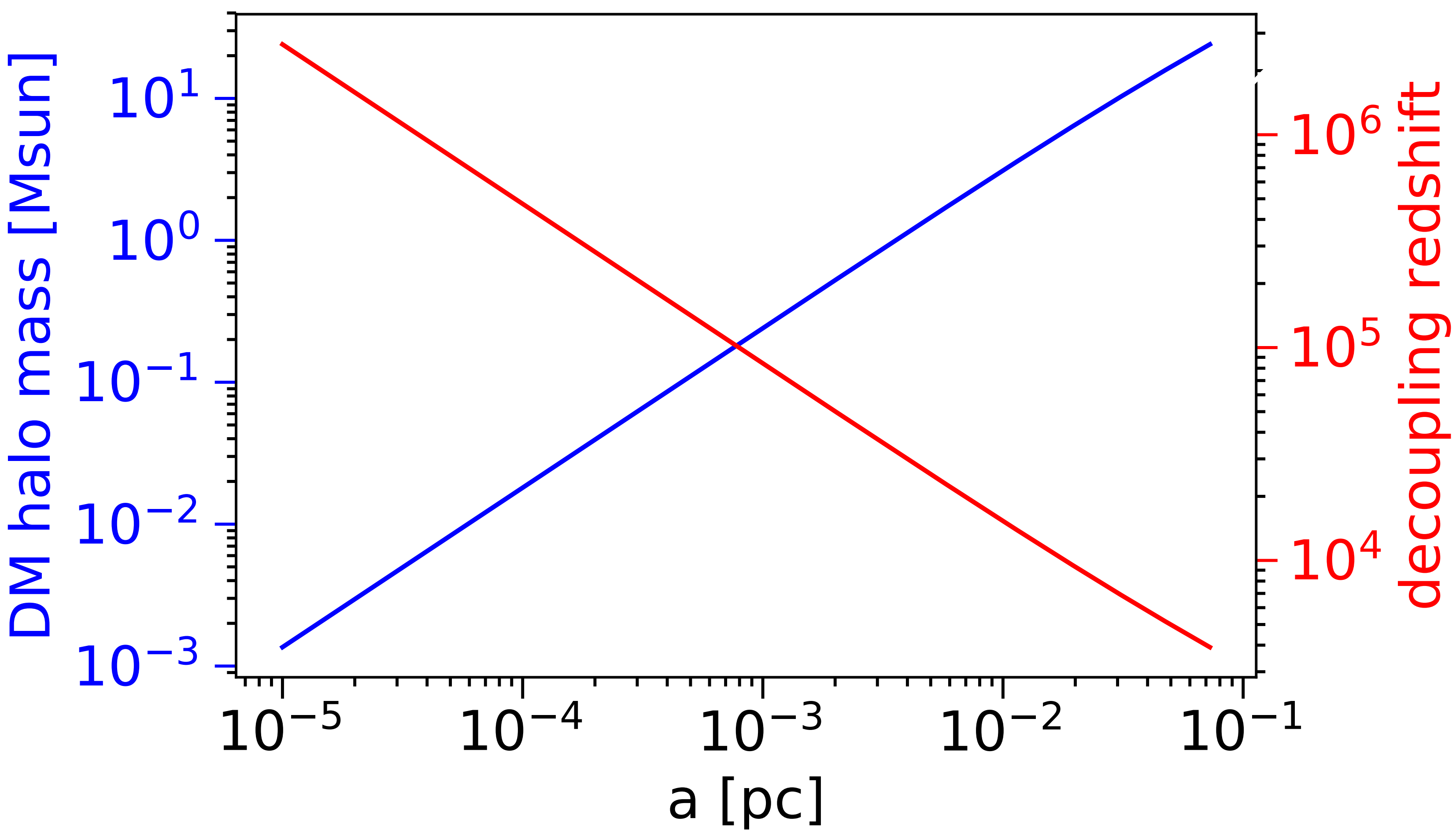


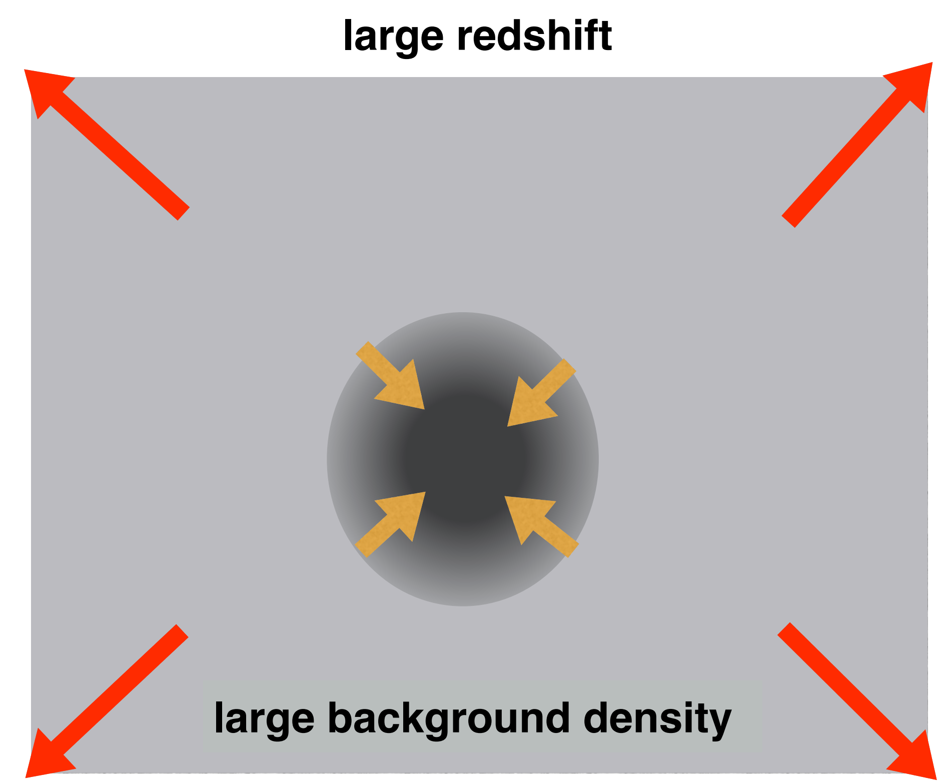
The merger rate of primordial BBHs *with dark dress*

- A sub-dominant population of PBHs immersed in a high-density DM-dominated environment, rapidly expanding and diluting

- Accretion of DM mini-halos governed by the **balance between gravitational pull and expansion of the universe**

$$R_{\rm tr}(z) = 1300\ \mathrm{AU}\left(\frac{M_{\rm eq}}{\mathrm{M}_\odot}\right)^{1/3}\left(\frac{1+z}{1+z_{\rm eq}}\right)^{-1}$$





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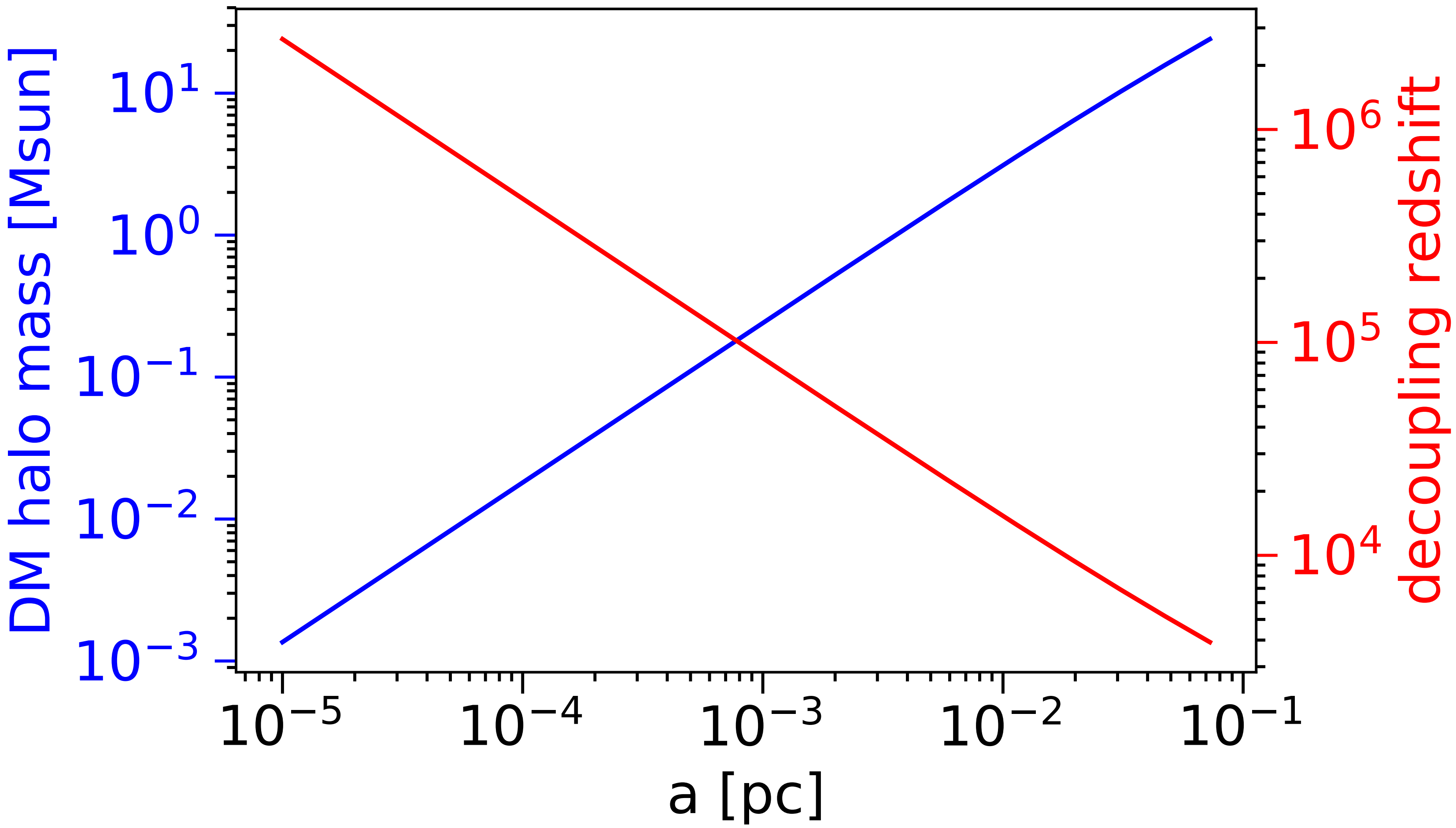
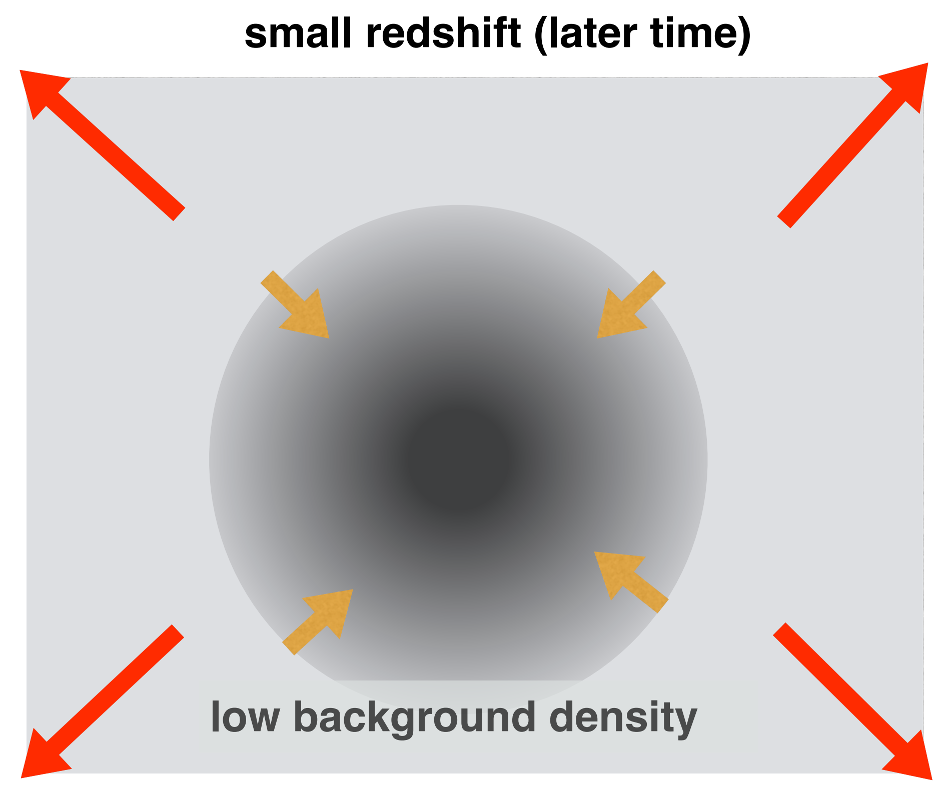
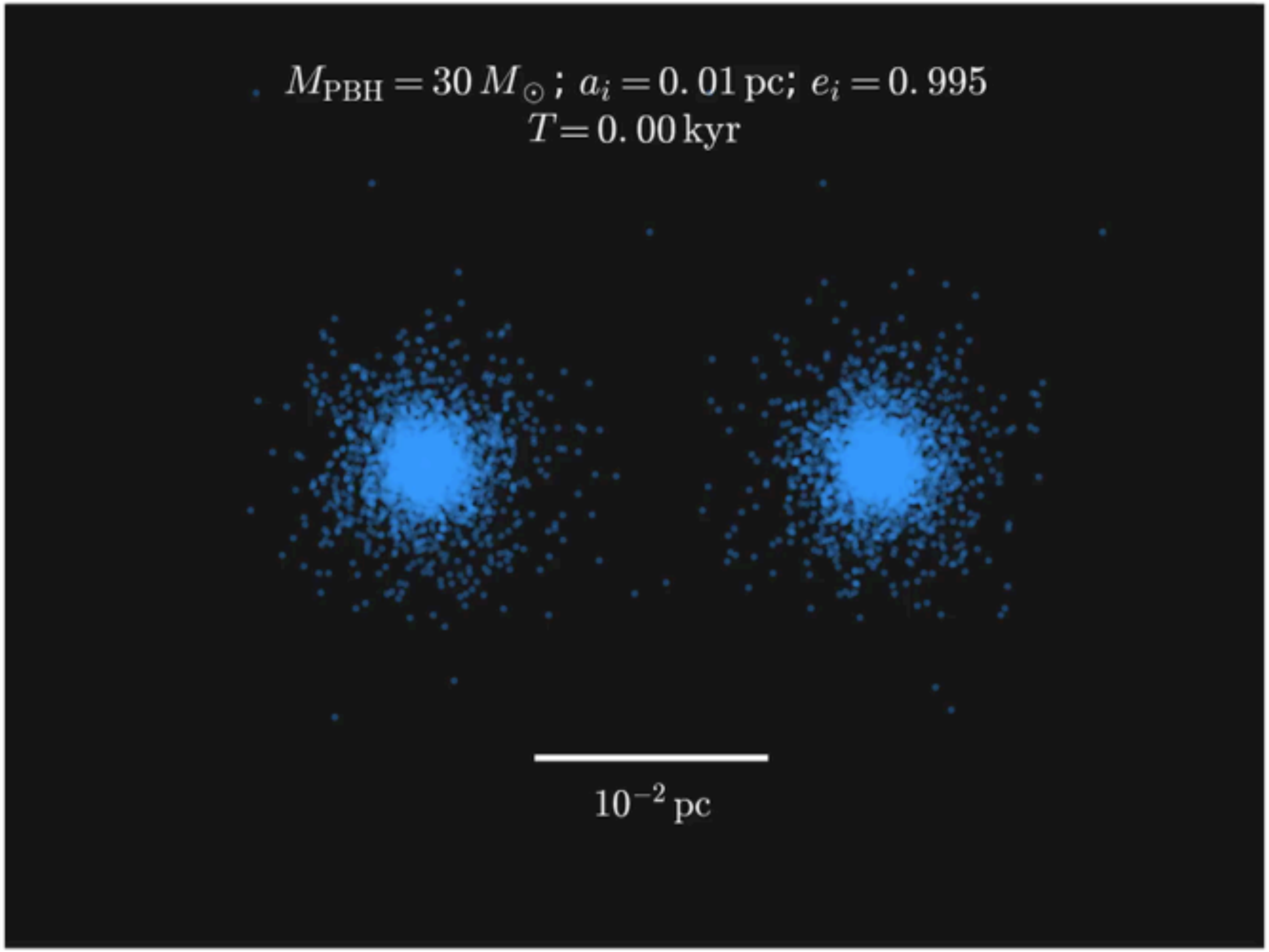


