Time Evolution of Operator Complexity Beyond Scrambling

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Entangle This: Chaos, Order and Qubits
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Work done in collaboration with *J.L.F. Barbon, E. Rabinovici and R. Shir*

Outline

A. Introduction and Motivation

B. K-Complexity

C. Dynamics of K-Complexity Growth

D. Operator Randomisation and K-entropy

E. Discussions and Conclusion

F. Open Questions

Motivation

- Chaotic quantum systems are known to scramble quantum information.
- Holographic quantum systems, typically with a large number of colour d.o.f, and dual to Black Holes, scramble information the fastest.
- In such systems, OTOCs provide an efficient way to characterise such scrambling behaviour.
  (See the talks by Pappadodimas, Sunderhauf for a review of OTOCs)

- OTOCs measure the growth of an initially ‘local’ operator, in terms of the commutator norm $||[W,V]||$. The growth rate is given by the “butterfly velocity” v_B . Here W and V are two local operators. Stanford, Shenker et al. ’13-’14

- In systems without spatial locality, such as the SYK model in 0+1- dimensions, the commutator norm involving initially local operators can grow exponentially fast.

$$C(x,t) \sim \frac{1}{N^2} e^{\lambda_L t}$$

- The growth rate is specified by a Lyapunov exponent λ_L .

- It has an upper bound which is saturated by the fastest scramblers such as Black Holes and the SYK model (at low temperatures). Maldacena, Shenker, Stanford ‘16

$$\lambda_L \leq \frac{2\pi}{\beta}$$

- In fact, this was one of the initial motivations for the SYK model to be considered a model for the BH. Kitaev ‘15; Maldacena, Stanford ‘16

- Fast scramblers scramble information at times of order

$$t_* \sim \frac{1}{\lambda_L}\log(S)$$ where S is the typical system size.

- Independent estimates for the randomisation of quantum states in the Hilbert space suggest timescales of order the Heisenberg time, $t_H \sim e^{O(S)}$

- Also the timescales for the onset of RMT-like behaviour in generic chaotic quantum systems such as the SYK model. Polchinski et al. '16 (However, see H. Gharibyan's talk)

- <u>Glaring shortcoming of the OTOCs</u>: No new information beyond scrambling time!

- Operator growth over this exponentially long timescale remains uncharacterised!
- To characterise dynamics beyond scrambling time, we invoke a new quantity called **K-complexity**. E. Altman et al. '18
- The definition of K-complexity coincides with the operator size upto scrambling times.
- The growth of K-complexity in the scrambling regime is in direct correspondence with the decay of the OTOCs. E. Altman et al. '18
- However, unlike OTOCs, the K-complexity continues to grow beyond scrambling times, since it relies on growth in the operator Hilbert space!

K-Complexity

The Krylov Basis

- In the Heisenberg picture, time evolution of an operator is:

$$\mathcal{O}_t = e^{itH}\, \mathcal{O}_0\, e^{-itH}$$

- **Krylov basis**: An operator basis with elements defined through nested commutators of the Hamiltonian with an initial local operator.

$$O_n \sim [\underbrace{H,[H,\ldots,[H}_{n},O_0],\ldots],]$$

- To make it into a basis, we perform the Gram-Schmidt orthogonalisation process,

$$b_n O_n = [H, O_{n-1}] - b_{n-1} O_{n-2}$$

where the b_n 's are called **Lanczos** coefficients.

- Orthogonality is defined through the following non-degenerate inner product in the operator algebra:

$$(\mathcal{O}|\mathcal{O}') = \frac{1}{\mathcal{N}} \mathrm{Tr}\,(\mathcal{O}^\dagger \mathcal{O}')\,, \qquad \mathcal{N} \equiv \mathrm{Tr}\,\mathbf{1}$$

- The adjoint action of the Hamiltonian invokes a linear operator in the operator vector space, called the **Liouvillian**.

- The Liouvillian acts on the basis elements in the following way,

$$\mathcal{L}|\mathcal{O}) \equiv |[\mathcal{H}, \mathcal{O}])$$

- In the operator basis then, the Liouvillian has a tri-diagonal representation in terms of the Lanczos coefficients owing to,

$$\left|[H, O_n]\right) = b_{n+1}\left|O_{n+1}\right) + b_n\left|O_{n-1}\right)$$

- Any operator O that can be represented in this basis of operators generated by H and O_0 , can be written as,

$$\left|O\right) = \sum_n i^n \varphi_n \left|O_n\right)$$

- Here φ_n's are time-dependent complex coefficients.

- One can set-up a differential equation for the φ_n's from the Heisenberg e.o.m. , $\partial_t O_t = i[H, O_t]$, given by

$$\partial_t \varphi_n = b_n \varphi_{n-1} - b_{n+1} \varphi_{n+1}$$

Solving this differential equation allows us to understand the operator growth in this basis, as well as calculate the K-complexity.

K-Complexity

- K-complexity is defined as the typical number of H-commutators required to build an operator $\mathcal{O}$.

$$C_K(O) = \left(O\middle|n\middle|O\right) = \sum_n n\,|\varphi_n|^2$$

- Same as the average operator size upto scrambling times!
- Physically, this object measures how the amplitude of projection of the initial operator spreads on the successive basis elements, corresponding to larger and larger operators.
- This growth slows down, once the growing front of the initial operator becomes as large as the operators which are typically the system size, in a finites size system. This happens around the scrambling timescales.
- This growth, however, continues since we can keep applying the Hamiltonian successively on the operator, for all times.
- In fact, replacing the operator $\mathcal{O}$ by the time evolved operator $\mathcal{O}_t$, we notice that this definition of K-complexity continues to hold even beyond scrambling timescales.

- Beyond scrambling time-scales, this object simply measures the increasing complexity of the time-evolving operator,

$$C_K(t) = \sum_n n\,|\varphi_n(t)|^2$$

- Physically, the operator now moves around in the sub-space of large operators, all of which are typically the size of the system. This growth is what we want to characterise!

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Some quick comments on K-Complexity

- Notice: K-complexity is different from usual complexity, since it doesn't require an arbitrary tolerance parameter in its definition. On the contrary, it has an upper bound given by the dimensionality of the operator Hilbert space. It is saturated when the operator "randomizes" over the entire Hilbert space.

- Important: This definition depends on a particular choice of the Hamiltonian and an initial operator, neither of which may not be the most optimal for the experiment. This is in contrast with random quantum circuits where one uses averaged Hamiltonians at each time-step, and therefore scramble d.o.f. in the most efficient way.

