On Entanglement Hamiltonians in 1D free lattice models

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Entangle This IV: Chaos, Order and Qubits

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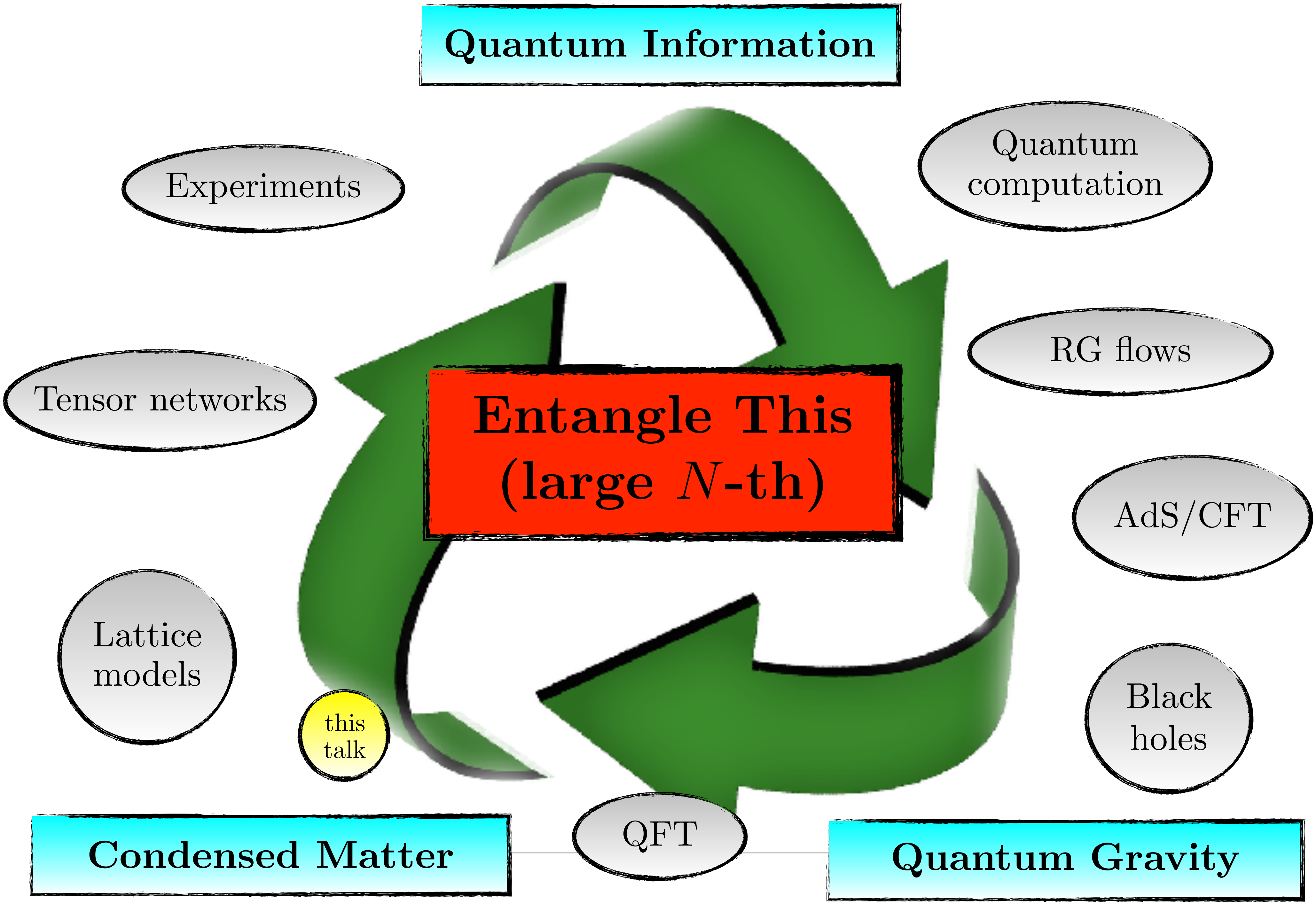
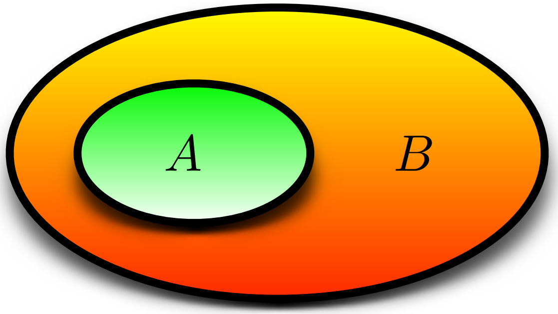
Entanglement: a crossroad of interests