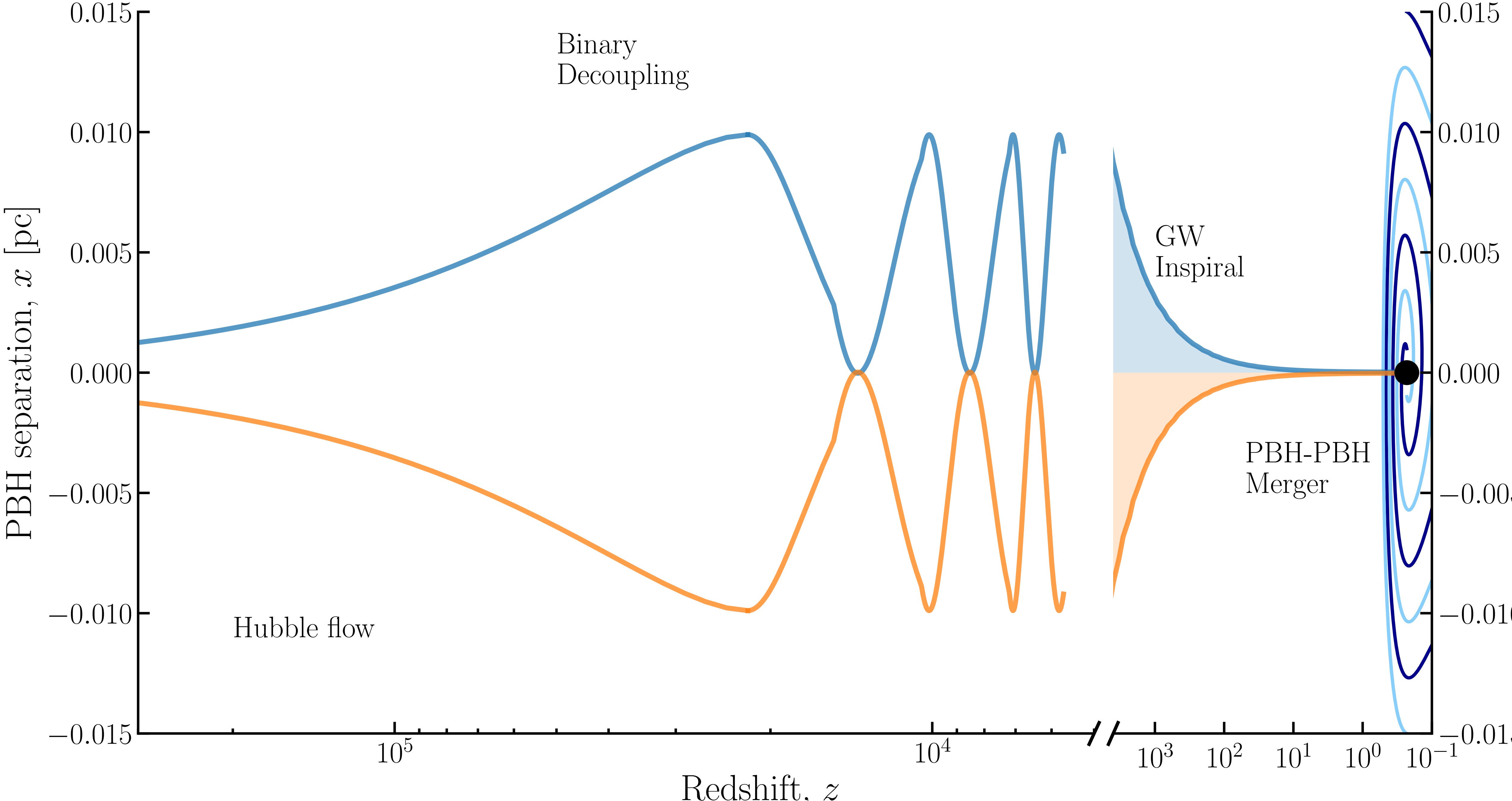


The merger rate of primordial BBHs *with dark dress*

- What is the impact of the Dark Dress on the evolution of the BBH system?



The life of a “naked” BBH system

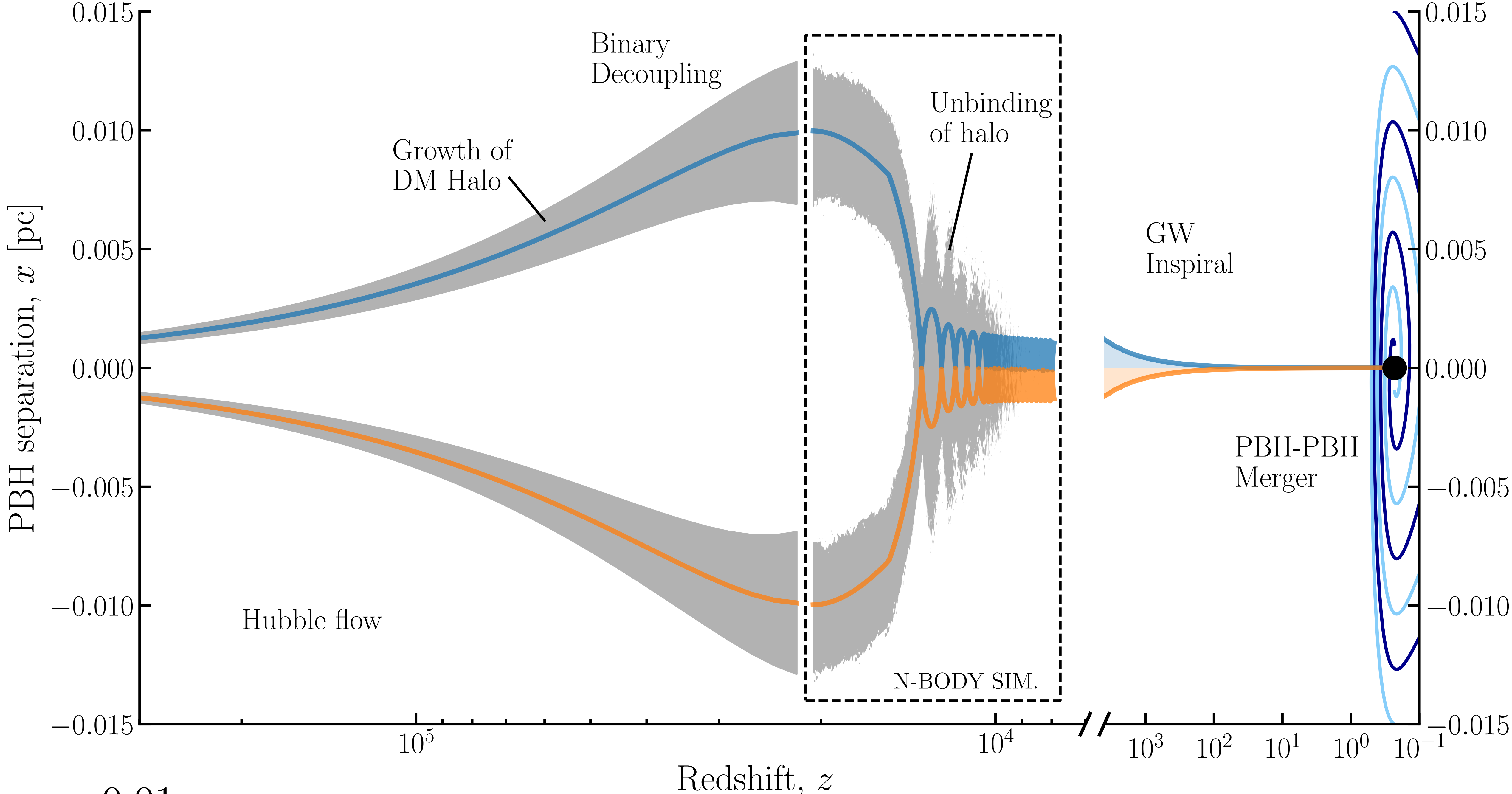


$a = 0.01\,\mathrm{pc}$

$e = 0.995$

- Courtesy of B. Kavanagh

The merger rate of primordial BBHs *with dark dress*



$a_i = 0.01\,\mathrm{pc}$

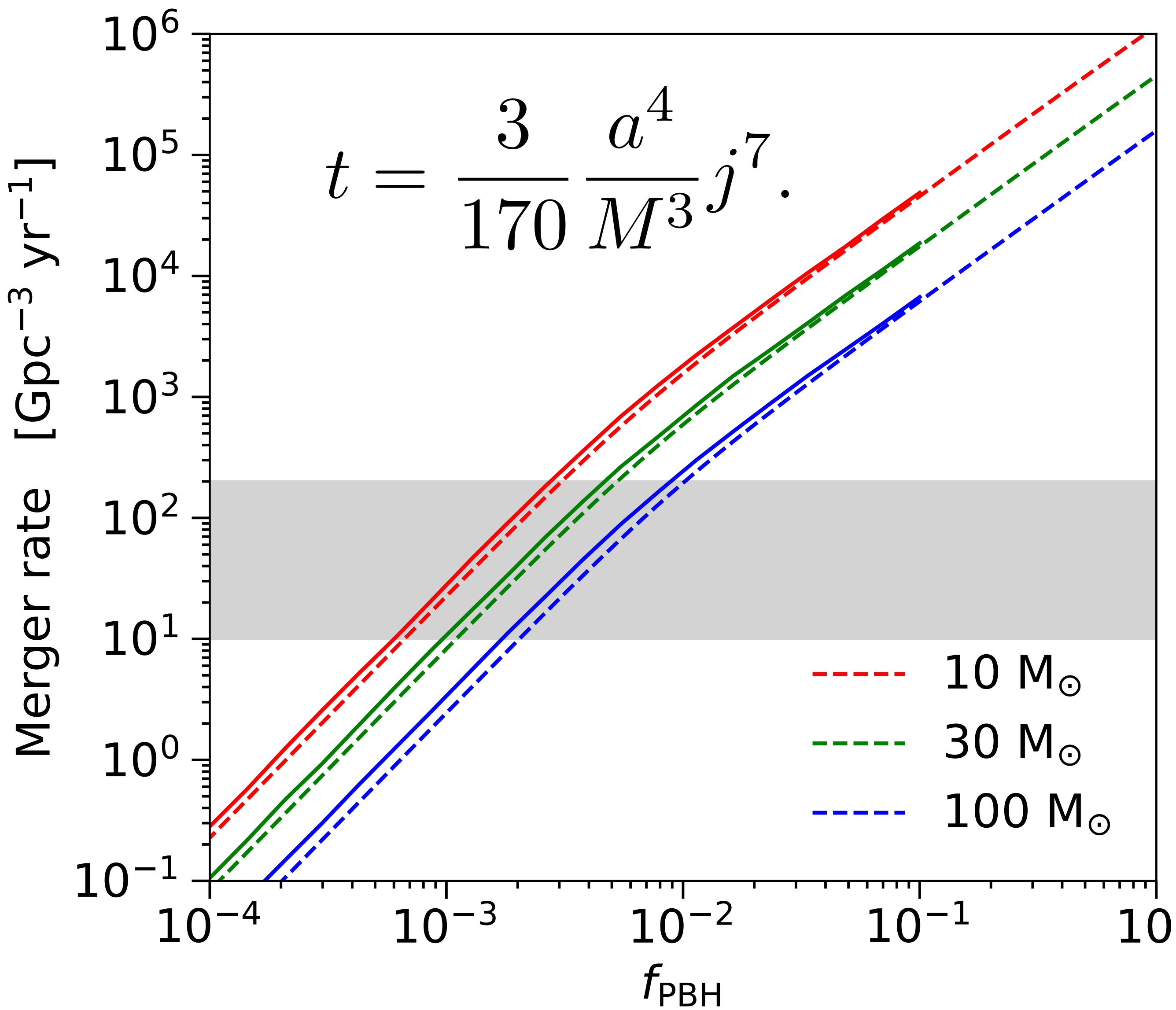
$e_i = 0.995$

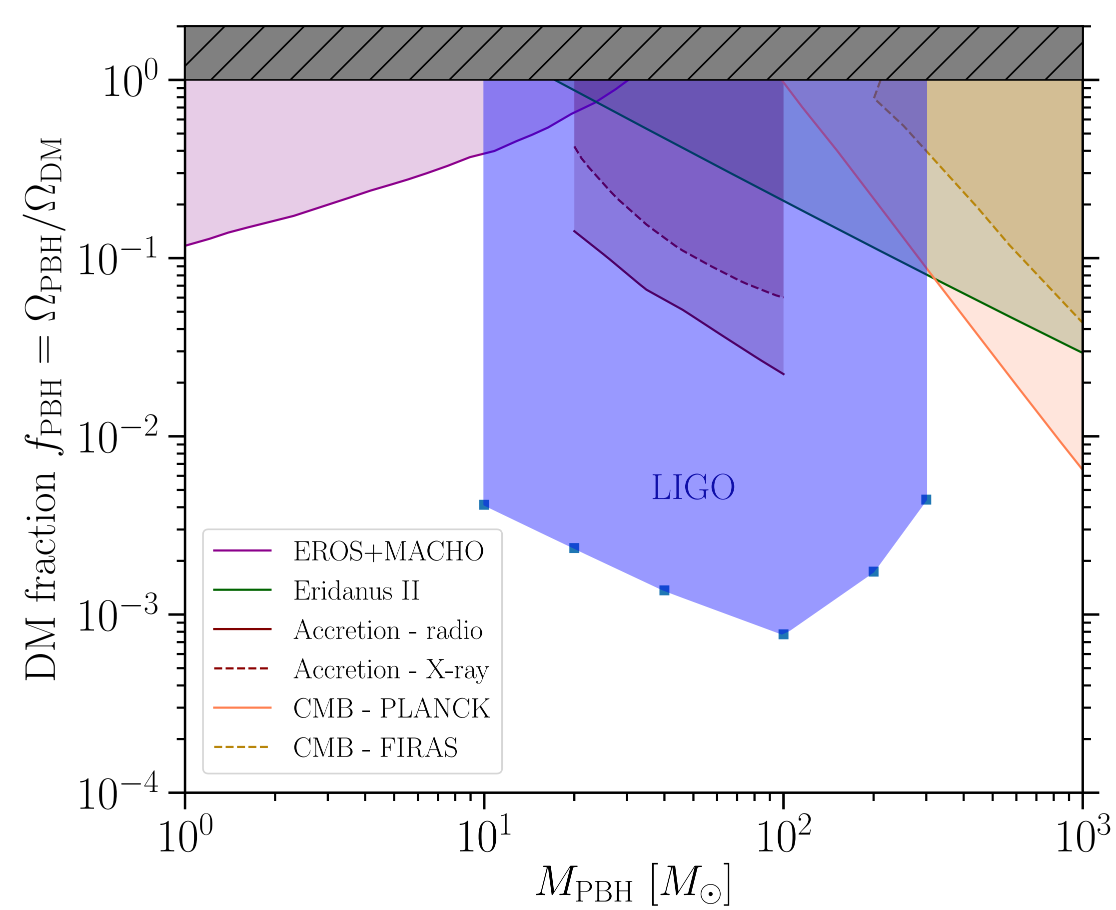
- Courtesy of B. Kavanagh

The merger rate of primordial BBHs *with dark dress*

- What is the **impact** of the Dark Dress on the **merger rate**? The rate is almost **conserved**!
- The binary shrinks and hardens
- The work done by dynamical friction heats and unbinds the DM halo

$$E^{\rm bind}(r_{\rm in}) = -4\pi G_N \int_{r_{\rm in}}^{\infty} \frac{M_{\rm enc}(r)}{r}\, r^2 \rho_{\rm DM}(r)\, \mathrm{d}r\,. \qquad E_i^{\rm orb} + 2\,E^{\rm bind}(r_{\rm min}/2) = E_f^{\rm orb}\,.$$