K-Complexity of Fast Scramblers

- Fast scramblers are defined as chaotic quantum systems that saturate the Lyapunov bound. Maldacena et al. '16

- For the SYK model, it was shown that in the thermodynamic limit, the Lanczos coefficients show a linear growth with *n* for asymptotically large *n*'s. Altman et al. '18

$$\boxed{b_n \approx \alpha n} \qquad , \; n \to \infty$$

- With the above value of b_n, the resulting K-complexity turns out to be exponential,

$$\boxed{C_K(t) \sim e^{2\alpha t}}$$

- **Conjecture**: The rate of K-complexity growth in the scrambling region provides an upper bound for the Lyapunov exponent in the OTOCs, and is saturated by the fastest scramblers. Altman et al. '18

$$\boxed{\lambda_L \leq 2\alpha}$$

- At scrambling time, $t_* \sim \frac{1}{\lambda_L}\log(S)$, the K-complexity is ,

$$\boxed{C_K(t_*) \sim n_* \sim S}$$

- However, for a finite system of size $O(S)$, the dimensionality of the operator Hilbert space is $\mathcal{N} \sim e^{O(S)}$.

- Since $S \ll e^{O(S)}$, it implies that there is still an enormous scope of growth for operator complexity!

Moments

- To understand the rate of complexity growth in the post-scrambling regime, and the corresponding form of the b_n 's, we must define the moments.

- A moment is defined through the Fourier transform of the correlation function as,

$$\mu_n = (\mathcal{O}_0|\mathcal{L}^n|\mathcal{O}_0) = \int \frac{d\omega}{2\pi}\, \omega^n\, \widetilde{G}(\omega)$$

- Here, $\widetilde{G}(\omega) = \int dt\, e^{-i\omega t}\, G(t)$

- Interestingly, the moments can be bounded from below using "not so simple" combinatorics, as

$$\mu_{2n} \geq b_1^2\, b_2^2 \cdots b_n^2$$

- For a non-decreasing sequence of the Lanczos coefficients, the moments can also be bounded from above as

$$\mu_{2n} \leq C_n\, b_1^2\, b_2^2 \cdots b_n^2$$

- Here Cn's are the Catalan numbers.

- To go beyond scrambling times in finite-size systems, we want $n \gg S$

- The rate of K-complexity growth in this region depends on the form of the b_n's.

- To estimate the form of the b_n's, we consider the spectral decomposition of the correlation functions,

$$\tilde{G}(\omega) = \frac{1}{N} \sum_{a,b} |O_{ab}|^2 \, 2\pi\delta(\omega - (E_a - E_b))$$

- Plugging this into the expression for the moments,

$$\mu_{2n} = \frac{1}{N} \sum_{a,b} (E_a - E_b)^{2n} |O_{ab}|^2$$

K-Complexity and ETH

- Now using ETH, one can estimate the matrix elements of the operator in the energy eigenbasis. For this, we get

$$\mu_{2n} \approx \frac{1}{N^2} \sum_{a,b} (E_a - E_b)^{2n} F(E_a - E_b)$$

where $F(E_a - E_b)$ is a smooth function of the energies.

- For $n \gg S$, it can be shown that the above sum is dominated by the largest possible energy differences, while the form factor can be conveniently ignored.

- Now, for a system which is extensive in the energy and where the UV cut-off is Λ, the largest energy differences are of order ΛS

Conjecture

- The moments then scale as $\mu_{2n} \sim \frac{1}{n^2}(\Lambda S)^{2n}$ $\quad n \gg S$
- If the b_n's asymptote to a particular value b_∞ for large-n, then the upper bound for the moments tells us that,

$$\mu_{2n} \sim (cb_\infty)^{2n+O(n)} \qquad c > 1 \qquad n \to \infty$$

- Comparing the result to the upper bound on the moments, we **<u>conjecture</u>** that the Lanczos coefficients, in the post-scrambling regime, would asymptote to a plateau of height,

$$\boxed{b_\infty \sim (\Lambda S)}$$

- It can be quickly verified that at scrambling time,

$$b_{n_*} \sim \alpha n_* \sim \alpha S$$

- For couplings of order unity in a strongly coupled system, the Lyapunov exponent is of order the characteristic frequency, $\lambda_L \sim 2\alpha \sim \Lambda$

- As a result, the Lanczos sequence grows with a linear slope λ_L between $0 < n < n_*$, beyond which it morphs into an approximate plateau which extends upto $n_{\max} \sim e^{O(S)}$

- This, in turn, results in a slower growth of K-complexity, which we shall characterise now.

The Lanczos coefficients' profile

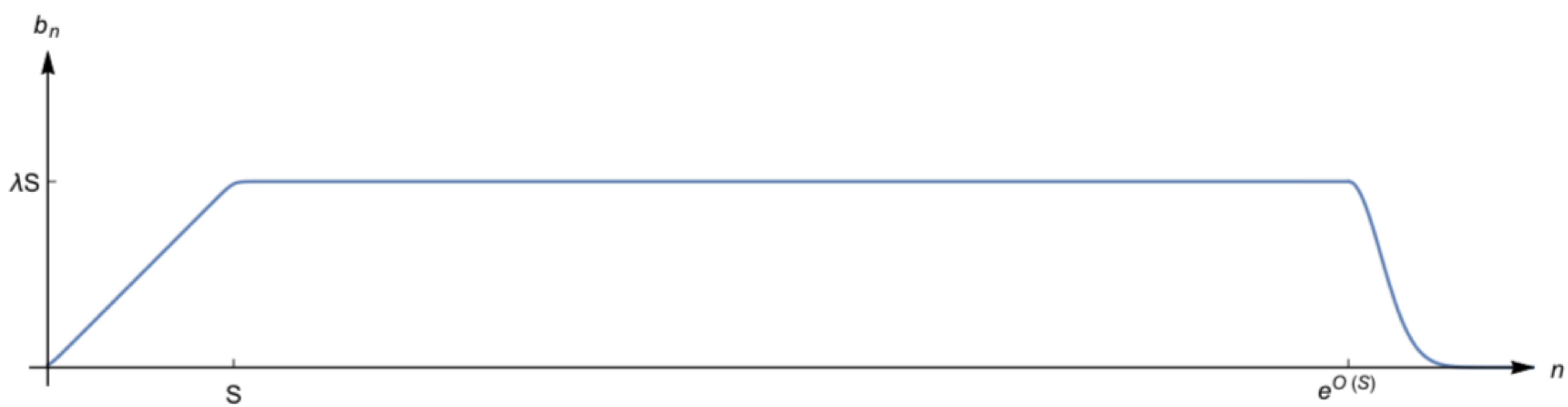


Figure 1: *Qualitative form of the Lanczos sequence for a fast scrambler with S degrees of freedom and Lyapunov exponent λ. It shows the linear growth characteristic of a fast scrambler, the long constant regime and a sharp turnoff at saturation.*

Dynamics of K-Complexity Growth

The Continuum Amplitudes

- To study the complexity growth profile in the post-scrambling region, we need to solve the differential equation (discrete in n) in $\varphi_n(t)$ using the conjectured growth of coefficients.

$$\partial_t \varphi_n = b_n \varphi_{n-1} - b_{n+1} \varphi_{n+1}$$

- However, to get a detailed matching between the scrambling and the post-scrambling regions, let us start with the continuum limit by introducing a coarse-graining scale ε .