Entanglement Entropy (EE), EH & Contour for EE

- Quantum system in its ground state: $\rho = |\Psi\rangle\langle\Psi|$
  Factorised Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



e.g.: spatial bipartition

- A's **reduced density matrix** $\rho_A = \mathrm{Tr}_B \rho$
  (normalisation: $\mathrm{Tr}\rho_A = 1$)

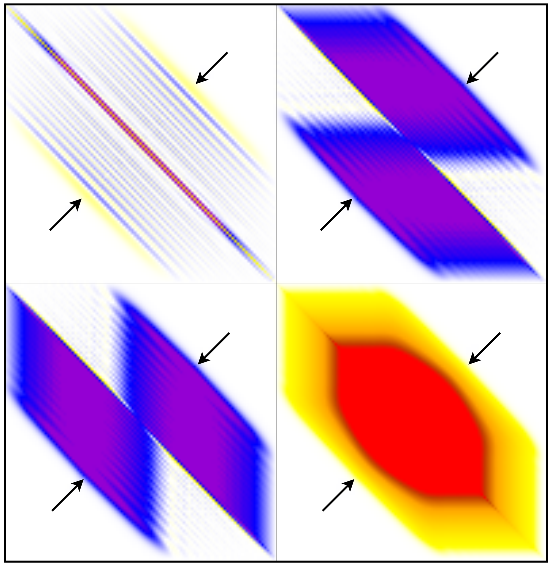
- Entanglement Entropy (EE) $S_A \equiv -\mathrm{Tr}_A(\rho_A \log \rho_A)$

- $\rho_A \propto e^{-\widehat{K}_A}$ Entanglement Hamiltonian (EH) $\widehat{K}_A$

- $S_A = \sum_{i \in A} s_A(i)$ Contour function for the EE $s_A(i)$

  $s_A(i) \geqslant 0$ (and other requirements)

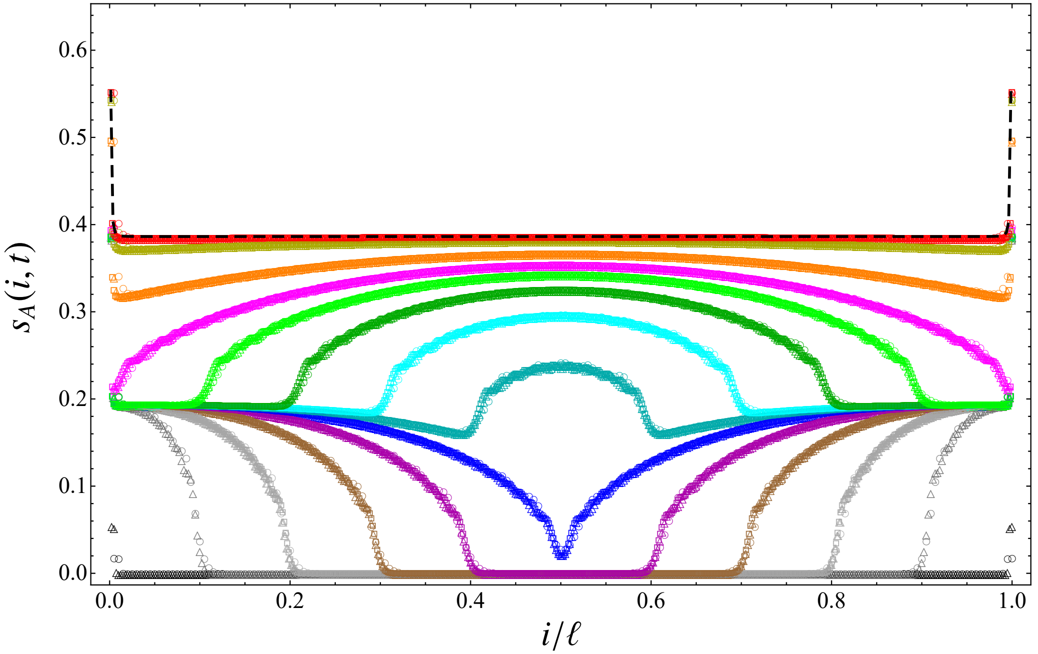
Outline



- Entanglement Hamiltonians (EH) in CFT
- EH and contours for the EE in
  - harmonic chains (HC)
  - free fermions chains (FFC)
- Continuum limit of the EH of an interval in a FFC [Eisler, E.T., Peschel, (2019)]