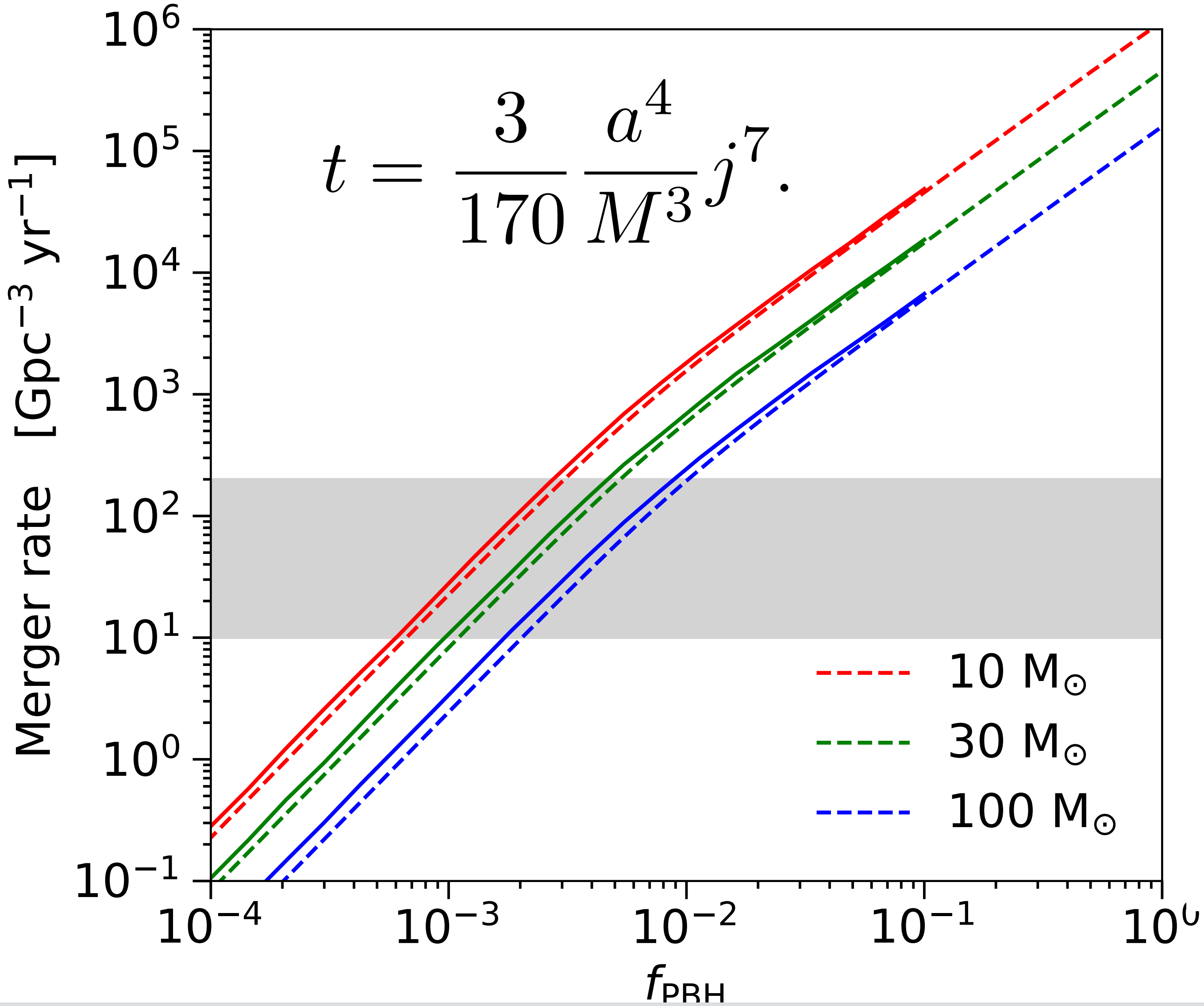


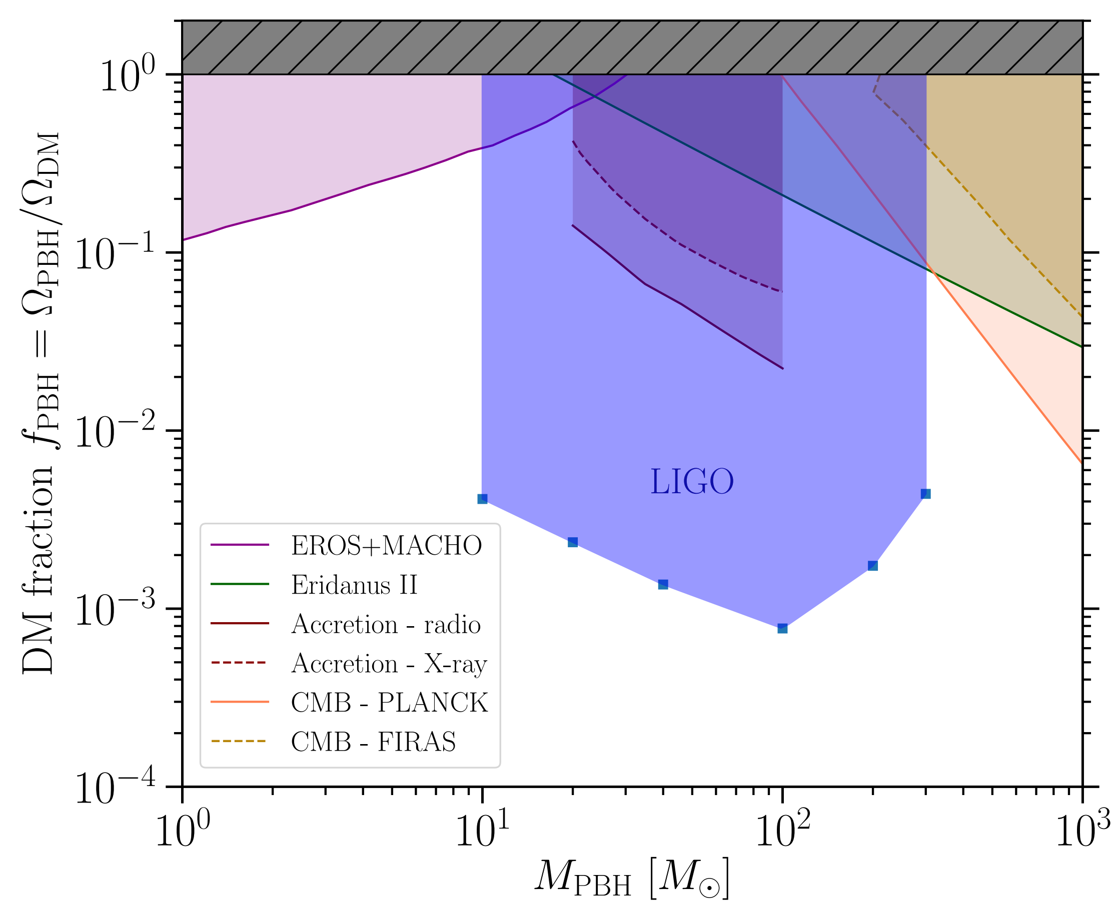


The merger rate of primordial BBHs *with dark dress*

- For very eccentric orbits there is very **little exchange of angular momentum** between the PBHs and the DM particles: For large eccentricity the orbits are almost radial
- Shrinking and hardening is not independent. **Merger rate roughly conserved**

$$a_f(a_i) = \frac{G_N M_f^2 a_i}{G_N M_{tot}^2 + 4 a_i E^{\rm bind}(r_{\rm in})}\,. \qquad j_f = \sqrt{\frac{a_i}{a_f}}\, j_i \qquad t_f = \sqrt{\frac{a_i}{a_f}}\, t_i\,,$$



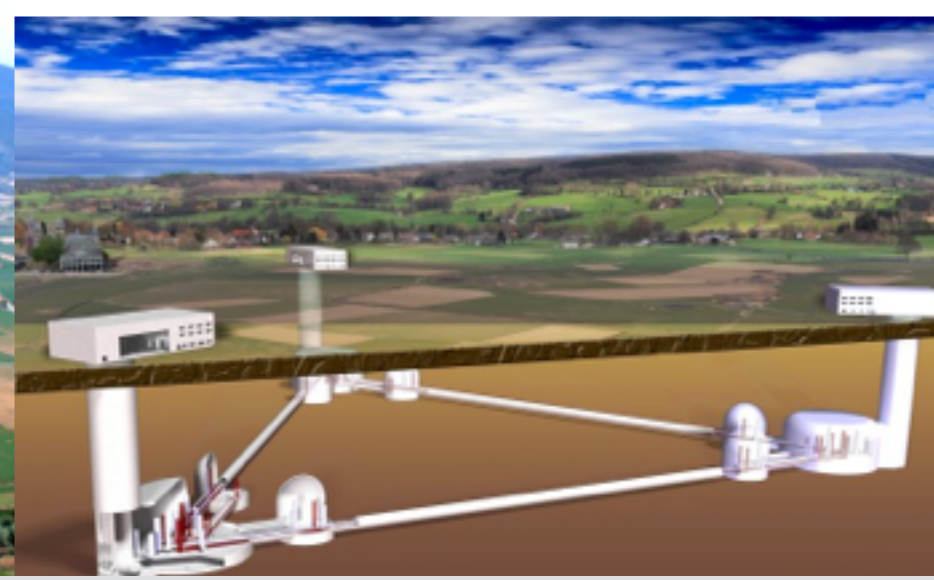


Searching for a (sub-dominant?) population of PBHs

- Goal: discover a population of PBHs in the 1 - 100 M_{Sun} window, even if sub-dominant with respect to the bulk of the Dark Matter.

Possible **roads towards a discovery**:

- Detection of GWs produced by the merger of O(10) M_{Sun} BHs at **large redshifts** ($z > 40$) with the Einstein telescope
- Detection of GWs produced by the merger of BHs with **mass $M < 1$ M_{Sun}** with LIGO/Virgo
- Detection of the **radio emission produced by the accretion** of gas onto 1–1000 M BHs


Astronomical searches in the radio/X-ray sky

Astronomical searches

- What about possible astronomical signatures of the presence of a population of PBHs?
- **Is it possible to detect them** even if a small sub-dominant component of the dark matter?
- **Some numbers:** If $\sim 30 M_\odot$ PBHs are the DM —> **$\sim 10^{11}$ objects of this kind in the Milky Way, and $\sim 10^{8}$ in the Galactic bulge**.
- **The idea:** these objects should **accrete interstellar gas in our Galaxy and emit a broad-band spectrum of radiation.**

  - **Results based on**
    - D. Gaggero, G. Bertone, F. Calore, R. Connors, M. Lovell, E. Storm, S. Markoff PRL 118 (2017)
    - J. Manshanden, D. Gaggero, G. Bertone, R. Connors, M. Ricotti, submitted to JCAP (2018)
  - **Ongoing further work at IFT in collaboration with the GRAPPA institute**