- This allows us to define, $x = \varepsilon n$, as well as a continuum form of the differential equation to first order in epsilon.

$$\partial_t \varphi = -v(x) \partial_x \varphi - \tfrac{1}{2} \partial_x v(x) \varphi + O(\varepsilon)$$

where $v(x) = 2\varepsilon b(\varepsilon n) \equiv 2\varepsilon b_n$, and $\varphi(x,t) \equiv \varphi_n(t)$

- This can be solved with a simple change of variables given by, $v(x)\partial_x = \partial_y$
- The corresponding rescaled amplitude is $\psi(y,t) = \sqrt{v(y)}\varphi(y,t)$
- In terms of these variables, the differential equation becomes,

$$(\partial_y + \partial_t)\psi(y,t) = 0 + \ldots$$

with the solution given by,

$\psi(y,t) = \psi_i(y-t)$ with initial condition $\psi_i(y) = \psi(y)$

The solution describes a simple ballistic motion of the initial amplitude in the y-space, towards positive values.

- The final solution is obtained simply by translating between the x and y space.

Note: The amplitudes can be shown to be normalised in the continuum limit.

K-Complexity

An expression for K-Complexity in the continuum variables is,

$$C_K(t) = \frac{1}{\varepsilon^2}\int dy\,|\psi_i(y)|^2\, x(y+t)$$

In the scrambling regime

For fast scrambling systems with $v(x)=\lambda x$, the solution in the x-space reads,

$$\varphi_{scr}(x,t)=e^{-\lambda t/2}\varphi_i(xe^{-\lambda t})$$

The K-complexity can be correspondingly shown to grow exponentially in time,

$$C_K(t)_{scr}\approx e^{\lambda(t-t_*)}C_K(t_*)$$

Note : For systems that scramble slower, $v(x)\sim x^{\delta}$ with $\delta<1$ the K-complexity grows as a power law $\sim(\alpha t)^{1/(1-\delta)}$

In the post-scrambling regime

With our conjectured behaviour of the Lanczos coefficients, the velocity turns out to be a constant in this regime. Hence, the x and y frames are simply proportional to each other $x = v_* y$

The amplitude then simply moves ballistically in x-space as

$$\varphi_{post-scr}(x,t) = \varphi(x - v(t - t_*), t_*)$$

It is easy to see then that the K-complexity increases linear in this regime as,

$$C_K(t)_{post-scr} \approx \lambda n_*(t - t_*) + C_K(t_*) \sim \lambda S(t - t_*) + O(S)$$

The time-scale for the amplitude to reach the reach $n_{\max} \sim e^{O(S)}$ is,

$$t_K \sim \frac{1}{\lambda S} e^{O(S)}$$

K-Complexity Curve

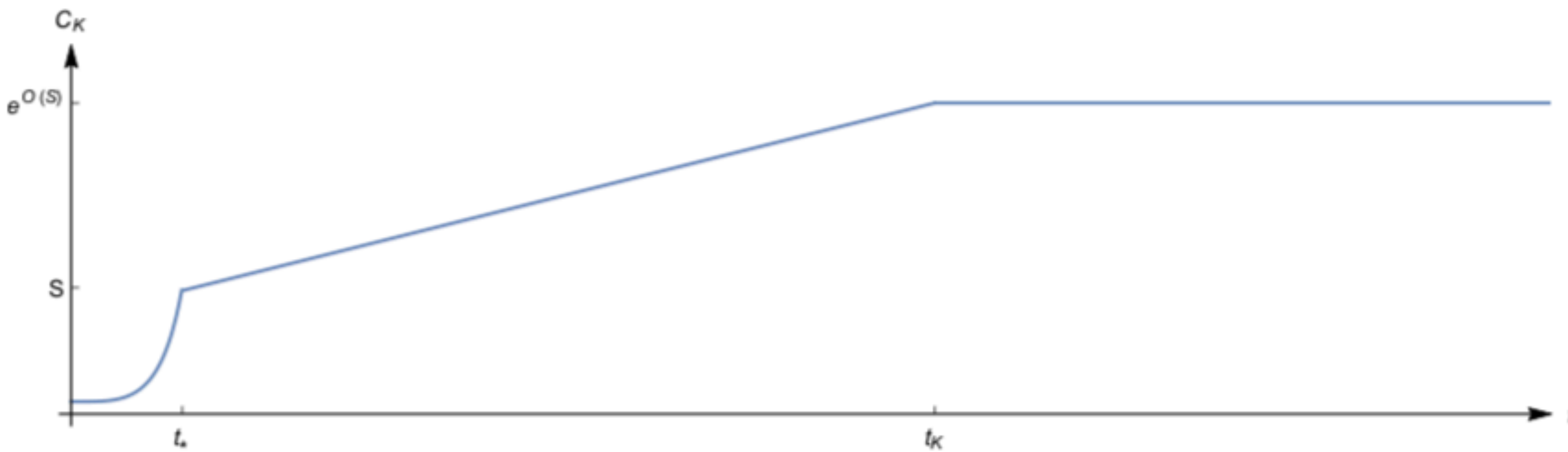


Figure 2: *Evolution of K-complexity for a fast scrambler of size S, featuring an exponential law in the pre-scrambling era $t < t_* \sim \lambda^{-1} \log S$, followed by a a linear law in the post-scrambling era, up to $t_K \sim e^{O(S)}/\lambda S$, when the complexity finally saturates.*

Note: The linear growth of complexity has been conjectured previously in the context of the volume and action proposals of complexity, and has been attributed to the growth of Black Hole interiors. **Susskind et al. '15-'19**

Operator Randomisation and K-Entropy

K-Entropy

- Until now we have characterised the operator growth in the scrambling and post-scrambling regimes, that had much to do with how the peak of the amplitude moves. However, that does not always tell us whether the amplitude randomises over the basis in an efficient way.

- To understand operator randomisation, we introduce a new quantity called the K-entropy.