[Di Giulio, Arias, E.T., (2019)]

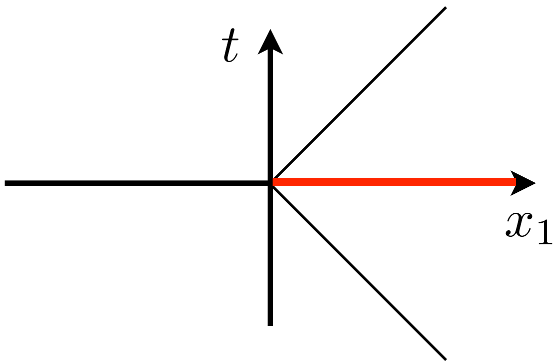
- Two global quantum quenches: temporal evolutions of
  - EH matrices
  - Contours for the EE
- Insights from CFT and from the quasi-particle picture



EH in field theories: Bisognano-Wichmann & CFT

- QFT in its ground state, A is the (right) half-space $x_1 > 0$ in $\mathbb{R}^{1,d-1}$

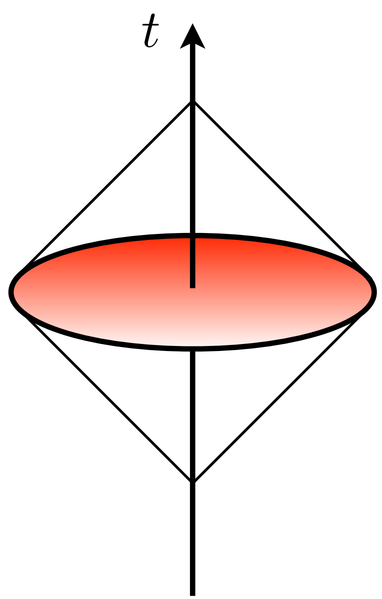
$$\widehat{K}_A = \int_A x_1 \, T_{00} \, d^{d-1}x$$



$\widehat{K}_A$ is the generator of the Lorentz boosts in the (right) Rindler wedge [Bisognano, Wichmann, (1975)]

- CFT$_d$ and A is a ball of radius R

$$\widehat{K}_A = 2\pi \int_A \frac{R^2 - r^2}{2R} \, T_{00} \, d^{d-1}x$$



The Rindler wedge can be conformally mapped into the causal diamond of A

[Hislop, Longo, (1982)] [Casini, Huerta, Myers, (2011)]

- CFT$_2$ and A is an interval (ground state)

$$\widehat{K}_A = 2\pi \int_0^{\ell} \frac{x(\ell - x)}{\ell} \, T_{00} \, dx$$

EH: harmonic chains (HC) & free fermions chains (FFC)

- Harmonic chain (periodic b.c.)

$$\widehat{H} = \sum_{i=0}^{L-1} \left(\frac{1}{2m} \hat{p}_i^2 + \frac{m\omega^2}{2} \hat{q}_i^2 + \frac{\kappa}{2} (\hat{q}_{i+1} - \hat{q}_i)^2 \right) \qquad \hat{\boldsymbol{r}} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix}$$

$$\widehat{K}_A = \frac{1}{2} \hat{\boldsymbol{r}}^{\mathrm{t}} H_A \hat{\boldsymbol{r}} \qquad H_A \equiv \begin{pmatrix} M & E \\ E^{\mathrm{t}} & N \end{pmatrix}$$

$$\gamma_A \equiv \begin{pmatrix} Q & R \\ R^{\mathrm{t}} & P \end{pmatrix}$$

Gaussianity and Wick theorem allow to write the EH matrix H_A in terms of the reduced covariance matrix $\gamma_A \equiv \mathrm{Re}\langle \hat{\boldsymbol{r}}\, \hat{\boldsymbol{r}}^{\mathrm{t}} \rangle |_A$