GRAPPA
GRavitation AstroParticle Physics Amsterdam

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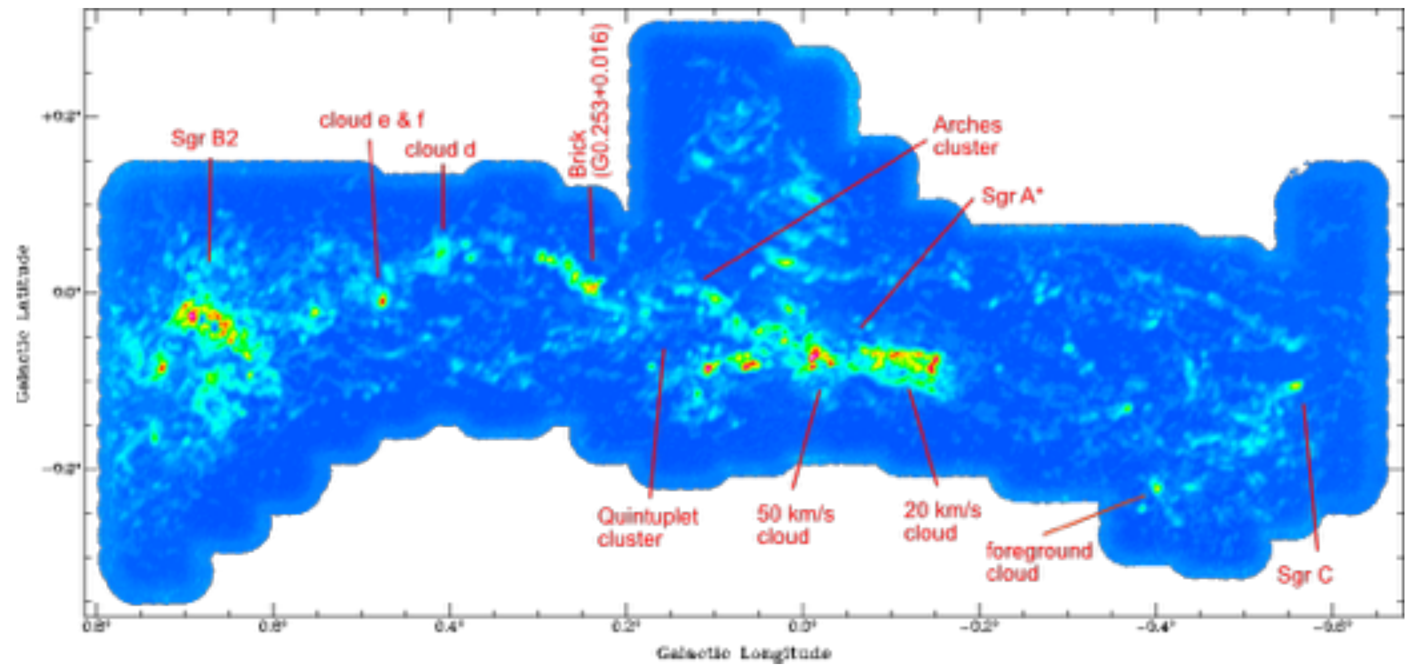
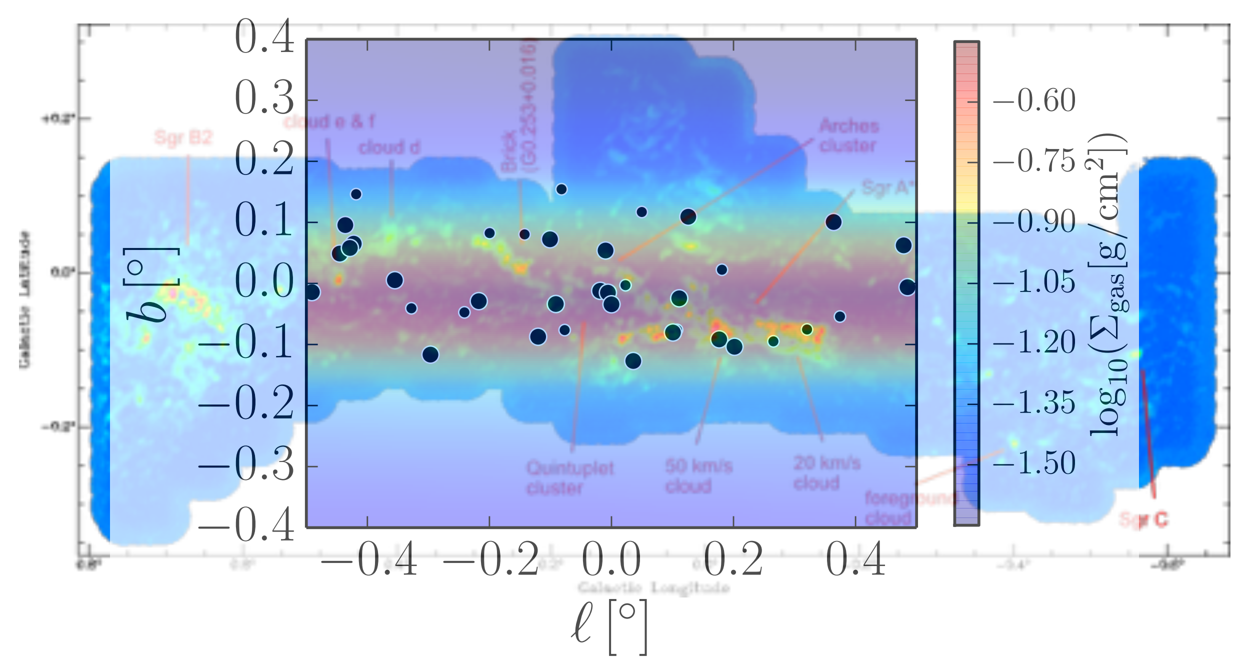
Astronomical searches: a recent simulation

- We set up a set of **Monte-Carlo simulations** to simulate the radio and gamma-ray emission from a population of PBHs that amount to a fraction of the Dark Matter in the Universe



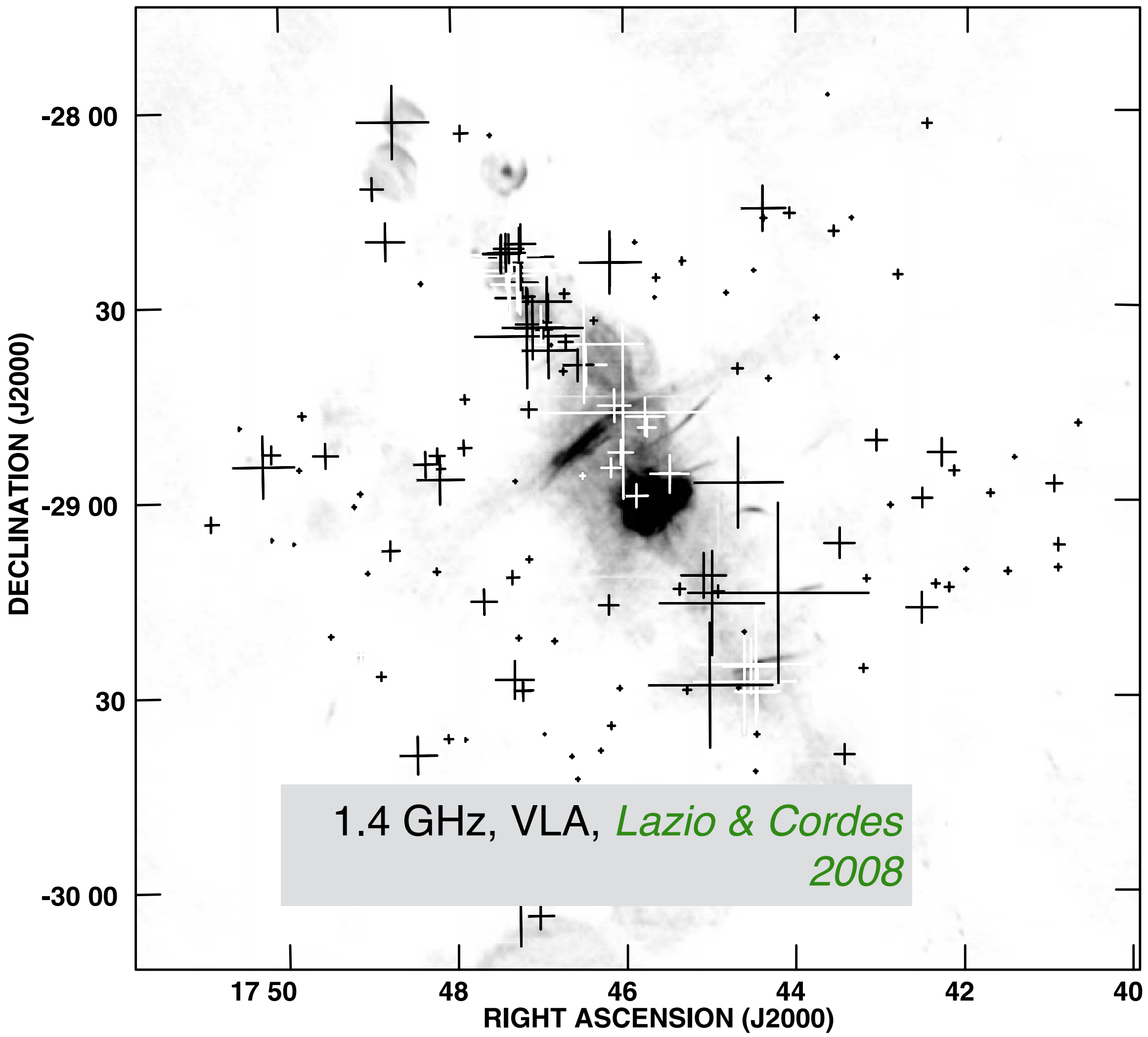
- We consider a state-of-the-art model for the **3D gas distribution** in the inner Galaxy

- **We compute simulated maps of the expected radio and X-ray sources** near the Galactic center region associated to the PBH population

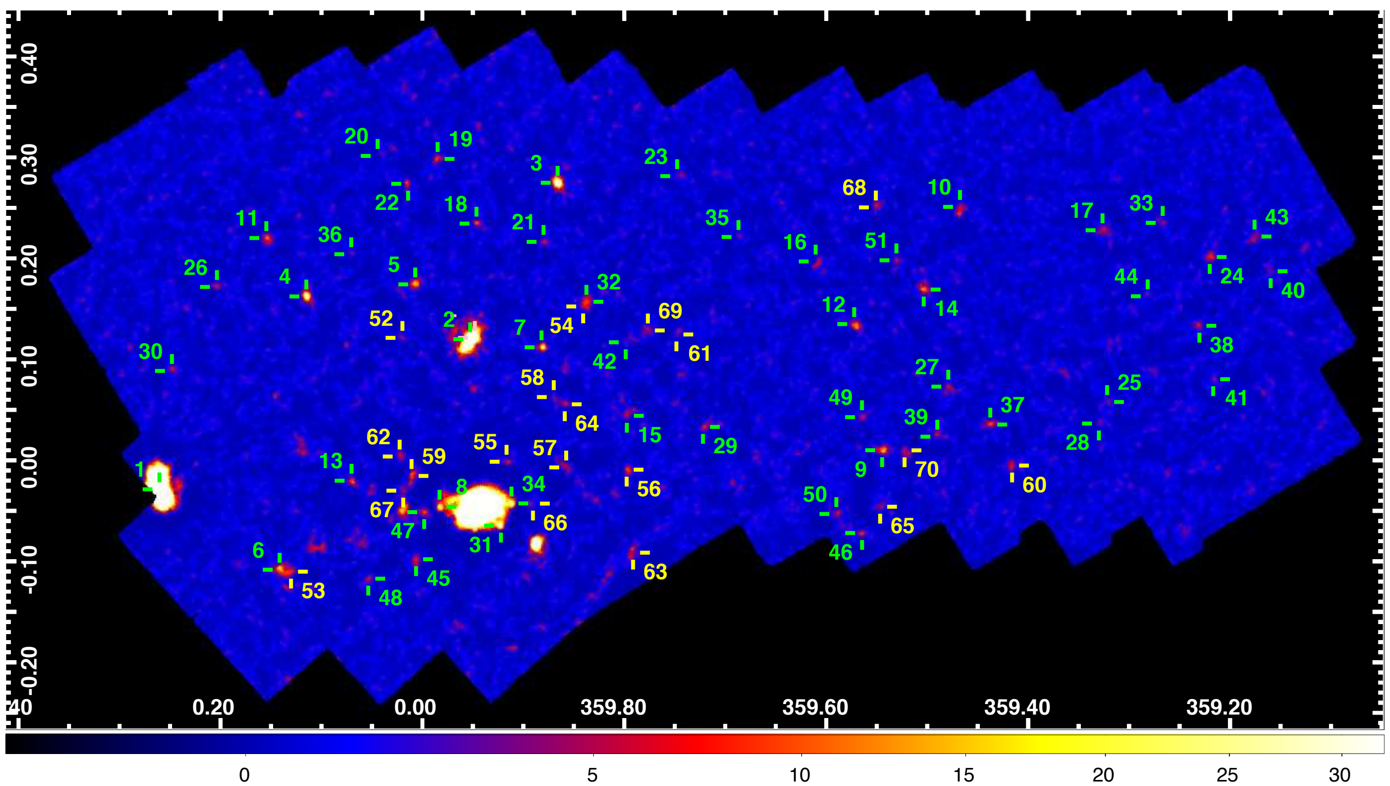


The data

based on: *D. Gaggero, G. Bertone, F. Calore, R. Connors, M. Lovell, S. Markoff, E. Storm, "Searching for Primordial Black Holes in the radio and X-ray sky", arXiv:1612.00457, PRL 2017*



1.4 GHz, VLA, *Lazio & Cordes 2008*

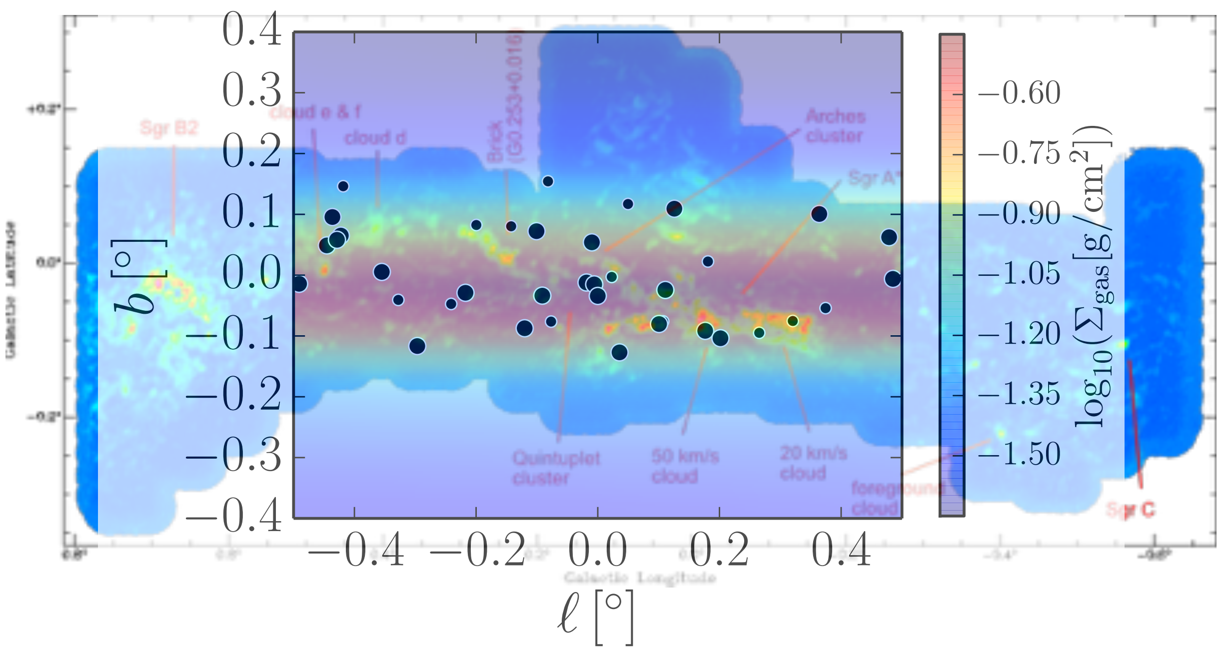


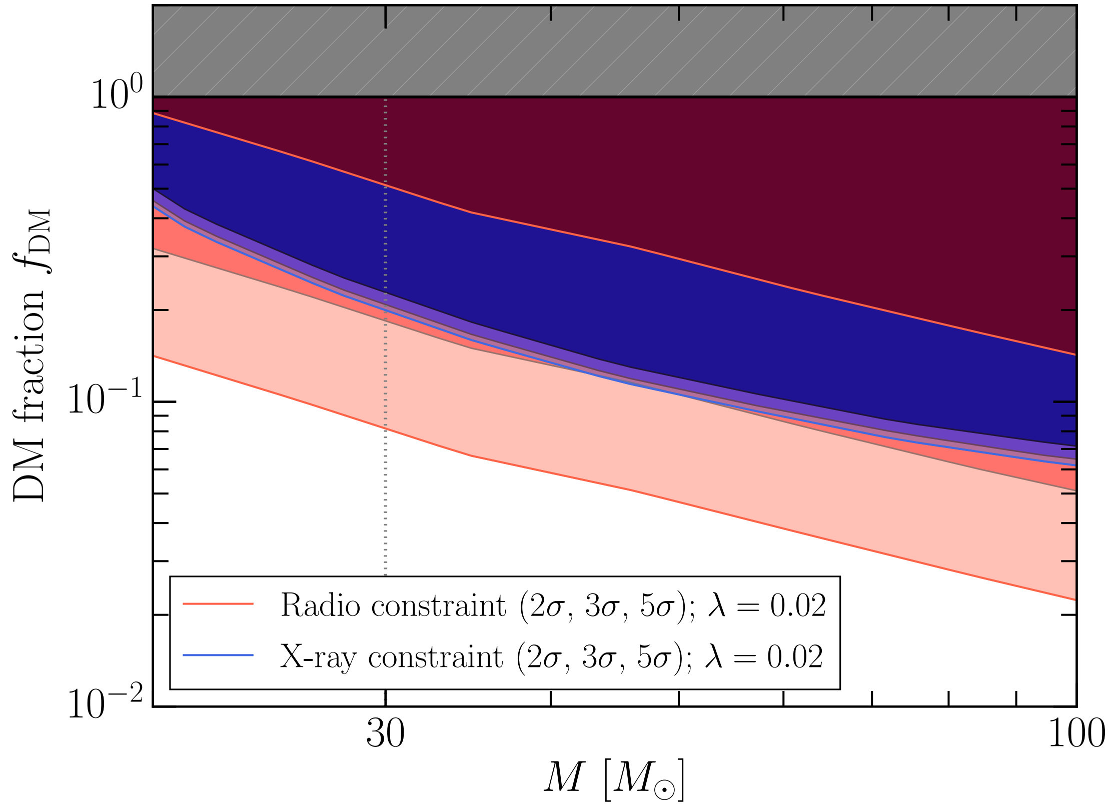
10-40 keV, NuStar catalog, *Hong et al. 2016*