- K-entropy or operator entropy is a von Neumann entropy defined through the amplitudes in the following way,

$$\boxed{S_K = -\sum_{n\geq 0} |\varphi_n|^2 \log|\varphi_n|^2}$$

- The K-entropy can effectively distinguish between a scrambled and a non-scrambled profile.

- If the amplitude is very peaked at a particular n, the K-entropy is small, whereas, if the entropy is uniform over an interval, $[0, n_M]$, the K-entropy is maximal, $\boxed{S_K = \log(n_M)}$

- In the continuum version, the K-entropy is

$$\boxed{S_K = -\frac{1}{\varepsilon}\int dx\, |\varphi(x,t)|^2 \log|\varphi(x,t)|^2}$$

- It is easy to see that in the scrambling region,

$$S_K = S_K(0) + \lambda t$$

- This means that there is entropy production leading to a linear rise in the K-entropy, which is captured by the differential equation to first order in epsilon.

- However, in the post-scrambling regime, no significant growth of the K-entropy is observed.

- Must go to higher orders in the derivative expansion!

Higher Orders in the Continuum Limit

The following terms appear at sub-leading order in the derivative expansion of the differential equation.

$O(\varepsilon)$

$$-\tfrac{1}{2}\varepsilon\,\partial_x v(x)\,\partial_x \varphi(x,t)$$

This corrects the velocity, has a small effect in the scrambling regime, completely ignorable in the post-scrambling regime

$O(\varepsilon^2)$

$$-\tfrac{1}{4}\varepsilon^2\,\partial_x v(x)\,\partial_x^2 \varphi(x,t) - \tfrac{1}{6}\varepsilon^2 v(x)\,\partial_x^3 \varphi(x,t)$$

The first is a diffusion term with the wrong sign, negligible effect.

The second, active throughout the post-scrambling era, leading correction!

The differential equation at $O(\varepsilon^2)$ in the y-frame is,

$$(\partial_t + \partial_y)\psi(y,t) = -\gamma\,\partial_y^3\psi(y,t)$$

where $\gamma = \frac{1}{6}\frac{\varepsilon^2}{v^2} \sim \frac{1}{(\lambda S)^2}$ is small.

This equation can be solved by going to Fourier space, and admits the following solution,

$$\psi(y,t) = \int dw\,\mathrm{Ai}(z-w)\,\psi_i[(3\gamma\Delta t)^{1/3}w]$$

where, $z = \frac{y-\Delta t}{(3\gamma\Delta t)^{1/3}}$, and the Airy function is convoluted with

the initial wave function profile.

For an initial Gaussian wave profile, $\psi_i(y,t_*) = \pi^{-1/4}\sqrt{\frac{\varepsilon}{\delta}}\exp\left(-\frac{(y-y_*)^2}{2\delta^2}\right)$

the solution is,

$$\psi(y,t) = \frac{\pi^{-1/4}\sqrt{2\varepsilon\delta}}{(3\gamma\Delta t)^{1/3}} e^B \mathrm{Ai}\left[\frac{\Delta y - \Delta t + C}{(3\gamma\Delta t)^{1/3}}\right]$$

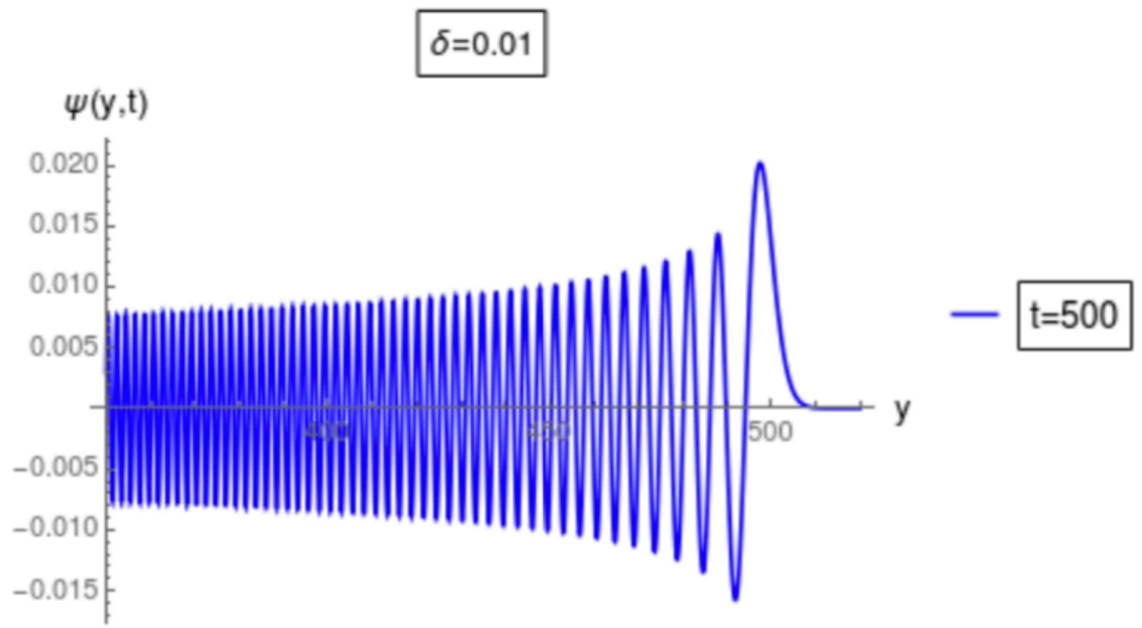
with,

$$B = -\frac{\delta^2}{6\gamma}\left(1-\frac{\Delta y}{\Delta t}\right) + \frac{\delta^6}{108\gamma^2\Delta t^2} \qquad C = \frac{\delta^2}{12\gamma\Delta t}$$