[Peschel, (2003)] [Casini, Huerta, (2009)] [Banchi, Braunstein, Pirandola, (2015)]

- Chain of free fermions

$$\widehat{H} = -\frac{1}{2} \sum_{n=-\infty}^{+\infty} \left(\hat{c}_n^\dagger \, \hat{c}_{n+1} + \hat{c}_{n+1}^\dagger \, \hat{c}_n \right)$$

$$\widehat{K}_A = \sum_{i,j=1}^{\ell} T_{i,j} \, \hat{c}_i^\dagger \, \hat{c}_j \qquad T^{\mathrm{t}} = \log(C_A^{-1} - \mathbf{1})$$

The EH matrix T can be written in terms of the correlation matrix C_A restricted to the subsystem A [Peschel, (2003)]

EH of an interval in the FFC: continuum limit

$$\widehat{K}_A = \sum_{i,j=1}^{N} T_{i,j}\, c_i^\dagger c_j \quad \xrightarrow{\text{continuum limit}} \quad \widehat{K}_A = 2\pi\ell \int_0^\ell \beta(x)\, T_{00}(x)\, dx$$

For an interval A:
$$\beta(x) = \frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)$$

For the Dirac fermion:

$$T_{00}(x) = \frac{1}{2}\left[\, \psi_{\mathrm{R}}^\dagger(x)\,(-\mathrm{i}\,\partial_x)\psi_{\mathrm{R}}(x) - \psi_{\mathrm{L}}^\dagger(x)\,(-\mathrm{i}\,\partial_x)\psi_{\mathrm{L}}(x) + \text{h.c.}\,\right]$$

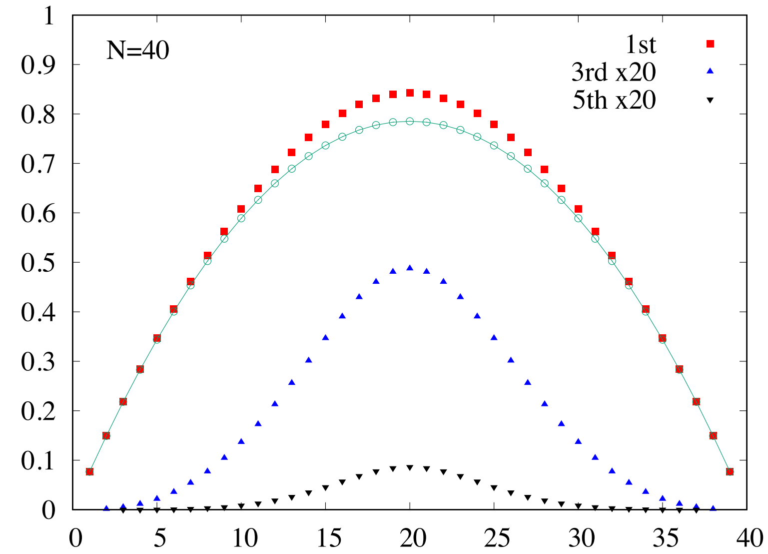
- Analysis based on discretisation of deltas [Arias, Blanco, Casini, Huerta, (2016)]
- Standard procedure for the continuum limit (long range hopping)
  $x = i\, s \qquad \ell = N s = \text{const}$ [Eisler, E.T., Peschel, (2019)]
  - Non trivial sums involving the hypergeometric functions must be performed
  - Analytic expressions for the higher derivatives (subleading) terms have been obtained