Astronomical searches: recent results

- **Result:** We predict **many more sources in the radio and X-ray bands** under consideration with respect to the ones compatible with being accreting BHs.
  - **X-rays**
    - Prediction: more than **3000 bright X-ray sources;** *observed sources in the ROI by Chandra:* ***~400*** (40% are cataclysmic variables)
  - **Radio**
    - Prediction: **40±6 bright radio sources** in the ROI; observed radio sources in the ROI: 170; *number of candidate black holes in the ROI:* ***0*** *assuming BHs obey the Fundamental Plane relation (i.e. no radio source in the ROI have a X-ray counterpart compatible with the FP relation they cannot be BHs accreting in the hard state)*

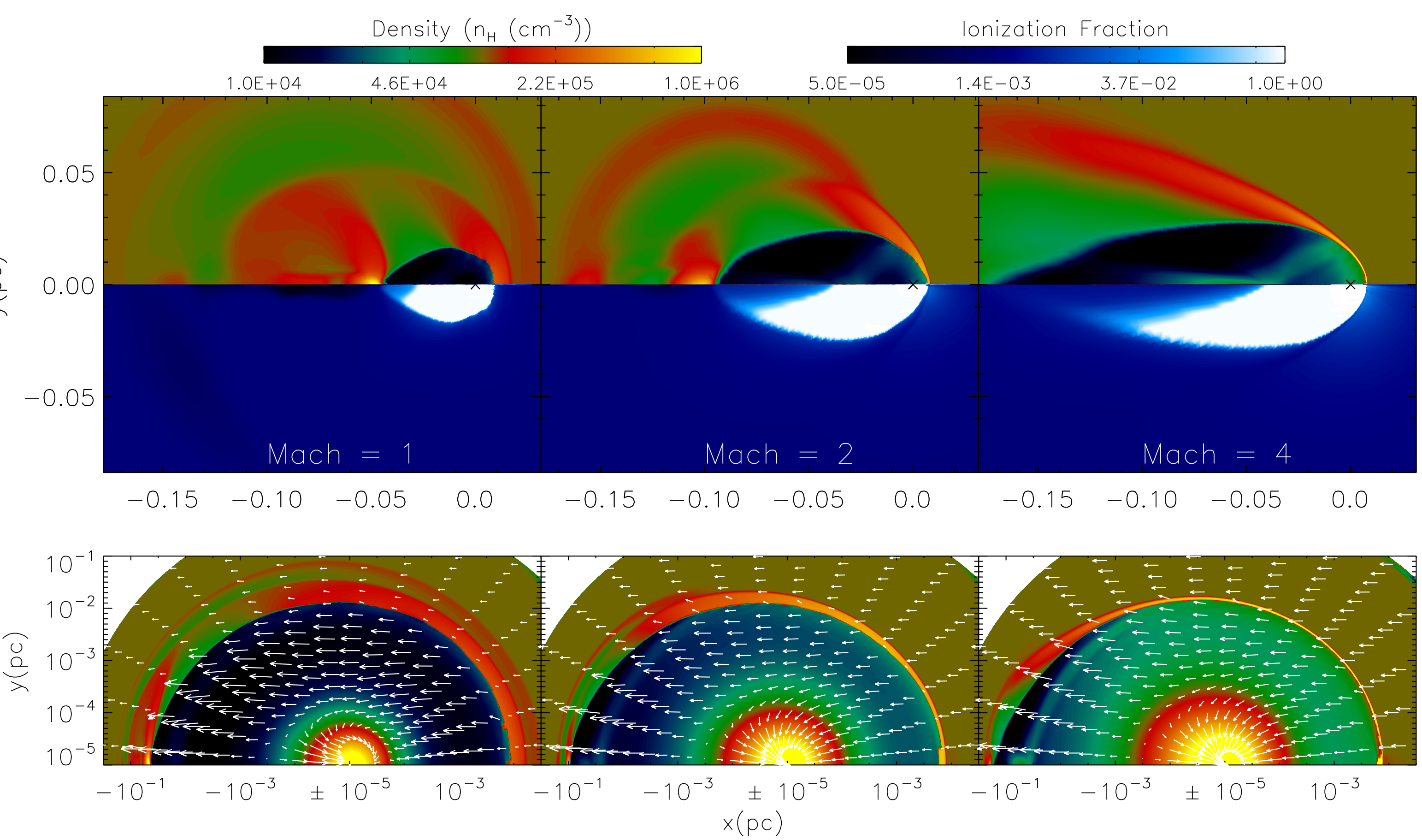
- We derive an **upper limit** on the abundance on the PBHs in the universe in the 10 - 100 M_{Sun} mass window



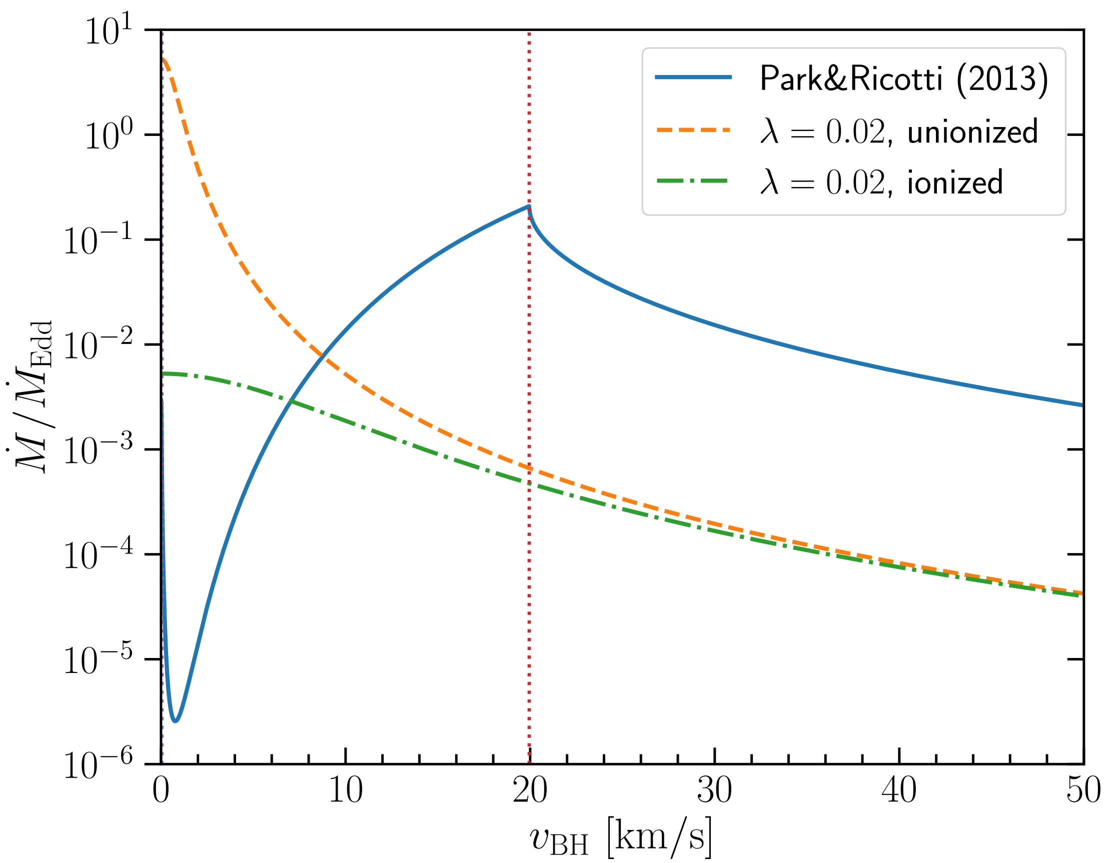


Intermezzo: gas accretion on compact objects

- The accretion of interstellar gas onto an isolated moving compact object is studied via *numerical simulations*
- Complex phenomenology: radiation feedback (the radiation emitted near the gravitational radius of the BH reduces the rate of gas supply; sound speed is altered); formation of a *bow shock.* Non trivial dependence of the accretion rate with respect to the BH speed.

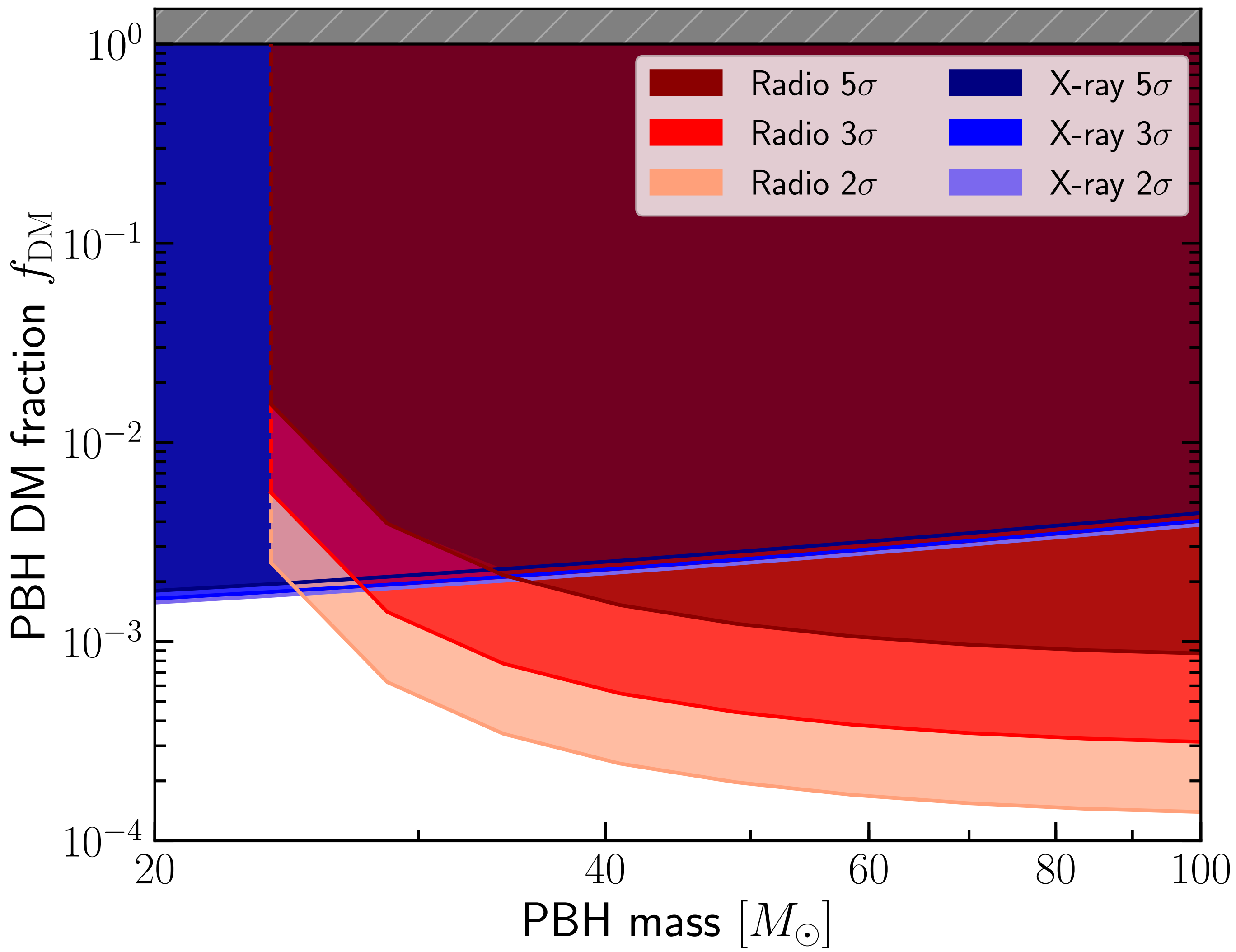


[Park and Ricotti, 2013]