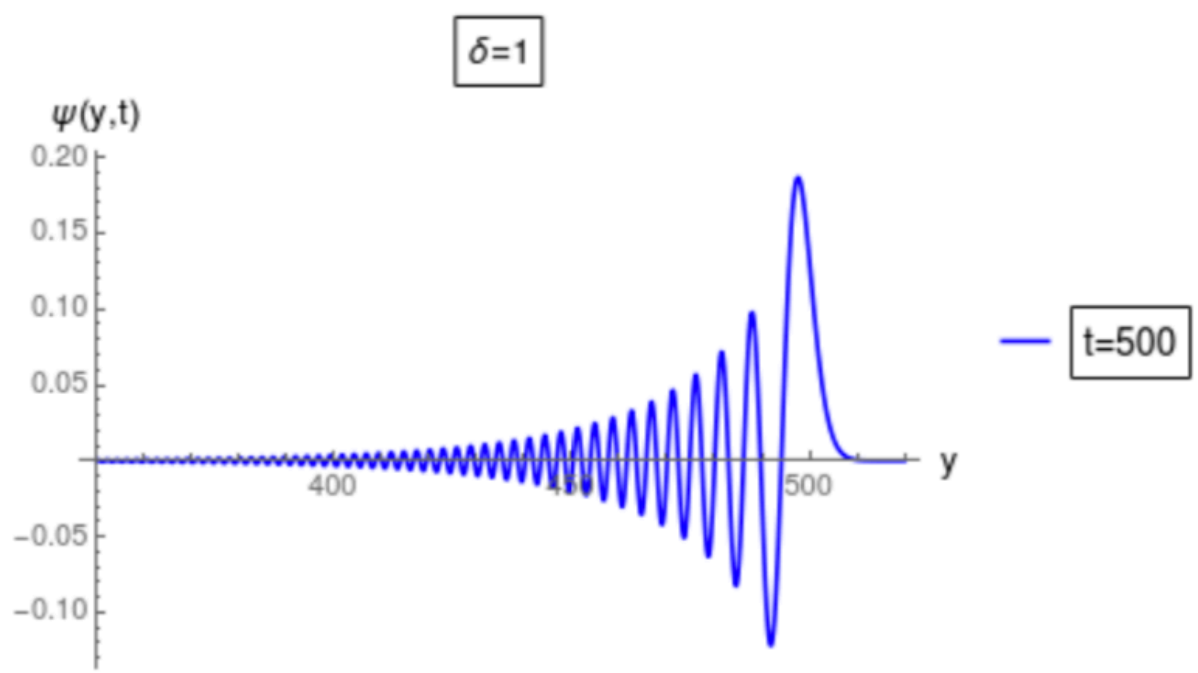
Looking at the long time tail, $\Delta y \ll \Delta t$, one finds

$$\varphi_{tail} \sim \sqrt{\delta\lambda n_*}\, e^{-(\delta\lambda n_*)^2} \frac{1}{\sqrt{2bt}} \times Osc_{[0,2bt]}$$

Exponential suppression, unless $\delta \sim 1/(\lambda S)$, for which there is very efficient randomisation!



Amplitude for a very narrow initial pulse, displaying efficient randomisation



Amplitude for a wide initial pulse, displaying an exponentially damped tail.

Quick Summary

- We solve for the third order equation, in terms of an Airy function convoluted with the initial wave function profile (typically a Gaussian).

- The tail is exponentially damped, unless the width is $\delta \sim 1/(\lambda S)$

- So, for the most physical scenario of a sufficiently broad Gaussian, the solution to this order seemingly fails to capture any randomisation of the operator in the K-basis.

- Efficient randomisation, however, occurs for $\delta \sim 1/(\lambda S)$ indicating that one could still hope to solve for the full discrete equation, in the hope of getting efficient randomisation over the K-basis.

The Discrete Amplitude

- To understand K-entropy production in the post-scrambling regime, let us look at the discrete equation.

- The discrete equation to be solved is: $\partial_t \varphi_n = b(\varphi_{n-1} - \varphi_{n+1})$

- It allows for a very simple solution, the Bessel function, which has the exact desired properties, the precise long tail we desire for effective randomisation.

$$\varphi_n(t) = J_n(2b_\infty t)$$

- However, it does not satisfy the correct boundary conditions, and hence can only be treated as a toy model.

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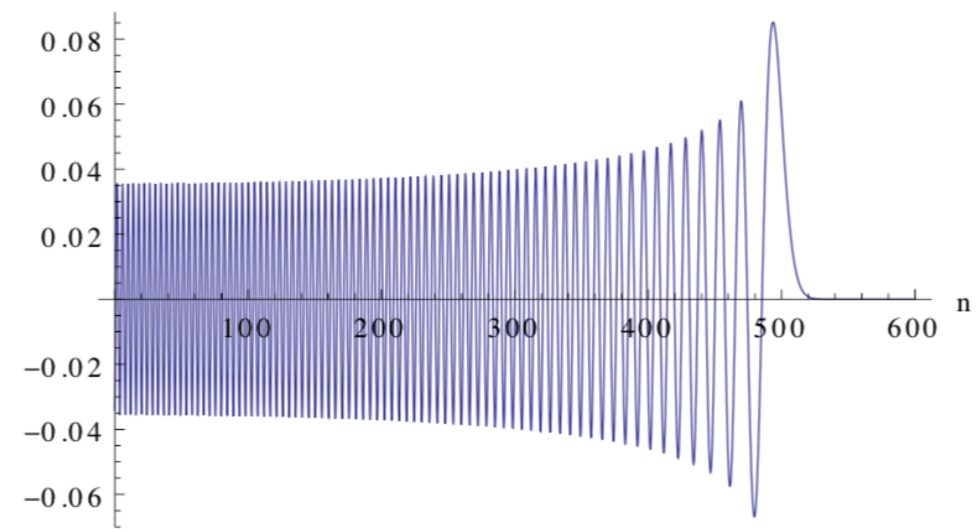

$$\varphi_n(t) = J_n(2b_\infty t)$$

Plot of the Bessel function $J_n(2bt)$ as a function of $n > 0$ for $2bt = 500$.

- However, it does not satisfy the correct boundary conditions, and hence can only be treated as a toy model.

- To get the right boundary conditions, we consider the following linear combination of the Bessel functions,

$$R_n(2b_\infty t) = J_n(2b_\infty t) + J_{n+2}(2b_\infty t) = \frac{(n+1)}{bt} J_{n+1}(2b_\infty t)$$

- This function, then satisfies the differential equation with the right boundary condition $\varphi_{-1}(t) = 0$

- However, it is still unphysical, since it has a zero right next to the peak value at t=0, and moreover, a delta function profile to start with!

- Moreover, any function in the post-scrambling regime is bound to have undergone a long phase of scrambling and randomisation, and hence has a significant width while it enters the region of constant b_n's!