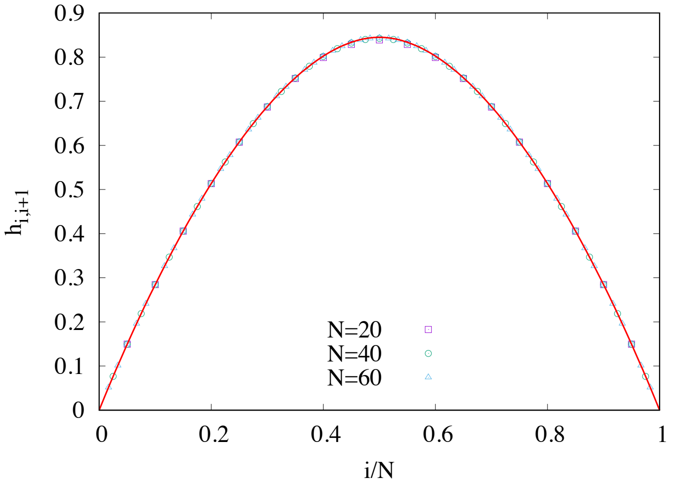
EH of an interval in the FFC

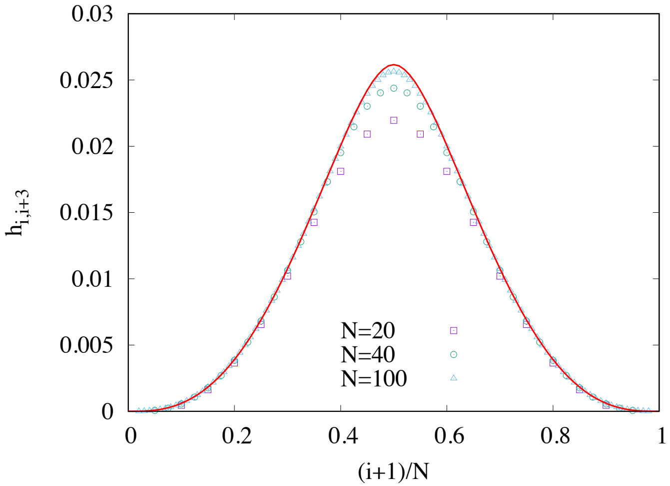
[Eisler, Peschel, (2017)]

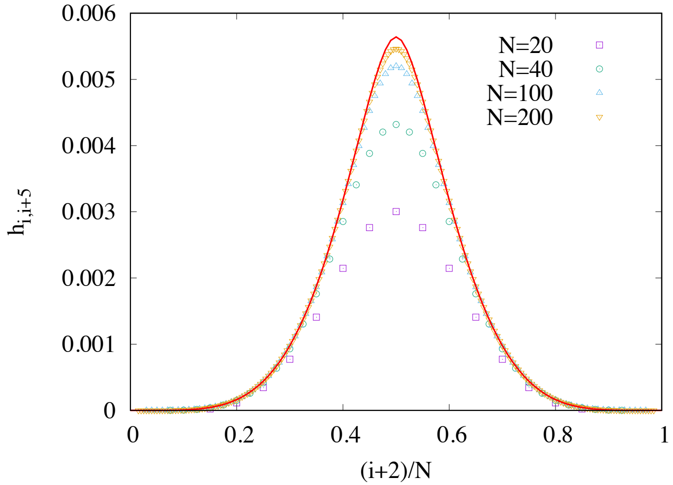
- Free fermions chain: analytic results for the interval A when the number of its sites diverges

$$-h_{i,j} \equiv \lim_{N\to\infty} \frac{T_{i,j}}{N}$$









- The result is written in terms of hypergeometric functions

$$h_{i,i+p} = \pi \, \frac{(2p-1)!!\,(4p-1)!!}{2^p\, p!\,(2p+1)!}\; z^{2p+1}\; {}_3F_2\!\left(p+\frac{1}{4}, p+\frac{1}{4}, p+\frac{3}{4}\,; p+1, 2(p+1)\,; (4z)^2\right) \qquad z \equiv \frac{i+p}{N}\left(1-\frac{i+p}{N}\right)$$

EH of an interval in the FFC: higher derivatives terms

[Eisler, E.T., Peschel, (2019)]

$$\widehat{K}_A = \sum_{m=0}^{\infty} \frac{1}{m!} \int_0^{\ell} \Big\{ F_m^{(+)}(x)\, \mathcal{H}_+^{(m)}(x) + F_m^{(-)}(x)\, \mathcal{H}_-^{(m)}(x) \Big\}\, dx$$