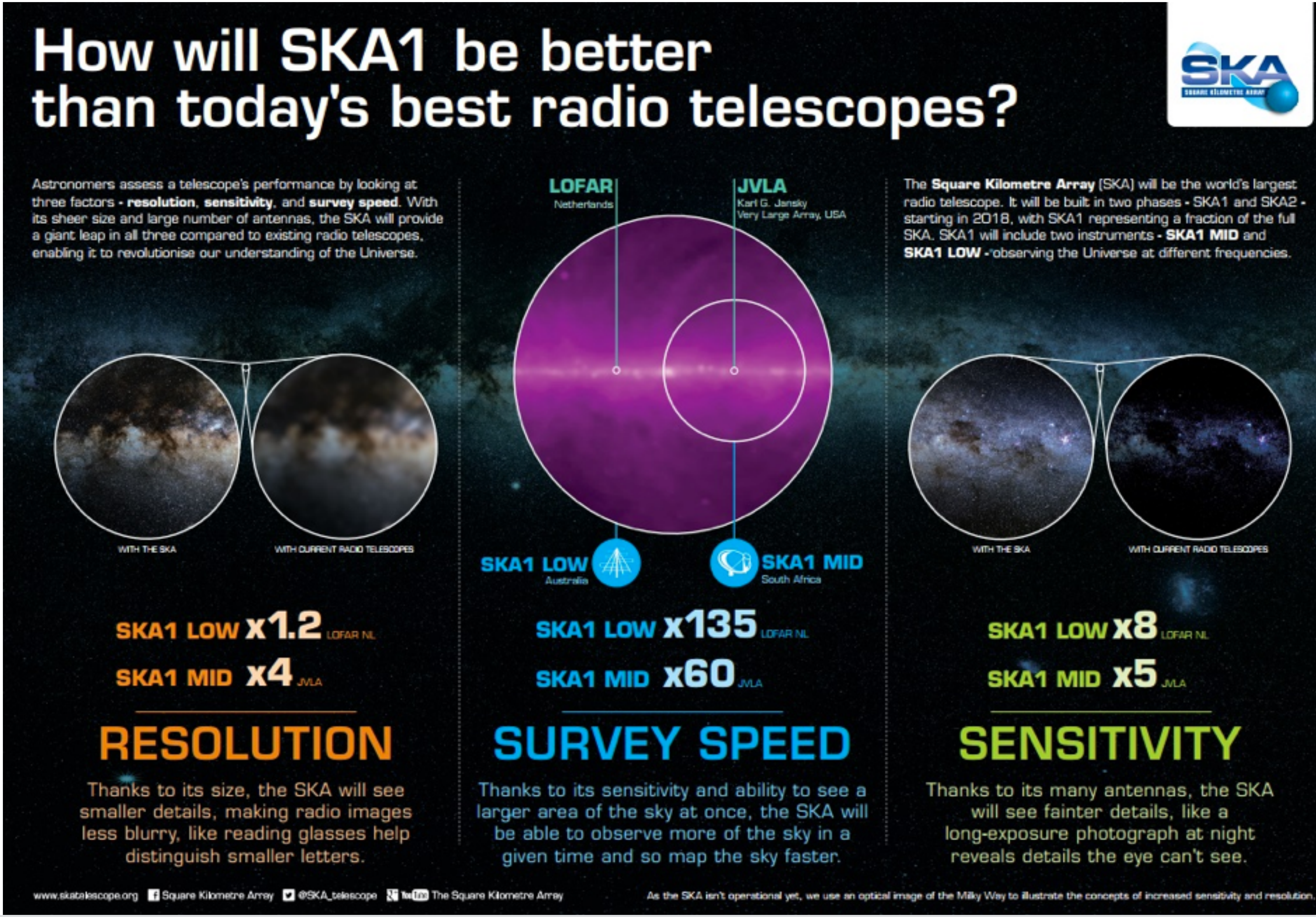


Astronomical searches: a new update

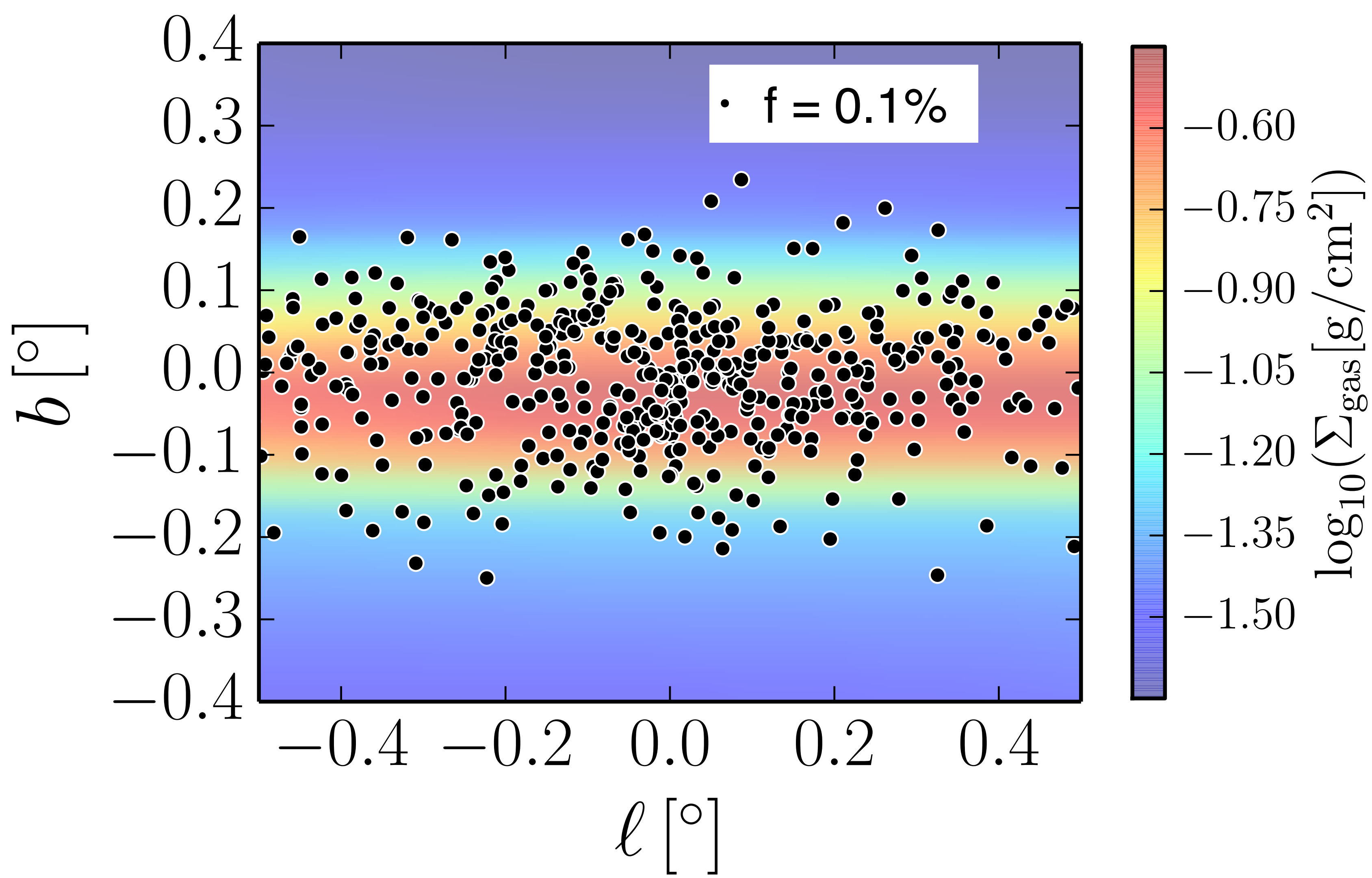
- Based on the accretion rate derived from numerical simulations, we derive an **upper limit** on the abundance on the PBHs in the universe in the 1 - 100 M_{Sun} mass window
- The limit is stronger because now we are sensitive to a wider portion of the phase-space, since the accretion rate does not decline steeply with the PBH speed!
- Caveats:
  - Clustering?
  - PBH distribution in the inner Galaxy



Astronomical searches: The future

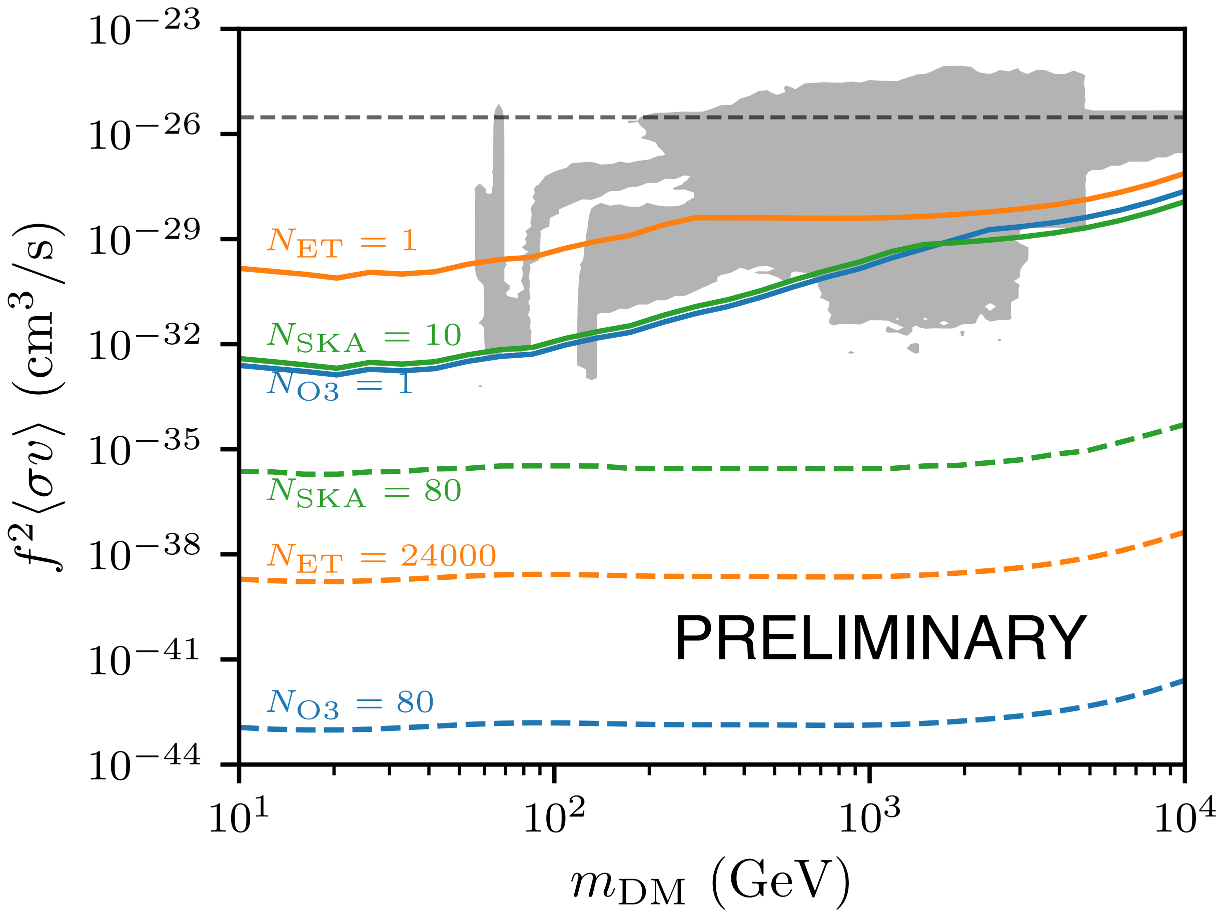


- During the next decade, the Square Kilometer Array will provide a huge increase in sensitivity
- SKA can detect even a sub-dominant population of PBHs that amount to a small fraction of the dark matter in the Universe, even O(0.1%) or smaller
- *How to distinguish astrophysical from primordial black holes?*



Multiple dark components: “almost all or almost nothing”

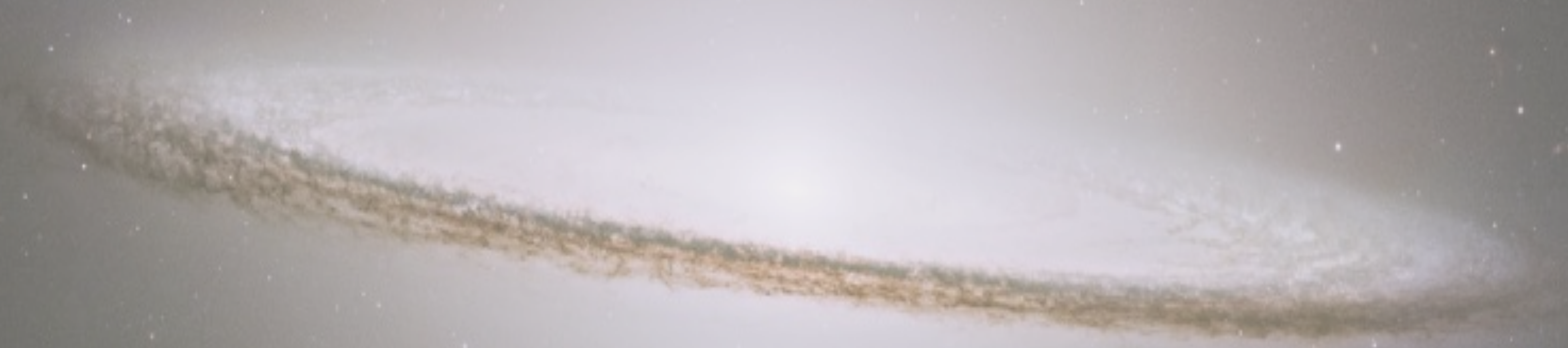
- If a small fraction of the DM consists of PBHs, DM halos with steep density profiles are expected to form.
- If the rest of the dark matter consists of weakly interacting massive particles (WIMPs), a detectable gamma-ray signal is expected to be produced (possible exceptions are, for instance, asymmetric DM models) [Lacki and Beacom 2010, Adamek+ 2019]
- **A firm detection of PBHs would imply that the remaining dark matter could not be WIMPs. Viceversa, if DM is made of WIMPs, the PBH density is highly constrained.**



Conclusions

- Primordial black holes may exist and constitute a portion of the dark matter in the universe
- Their discovery would be of paramount importance and has consequences on our understanding of the dark matter even if they make up a little portion of it
- Primordial black holes may shine in the radio and X-ray sky because of gas accretion in the inner Galaxy
- The combination of future data from current and future gravitational wave observatories and radio/X-ray facilities may provide a promising road to a discovery

Thank you for your attention!



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Primordial black hole formation

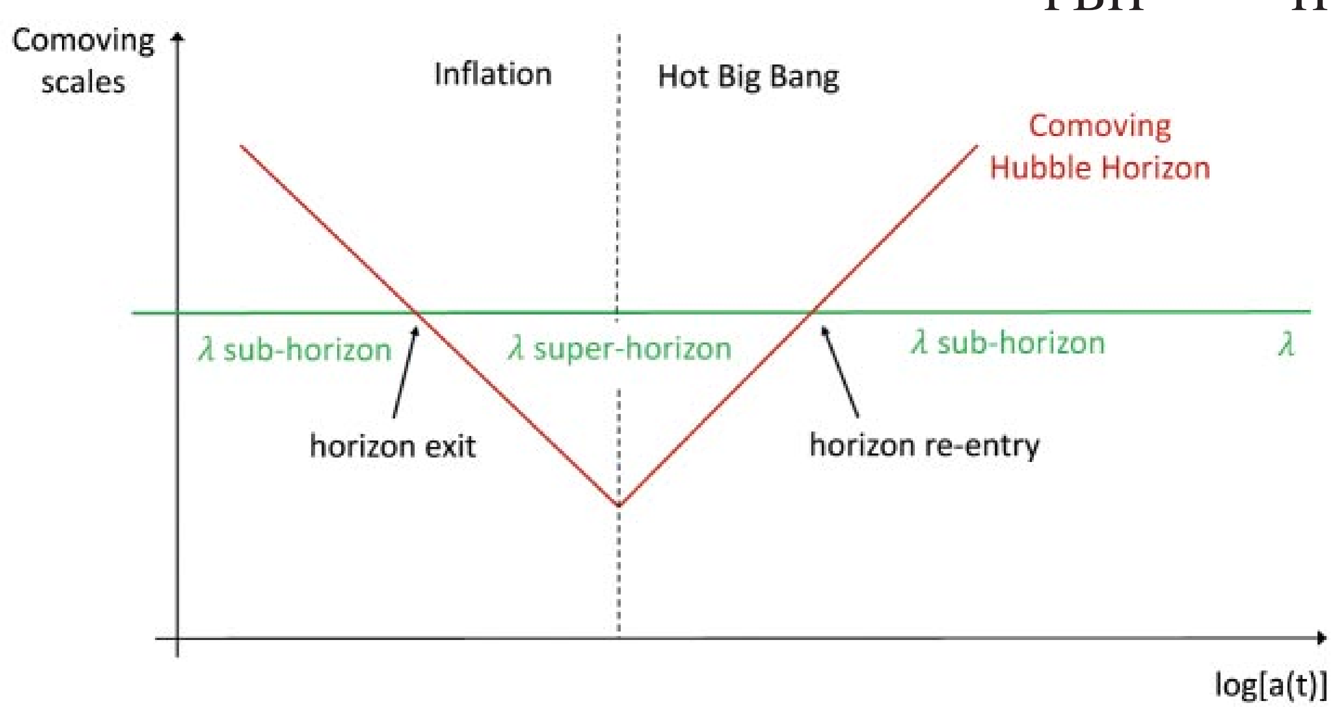
- Given some over-density at the corresponding scale, **it is possible to form PBHs in a wide range of masses** deep in the radiation-domination era before big-bang nucleosynthesis.