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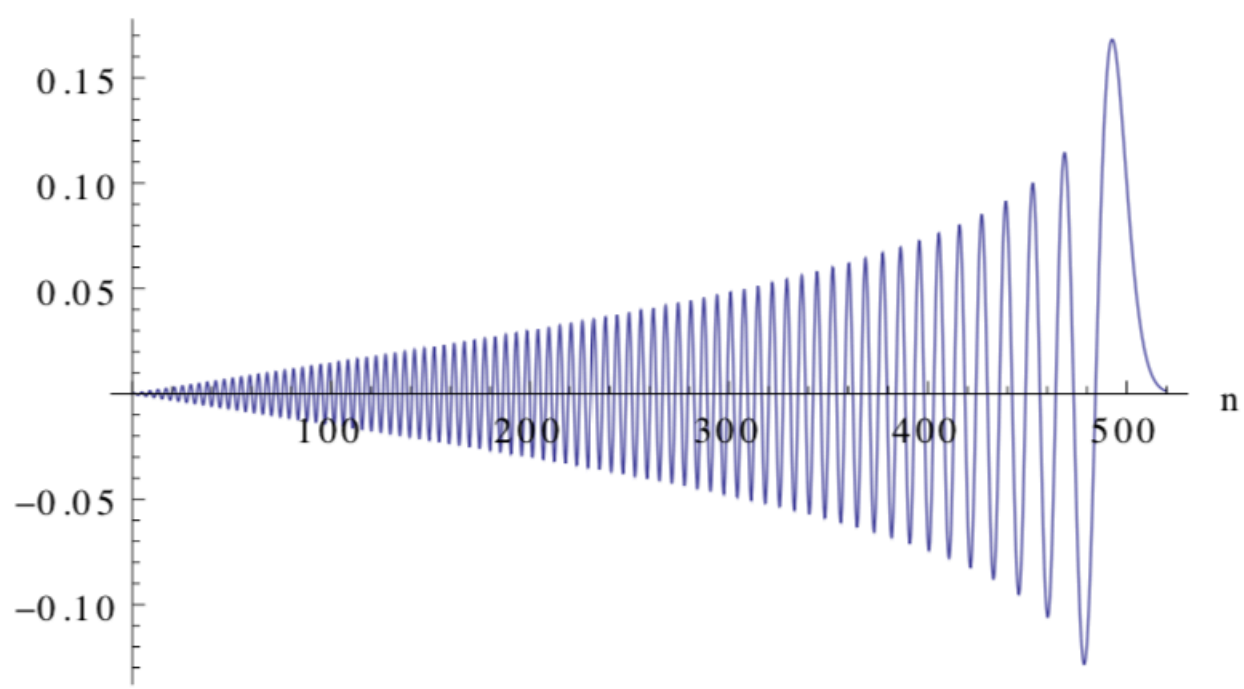
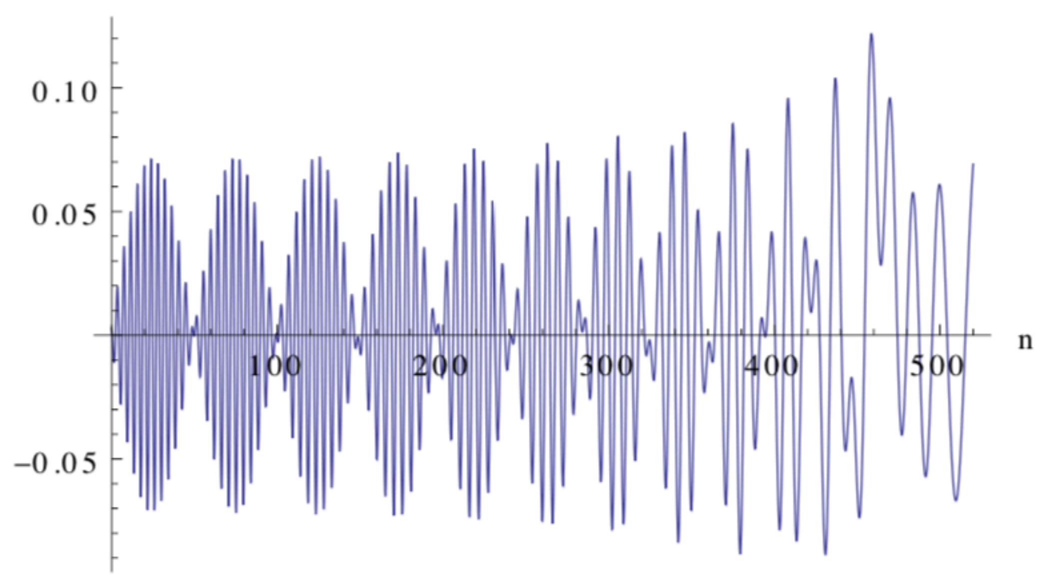


Plot of $R_n(2bt)$ as a function of $n > 0$ for $2bt = 500$.

- To solve the problem of the zero next to the peak, and simulate a more physical initial amplitude, we first define a new function of the following kind:

$$R_n^{(k)}(2b_\infty t) = J_{n-k}(2b_\infty t) + (-1)^k J_{n+k+2}(2b_\infty t)$$

- With this function, we can manufacture a zero at any *n=k>0* at will, and arbitrarily far away from n=0.



Plot of $R_n^{(k)}(2bt)$ *as a function of* $n > 0$ *for* $k = 30$ *and* $2bt = 500$.

- To solve the second problem of an amplitude with a significant width, we can take a linear combination of the above function and define a general function that solves the differential equation, satisfies the boundary conditions, and simulates a more realistic scenario.

$$\varphi_n(t) = \sum_{k=0}^{K_0-1} \alpha_k R_n^{(k)}(2b_\infty t) \qquad \text{with} \qquad \sum_{k=0}^{K_0-1} |\alpha_k|^2 = 1$$

where K_0 is the signal width.

- For a signal of width $K_0 \sim S$, the above amplitude simulates a signal prepared by a previous period of fast-scrambling.

- From the plot with a square pulse, one can estimate that with an average tail of height $1/\sqrt{2bt}$ and a width of order $2bt$, efficient randomisation occurs in order of magnitude.