- Operators:

$$\mathcal{H}_{\pm}^{(m)} \equiv \sum_{k=0}^{m} \binom{m}{k} \left(-\frac{1}{2}\, \partial_x \right)^{m-k} \Psi_{\pm}^{(k)}$$

$$\Psi_+^{(k)} \equiv -\frac{1}{2} \left(\psi_{\mathrm{R}}^\dagger\, \psi_{\mathrm{R}}^{(k)} + \psi_{\mathrm{L}}^\dagger\, \psi_{\mathrm{L}}^{(k)} \right) + \text{h.c.} \qquad \Psi_-^{(k)} \equiv -\frac{\mathrm{i}}{2} \left(\psi_{\mathrm{R}}^\dagger\, \psi_{\mathrm{R}}^{(k)} - \psi_{\mathrm{L}}^\dagger\, \psi_{\mathrm{L}}^{(k)} \right) + \text{h.c.}$$

- Weight functions:

$$\begin{cases} F_m^{(+)}(x) \;\equiv\; \delta_{m,0}\, t_0(x) + 2\, s^m \displaystyle\sum_{r=1}^{\infty} r^m \cos(r q_F s)\, t_r(x) \\[2ex] F_m^{(-)}(x) \;\equiv\; 2\, s^m \displaystyle\sum_{r=1}^{\infty} r^m \sin(r q_F s)\, t_r(x) \end{cases}$$

- $m = 0$ $\begin{cases} \mathcal{H}_-^{(0)} = 0 \\ F_0^{(+)}(x) = 0 \end{cases}$ $\qquad m = 1$ $\begin{cases} \mathcal{H}_+^{(1)} = 0 \\ \mathcal{H}_-^{(1)} = T_{00} \qquad F_1^{(-)}(x) = 2\pi\ell\, \beta(x) \end{cases}$

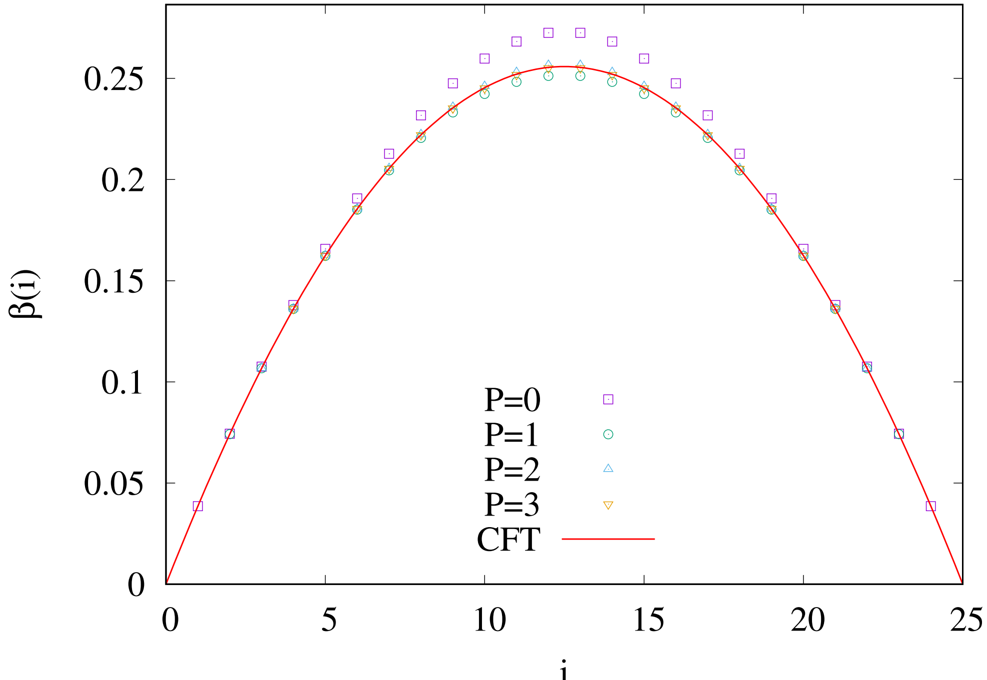
EH of an interval in the FFC: finite size & finite T