- With a more formal treatment [see e.g. the review Sasaki et al. arXiv:1801.05235], it is possible to obtain a more accurate criterion for PBH formation:

- The condition is that the comoving slice density contrast at the time when the scale of interest re-enters the Hubble horizon is greater than $\Delta_c = c_s^2 = \frac{1}{3}$

- **The mass of the formed PBH is equal to the horizon mass at the time of formation:**

$$M_{\rm PBH} \sim M_{\rm H} \sim \frac{c^3 t}{G} \sim 10^{15} \left(\frac{t}{10^{-23}\,{\rm s}}\right) {\rm g}\,.$$



Primordial black hole formation

- The amplitude of primordial fluctuations is much smaller than order 0.1 - 1 at the scales corresponding to the large-scale structure!
- At scales probed by CMB: O(10^4) - O(10^{-1}) Mpc the amplitude of the fluctuations is $\delta \sim 10^{-5}$
- However, **the amplitude of primordial fluctuations is basically unconstrained at scales possibly associated to PBHs**

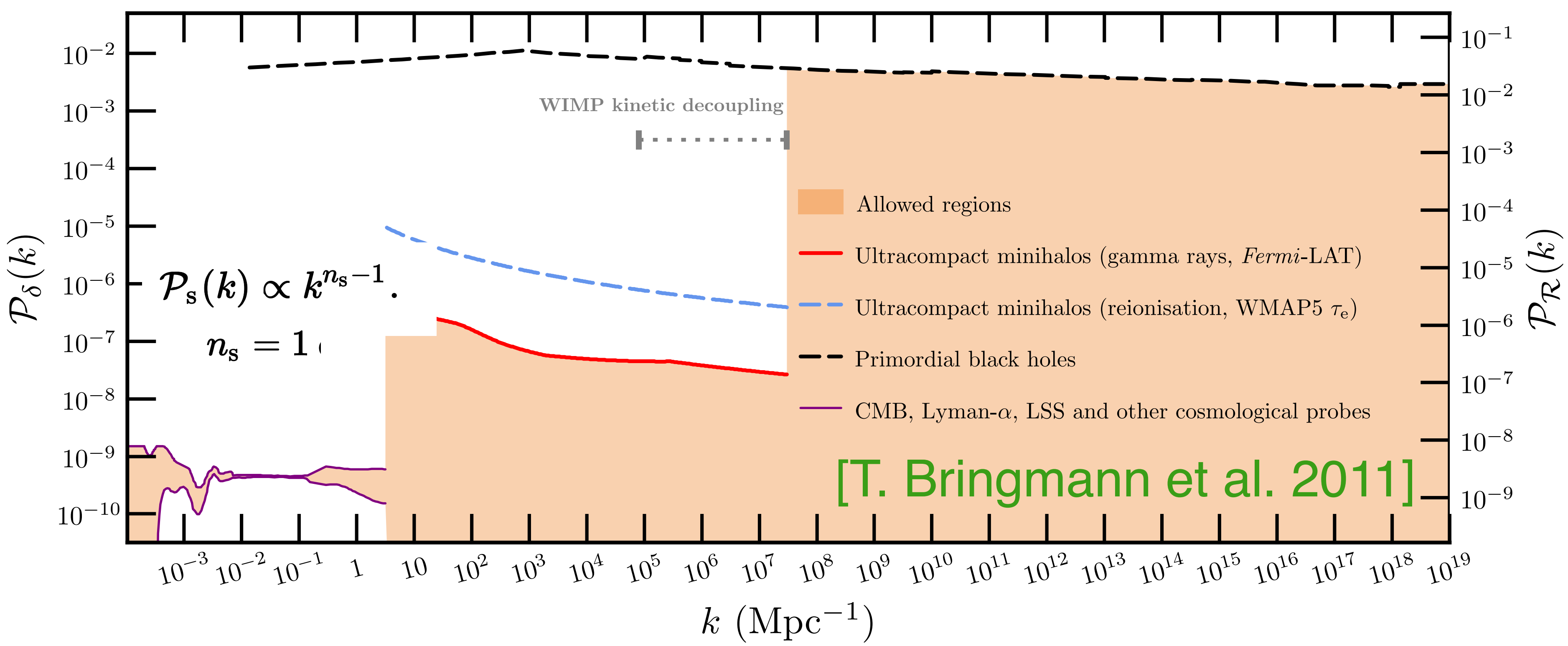


FIG. 6: Constraints on the allowed amplitude of primordial density (curvature) perturbations $\mathcal{P}_\delta$ ($\mathcal{P}_\mathcal{R}$) at all scales. Here we give the combined best measurements of the power spectrum on large scales from the CMB, large scale structure, Lyman-α observations and other cosmological probes [152, 153, 156]. We also plot upper limits from gamma-ray and reionisation/CMB searches for UCMHs, and primordial black holes [43]. For ease of reference, we also show the range of possible DM kinetic decoupling scales for some indicative WIMPs [74]; for a particle model with a kinetic decoupling scale $k_{\rm KD}$, limits do not apply at $k > k_{\rm KD}$. Note that for modes entering the horizon during matter domination, $\mathcal{P}_\delta$ (but not $\mathcal{P}_\mathcal{R}$) should be multiplied by a further factor of 0.81.

$$\delta(\vec{x}) \stackrel{\text{def}}{=} \frac{\rho(\vec{x})}{\bar{\rho}} - 1 = \int \mathrm{d}k\, \delta_k\, e^{i\vec{k}\cdot\vec{x}},$$

$$\langle \delta_k \delta_{k'} \rangle = \frac{2\pi^2}{k^3}\, \delta(k - k')\, \mathcal{P}(k).$$

$$\mathcal{P}_s(k) = |\delta_R|^2.$$

Primordial black holes and Dark Matter

- **Primordial Black Holes, if massive enough, are testable dark matter candidates!**

- If the mass is too low, the PBHs have enough time to evaporate (Hawking-Bekenstein radiation)

$$t_{\text{evaporation}}[\mathrm{s}] = 10^{71}\left(\frac{M}{M_\odot}\right)^3$$

- Chapline was among the first to suggest explicitly PBHs as a DM candidate [G. F. Chapline, Nature **253**, 251 (1975)].
- However, Hawking already noticed in 1971 that *"…it is tempting to suppose that the major part of the mass of the Universe is in the form of collapsed objects. This extra density could stabilize clusters of galaxies which, otherwise, appear mostly not to be gravitationally bound."*

- Given the evaporation time scale and the age of the Universe, the **typical ranges** for a **PBH as DM candidate** are:
  - M ~ 10^{16} g (10^{-17} $M_\odot$) — 10^{39} g (10^{5} $M_\odot$)
  - size ~ 10^{-13} cm — 10^{10} cm
  - number in our Galaxy ~ 10^{29} — 10^{6}

Merger rate from BBH formed in halos

$$\sigma = \pi \left(\frac{85\,\pi}{3}\right)^{2/7} R_s^2 \left(\frac{v_{\rm pbh}}{c}\right)^{-18/7}$$
$$= 1.37 \times 10^{-14}\, M_{30}^2\, v_{\rm pbh-200}^{-18/7}\, {\rm pc}^2$$

$$\mathcal{R} = 4\pi \int_0^{R_{\rm vir}} r^2 \frac{1}{2} \left(\frac{\rho_{\rm nfw}(r)}{M_{\rm pbh}}\right)^2 \langle \sigma v_{\rm pbh} \rangle \, dr$$

$$\mathcal{V} = \int (dn/dM)(M)\, \mathcal{R}(M)\, dM.$$

$$\mathcal{V} = 2\, f (M_c/400\, M_\odot)^{-11/21}\, {\rm Gpc}^{-3}\, {\rm yr}^{-1}$$

Simeon Bird, Ilias Cholis, Julian B. Muñoz, Yacine Ali-Haïmoud, Marc Kamionkowski, Ely D. Kovetz, Alvise Raccanelli, Adam G. Riess, Phys. Rev. Lett. ***116****, 201301 (2016)*

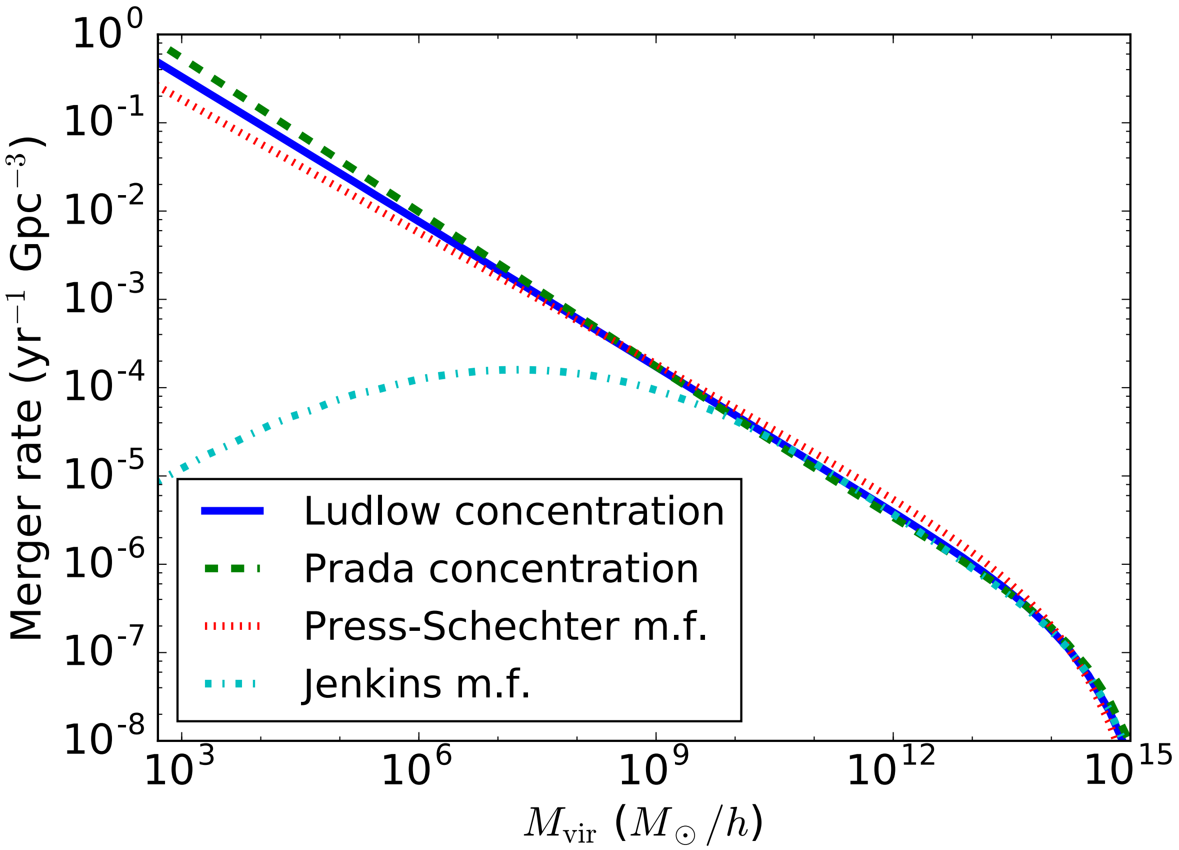


FIG. 2. The total PBH merger rate as a function of halo mass. Dashed and dotted lines show different prescriptions for the concentration-mass relation and halo mass function.

The merger rate of primordial BBHs

- If PBHs make up all of the DM, most of the pairs decouple from the Hubble flow before matter-radiation equality and form bound systems. If $f_{PBH} << 1$ only rare pairs with small separation form binaries.

- Binary decoupling described by the equation of motion of two point sources subject to gravitational pull and Hubble flow

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{2G_N M_{\mathrm{PBH}}}{r^2}\frac{r}{|r|} + (\dot{H}(t) + H(t)^2)\, r\,,$$

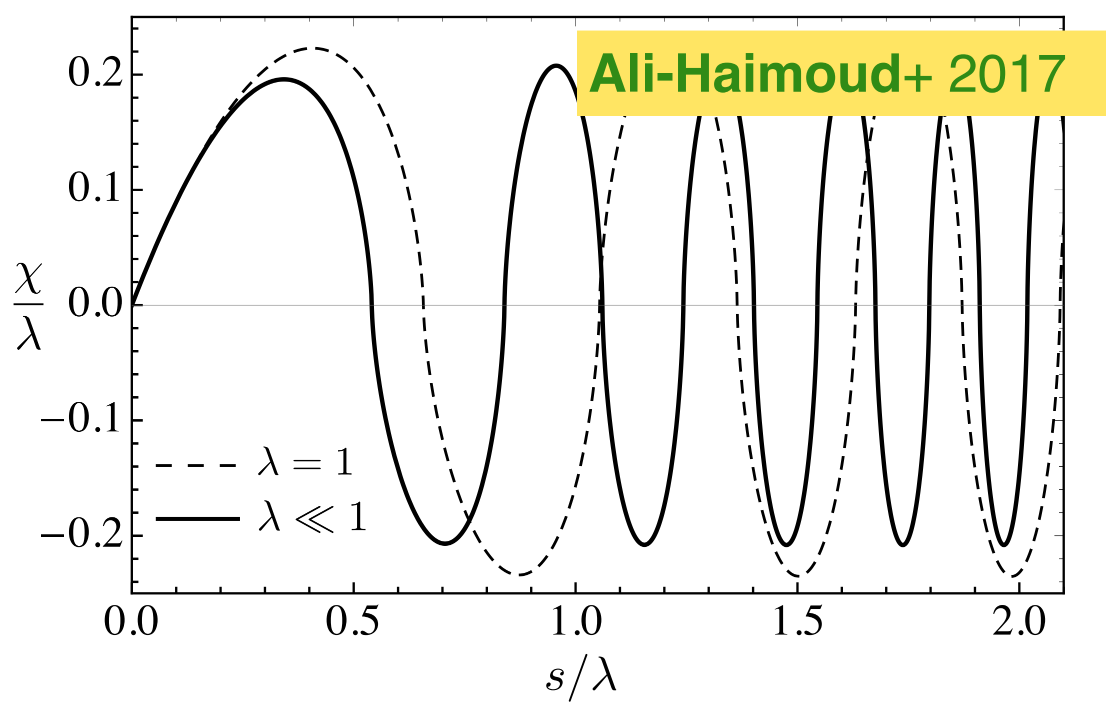
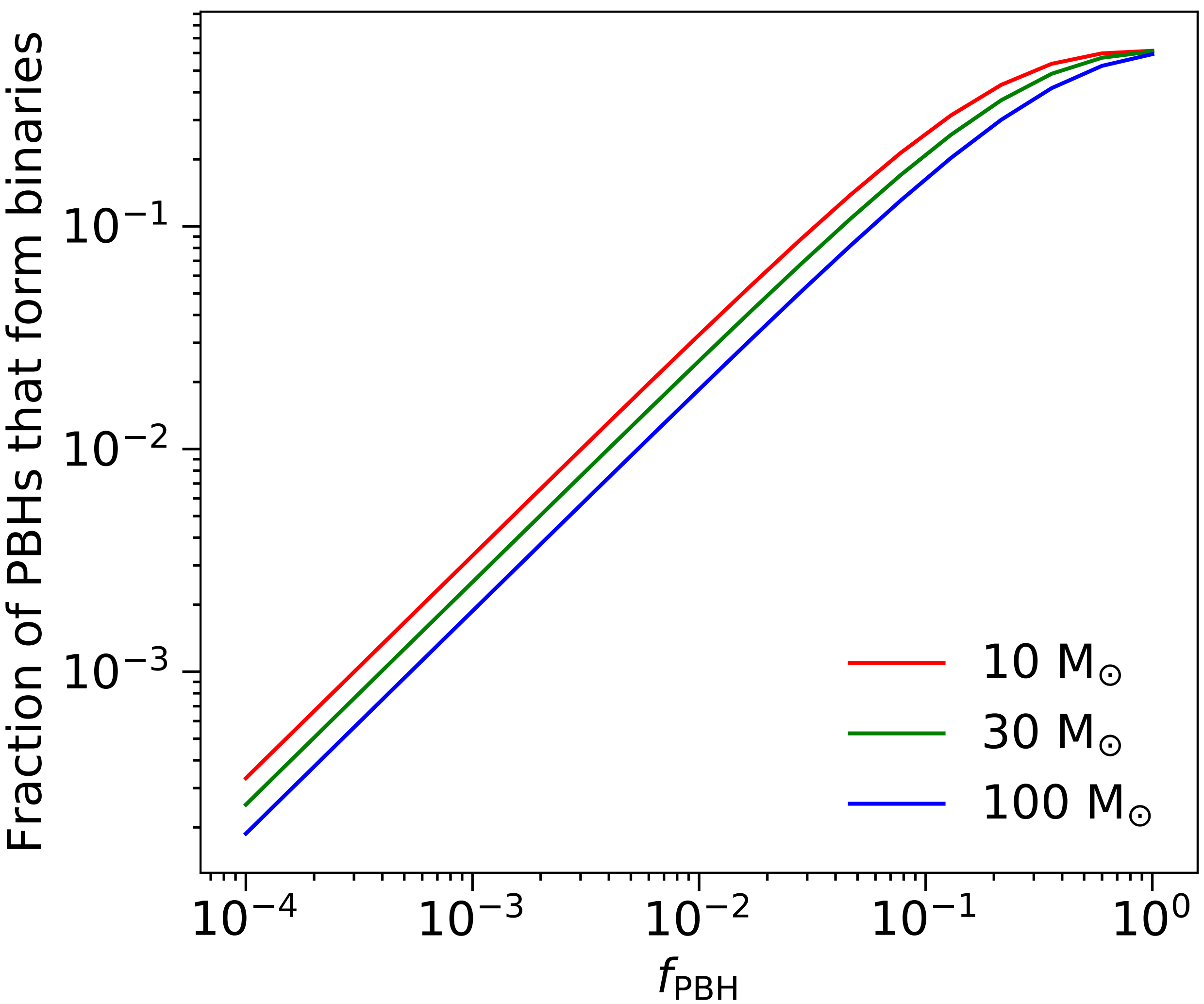