- As a result, the K-entropy grows as $$S_K(t) \sim \log(2b_\infty t)$$

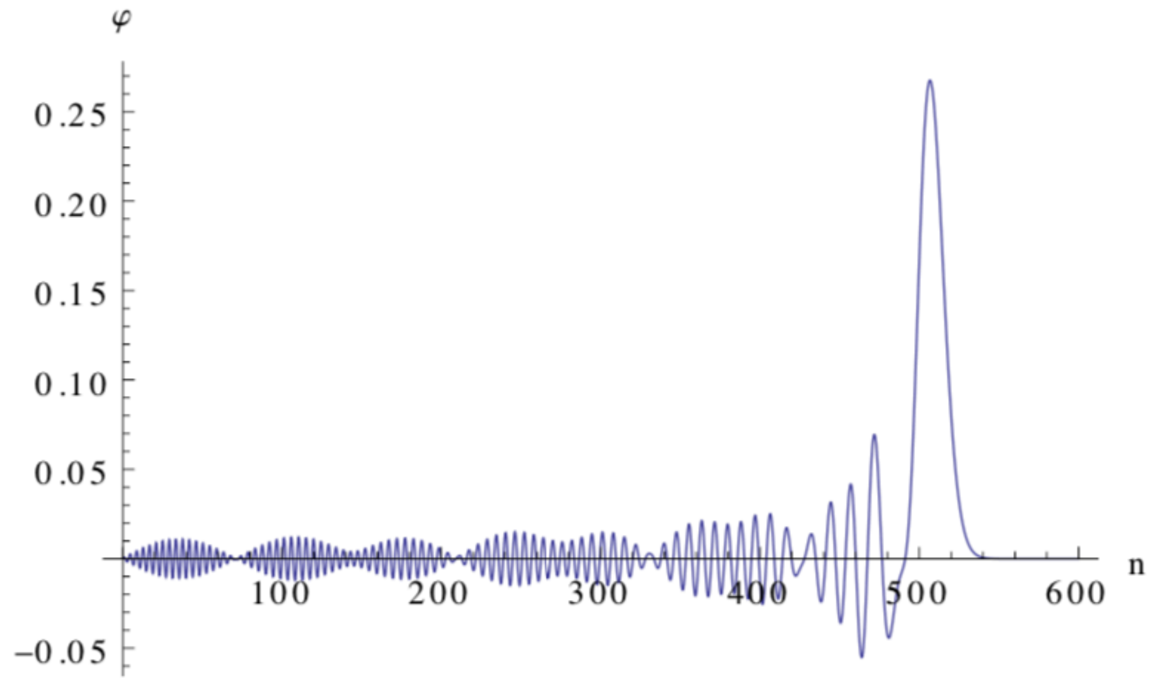


Figure 9: *Plot of the amplitude (77) as a function of* $n > 0$ *for* $2bt = 500$ *and an initial square pulse of width* $K_0 = 20$.

- From the plot with a square pulse, one can estimate that with an average tail of height $1/\sqrt{2bt}$ and a width of order $2bt$, efficient randomisation occurs in order of magnitude.
- As a result, the K-entropy grows as $S_K(t) \sim \log(2b_\infty t)$

K-Entropy Curve

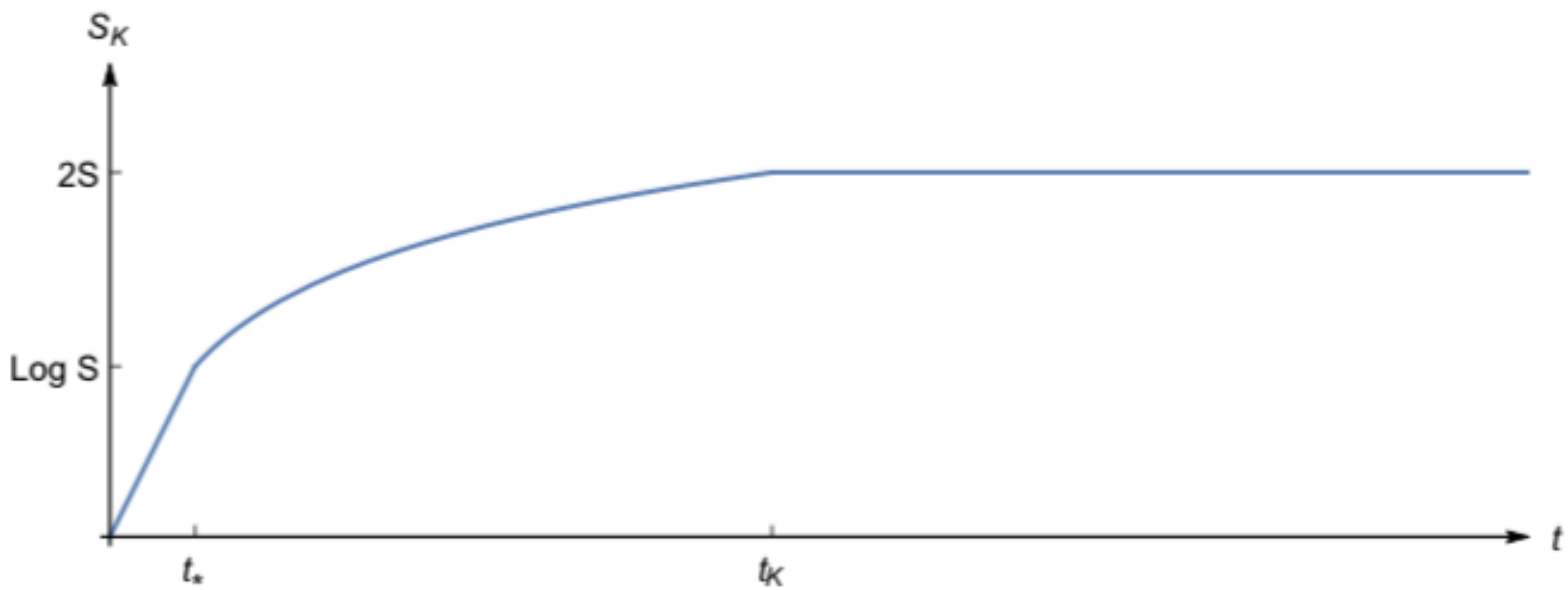
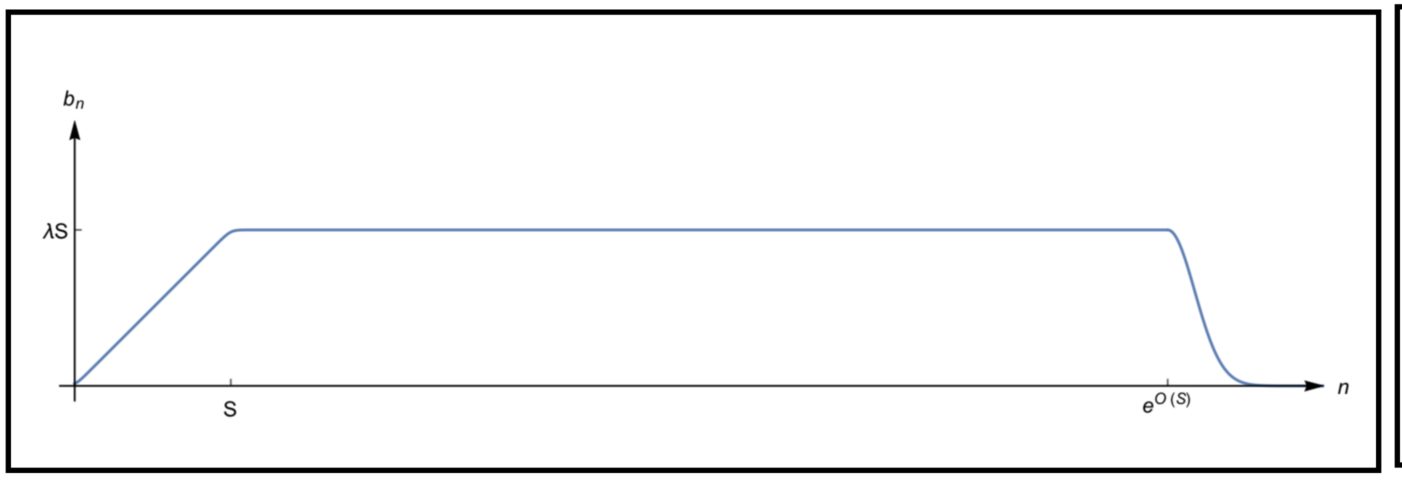
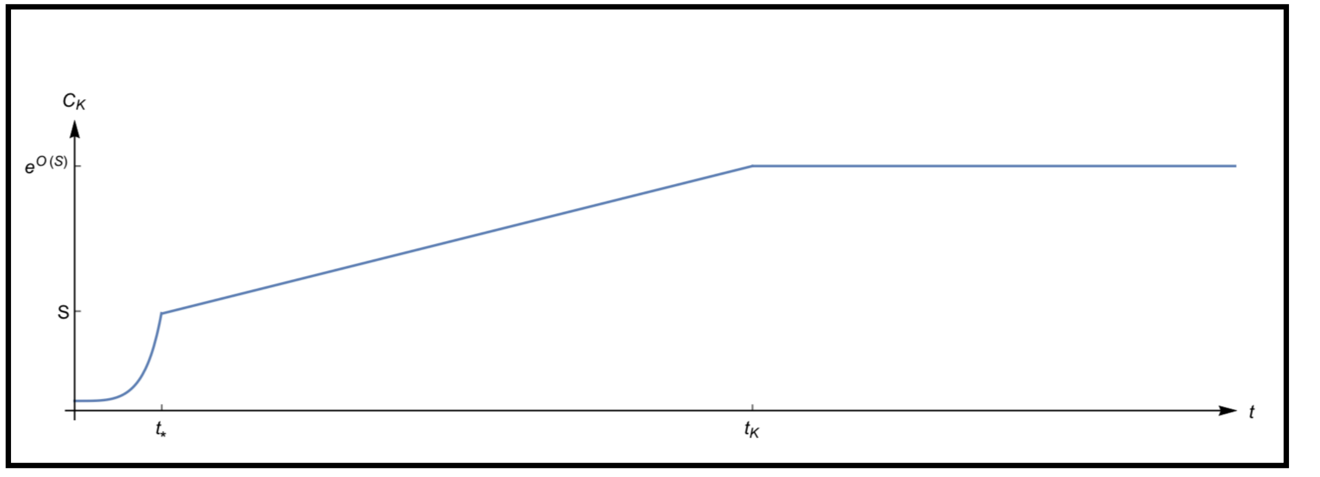