- At finite size and finite T, we observe numerically that

$$\beta(i) = \frac{1}{\pi N} \sum_{p=0}^{P} (-1)^p (2p+1) H_{i-p,i+p+1}$$

- Periodic chain of length L

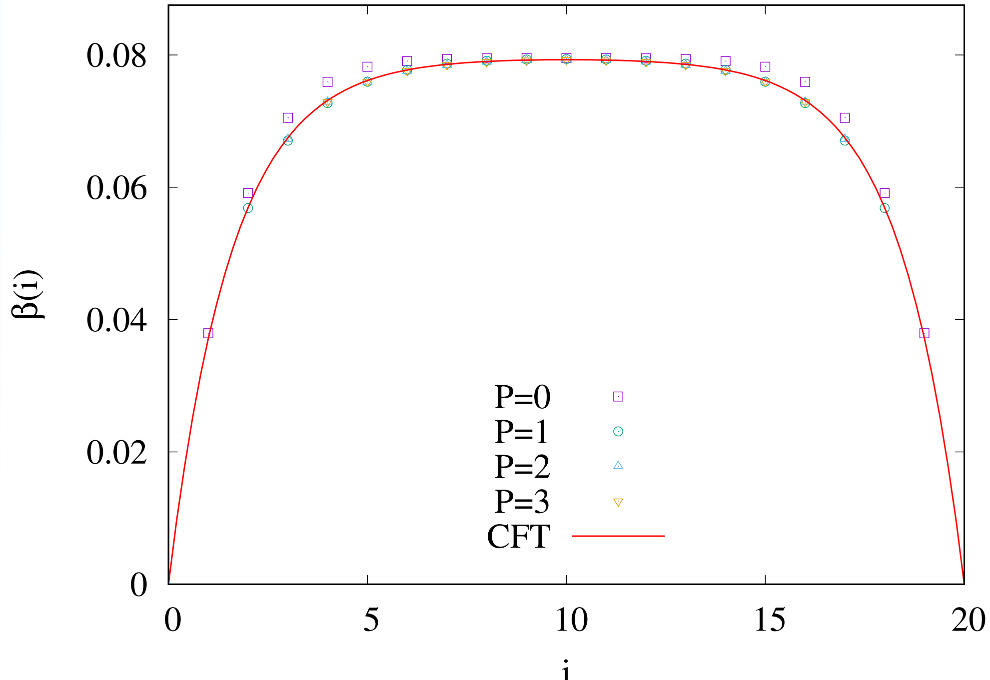
$$\beta(x) = \frac{L}{\pi\,\ell}\,\frac{\sin(\pi\,x/L)\sin(\pi(\ell-x)/L)}{\sin(\pi\,\ell/L)}$$



$N = 25$ $\qquad$ $\ell/L = 1/6$

- Infinite chain at temperature $1/\beta$

$$\beta(x) = \frac{\beta}{\pi\,\ell}\,\frac{\sinh(\pi\,x/\beta)\sinh(\pi(\ell-x)/\beta)}{\sinh(\pi\,\ell/\beta)}$$



$N = 20$ $\qquad$ $\beta = 10$

(Global) Quantum quenches

[Calabrese, Cardy, (2005), (2007)]

- Quantum quench:
  - System prepared in the ground state $|\psi_0\rangle$ of H_0
  - At $t = 0$ a sudden change is performed

Unitary evolution: $|\psi(t)\rangle = e^{-\mathrm{i}Ht}|\psi_0\rangle$ $\quad \rho(t) = |\psi(t)\rangle\langle\psi(t)|$