FIG. 1. Dimensionless separation $\chi = r/x$ of two point masses, rescaled by the parameter $\lambda = \frac{1}{f}(x/\overline{x})^3$, as a function of the rescaled scale factor s/λ, in the limit $\lambda \ll 1$ (solid) and for $\lambda = 1$ (dashed).



The merger rate of primordial BBHs *with dark dress*

- Let's consider the case of a sub-dominant population of PBHs immersed in a high-density DM-dominated environment, rapidly expanding and diluting

- Accretion of DM mini-halos governed by the balance between gravitational pull and expansion of the universe [Mack, Ostriker, Ricotti arXiv:0608642; Ricotti arXiv:0706.0864]

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{G M_{\mathrm{PBH}}}{r^2} + (\dot{H} + H^2) r \,,$$

- A PBH can accrete a DM halo with $M_{Halo} = M_{PBH}$ at the end of the radiation era ($z = z_{eq}$)

- Density profile studied by a simple analytical model of radial infall in a Einstein-De Sitter Universe in [Bertschinger ApJS 1985]

$$\rho(r) \propto r^{-3/2}.$$

- Similar profiles recently found in UCMHs [Sten Delos et al. 1712.05421, Gosenca et al. 1710.02055]
- In UCMHs, steeper profiles (α = -9/4) can form in idealized conditions (self-similarity, isolation). DM minihalos can thus be even more compact if PBHs are a small fraction of the DM [Adamek et al. 1901.08528]

The merger rate of primordial BBHs *with dark dress*

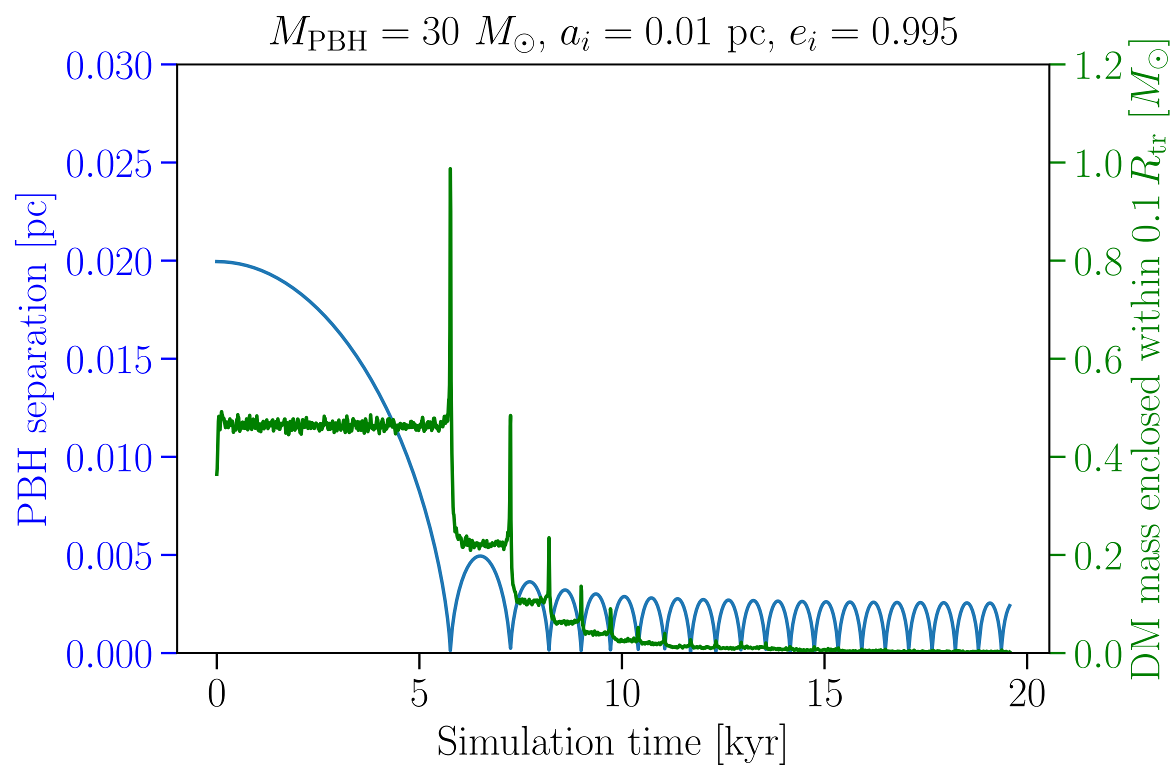


FIG. 6. **PBH separation and retained DM halo mass during a single simulation.** In blue (left axis), we show the separation of the PBHs during a single simulation while in green (right axis) we show the DM mass enclosed within 10% of the halo truncation radius, $R_{\rm tr}$. Here, we simulate $M_{\rm PBH} = 30\,M_\odot$ and initial orbital elements $a_i = 0.01\,$pc and $e_i = 0.995$. The truncation radius is $R_{\rm tr} \approx 4 \times 10^{-3}\,$pc and the total DM mass per halo is $3.1\,M_\odot$.

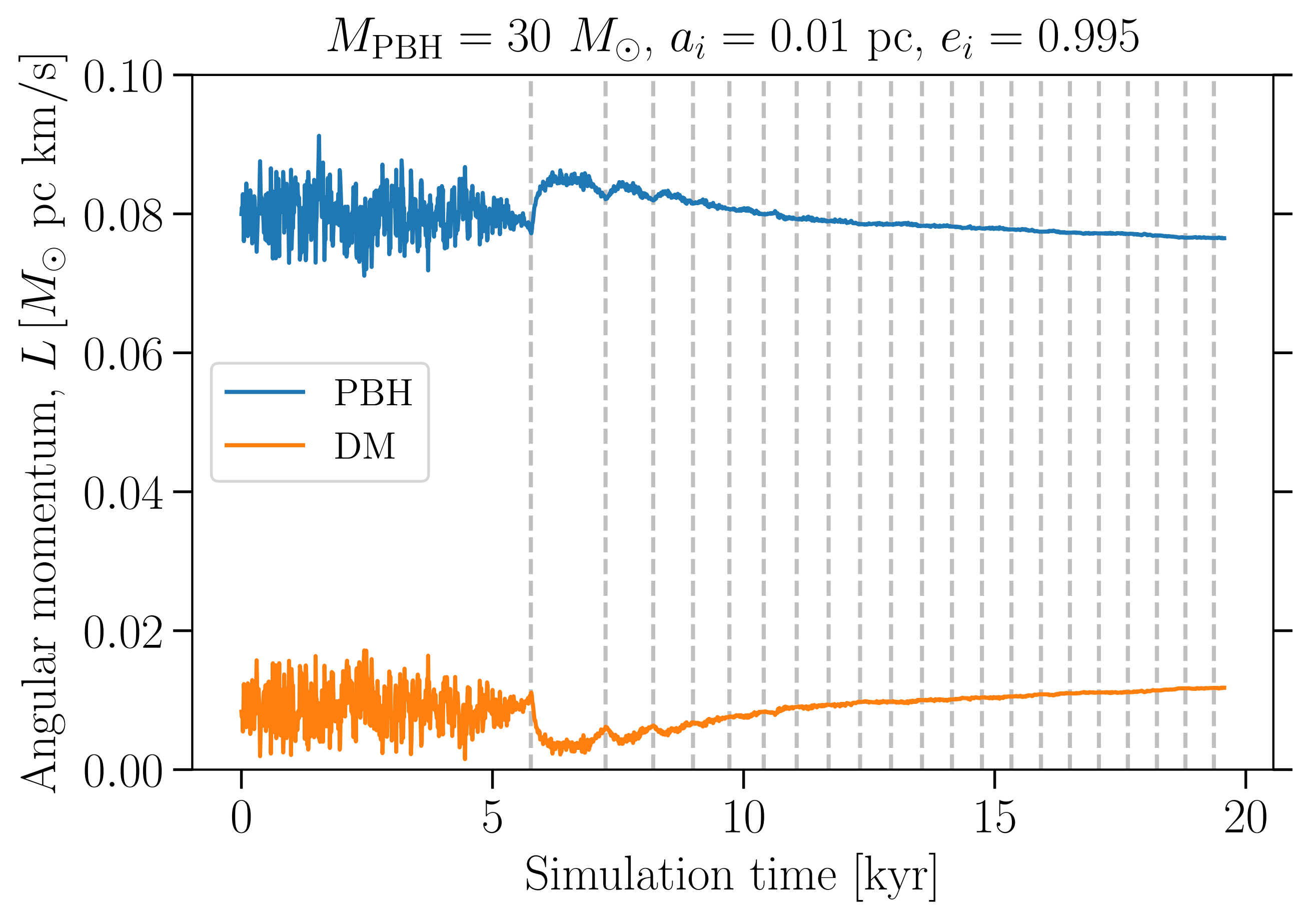


FIG. 7. **Angular momentum of PBHs and DM during a single simulation.** The total angular momentum of the PBH (DM) particles in the simulation is shown in blue (orange). The simulation parameters are as in Fig. 6. The times at which the PBHs undergo a close passage are marked by grey dashed lines.

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The merger rate of primordial BBHs *with dark dress*

$$t_{\rm merge} = \frac{3\,c^5}{170\,G_N^3}\,\frac{a^4 j^7}{M_{\rm PBH}^3}\,.$$

$$E^{\rm bind}(r_{\rm in}) = -4\pi G_N \int_{r_{\rm in}}^{\infty} \frac{M_{\rm enc}(r)}{r}\, r^2 \rho_{\rm DM}(r)\, {\rm d}r\,.$$

$$L^2 = \frac{1}{2} G_N M_{\rm PBH}^3\, a\, j^2\,,$$

$$E_i^{\rm orb} + 2\,E^{\rm bind}(r_{\rm min}/2) = E_f^{\rm orb}\,.$$

$$j_f = \sqrt{\frac{a_i}{a_f}}\, j_i$$

$$a_f(a_i) = \frac{G_N M_f^2 a_i}{G_N M_{tot}^2 + 4 a_i E^{\rm bind}(r_{\rm in})}\,.$$

$$t_f = \sqrt{\frac{a_i}{a_f}}\, t_i\,,$$

Intermezzo: gas accretion on compact objects

- A compact object that crosses a gas-rich region accretes gas with a rate that depending on the properties of the medium and on its speed.

- Simple “textbook” approach: **Bondi-Hoyle-Littleton formula**

$$\dot{M}_{\rm BHL} = 4\pi \frac{(GM)^2 \rho_\infty}{(v^2 + c_\infty^2)^{3/2}}$$

- In the v = 0 limit it is based on the key assumptions of *steady flow* and *spherical symmetry*.
- The derivation only relies on fluid mechanics and thermodynamics equations: *continuity equation*, *Euler equation*, and the definition of the sound speed

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \quad \rightarrow \quad \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \rho v \right) = 0.$$

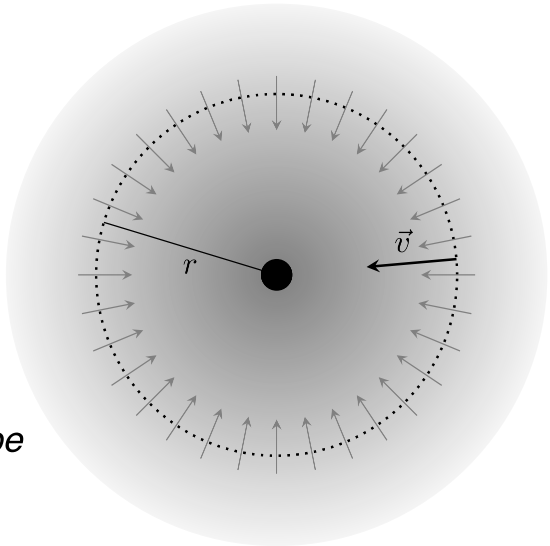
$$\dot{M} = 4\pi r^2 \rho (-v)$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} = -\vec{\nabla} P + \vec{f} \quad \rightarrow \quad \rho v \frac{dv}{dr} = -\frac{dP}{dr} - \frac{GM\rho}{r^2}$$

$$R_s = \frac{2GM}{c_S^2}$$

- Sonic radius: $c_s = v_{escape}$

$$c_s = (dP/d\rho)_0^{1/2}$$



[credit for the pic: J. Manshanden, M.Sc. thesis]