Figure 11: *Sketch of the K-entropy dynamics in a fast scrambler with S degrees of freedom and Lyapunov exponent λ. A linear growth proportional to λt during scrambling is followed by a logarithmic increase in the post-scrambling era, according to a scaling* $\log(2S\lambda t)$*, and a final saturation beyond times of order t_K.*

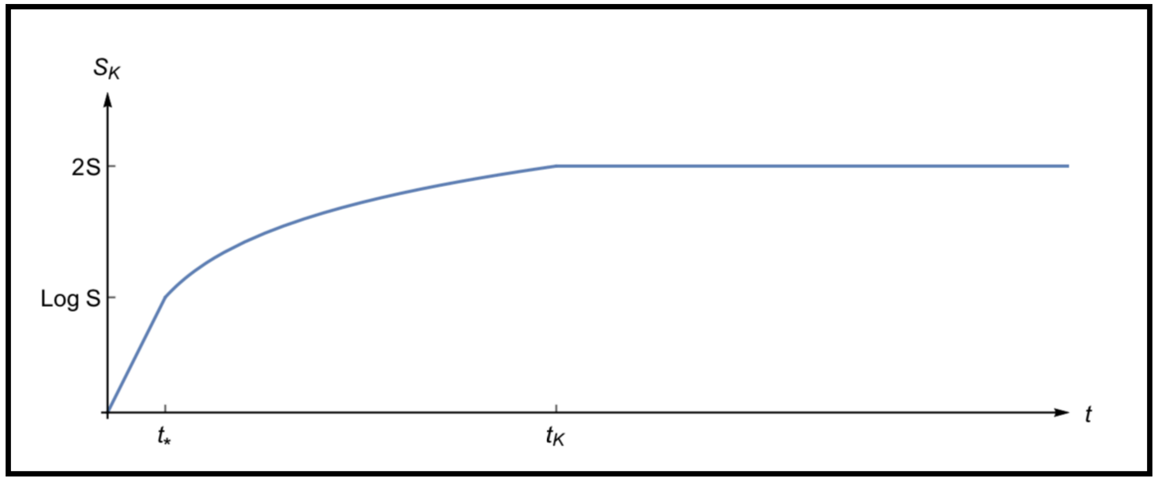
Summary



Growth of Lanczos coefficients



Growth of K-Complexity



Growth of K-entropy

Discussions and Conclusion

- We defined an algebraic notion of operator complexity, called K-complexity, that relies on the effective dimensionality of the linear subspace generated under the operator's time evolution.

- The time dynamics of this quantity was already known, upto scrambling times, and had been used as an effective characterisation of chaotic behaviour, being governed by the same Lyapunov exponent as in OTOCs.

- Using ETH, we conjectured its post-scrambling behaviour to correspond to a linear growth with a rate that is extensive in the system size *S*.

- We simultaneously argued that the linear growth of the K-complexity must saturate to a constant value of about $e^{O(S)}$, at time-scales also of about $e^{O(S)}$ and remain constant thereafter upto Poincare recurrence times, which are of order $t_P \sim e^{e^{O(S)}}$.

- Upto scrambling times, the operator grows in size, while simultaneously randomising over the operator space.

- To characterise whether the operator randomises post-scrambling, we invoke the notion of K-entropy.

- The K-entropy effectively measures the degree of uniformity of the φ_n's.

- It grows during the linear growth of K-complexity, post-scrambling, signifying randomisation in operator space.

- When K-complexity saturates, the K-entropy saturates to $S_K(t) \sim \log(2b_\infty t)$ signifying complete randomisation in order of magnitude.

Open Questions

- What is the effect of this randomisation on correlation functions? Does it pave the way for an RMT description of the system beyond Heisenberg times?

- During scrambling, K-complexity is equivalent to operator size. Under the holographic dictionary, increasing operator size, corresponds to a free falling particle towards the horizon. Does it imply that the post-scrambling linear growth of operator complexity effectively describes particle motion in the black hole interior?