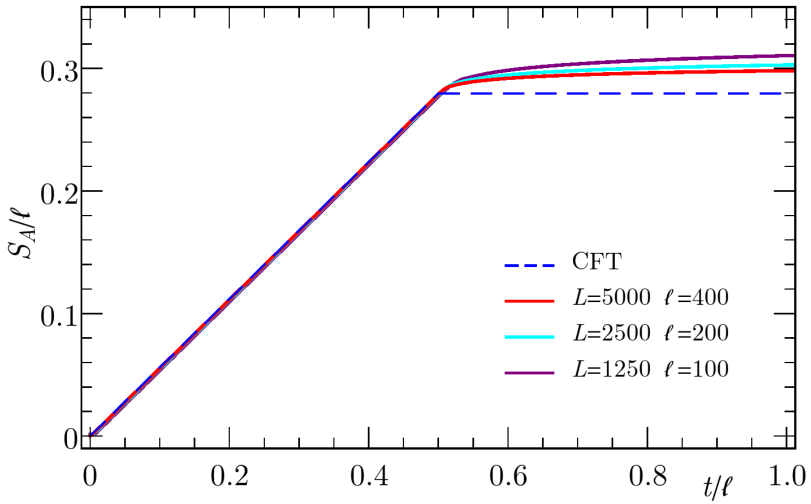
Global quench

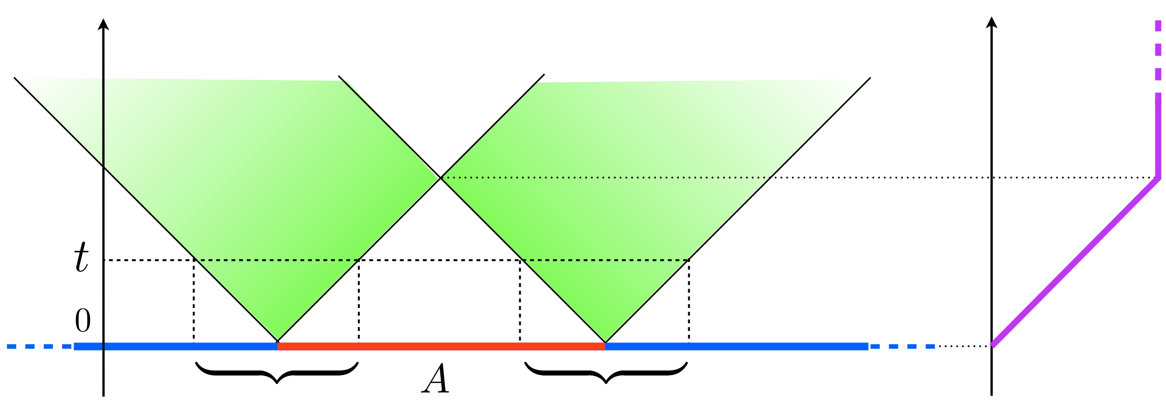
A parameter of H_0 is modified
(e.g. mass switched off)

Local quench

Interaction modified in one point
(e.g. two half-lines joined)

→ $S_A(t)$ when H is the Hamiltonian of a CFT $\quad S_A \simeq 2\pi c t/(3\tau_0) \quad t/\ell < 1/2$





Figures from [Coser, E.T., Calabrese, (2014)]

EH of a semi-infinite line after a global quench in CFT

[Cardy, E.T., (2016)]

- Conformal symmetry and a proper analytic continuation provides $\widehat{K}_A$ when A is a semi-infinite line after a global quench

$$\widehat{K}_A = \frac{\tau_0}{\pi} \int_{-t}^{\infty} \frac{\sinh(\pi[x+t]/\tau_0)\,\cosh(\pi[x-t]/\tau_0)}{\cosh(2\pi t/\tau_0)}\, T(x)\, dx + \frac{\tau_0}{\pi} \int_{t}^{\infty} \frac{\sinh(\pi[x-t]/\tau_0)\,\cosh(\pi[x+t]/\tau_0)}{\cosh(2\pi t/\tau_0)}\, \overline{T}(x)\, dx$$

  - The expected linear growth of S_A is recovered.

- Gaps in the entanglement spectrum

$$g_{a,0} \simeq \frac{\pi \tau_0\, \Delta_a}{2\, t}$$

- This $\widehat{K}_A$ provides a natural candidate for the contour function $s_A(x,t)$

- Further analysis in [Wen, Ryu, Ludwig, (2018)]

Global quenches protocols

- Harmonic chain: quench of the frequency parameter $\omega_0 \to \omega$
  [Calabrese, Cardy, (2007)]
  We mainly considered the quench having $\omega_0 = 1$ and $\omega = 0$

- Free fermions chain: [Eisler, Peschel, (2007)]
  the initial state is the ground state of a dimerised chain

$$\widehat{H}_0 = -\frac{1}{2}\sum_{n=-\infty}^{+\infty} t_n \big(\hat{c}_n^\dagger\, \hat{c}_{n+1} + \hat{c}_{n+1}^\dagger\, \hat{c}_n\big) \qquad \begin{cases} t_{2n} = 1 \\ t_{2n+1} = 0 \end{cases}$$

At $t = 0$ the inhomogeneity is removed

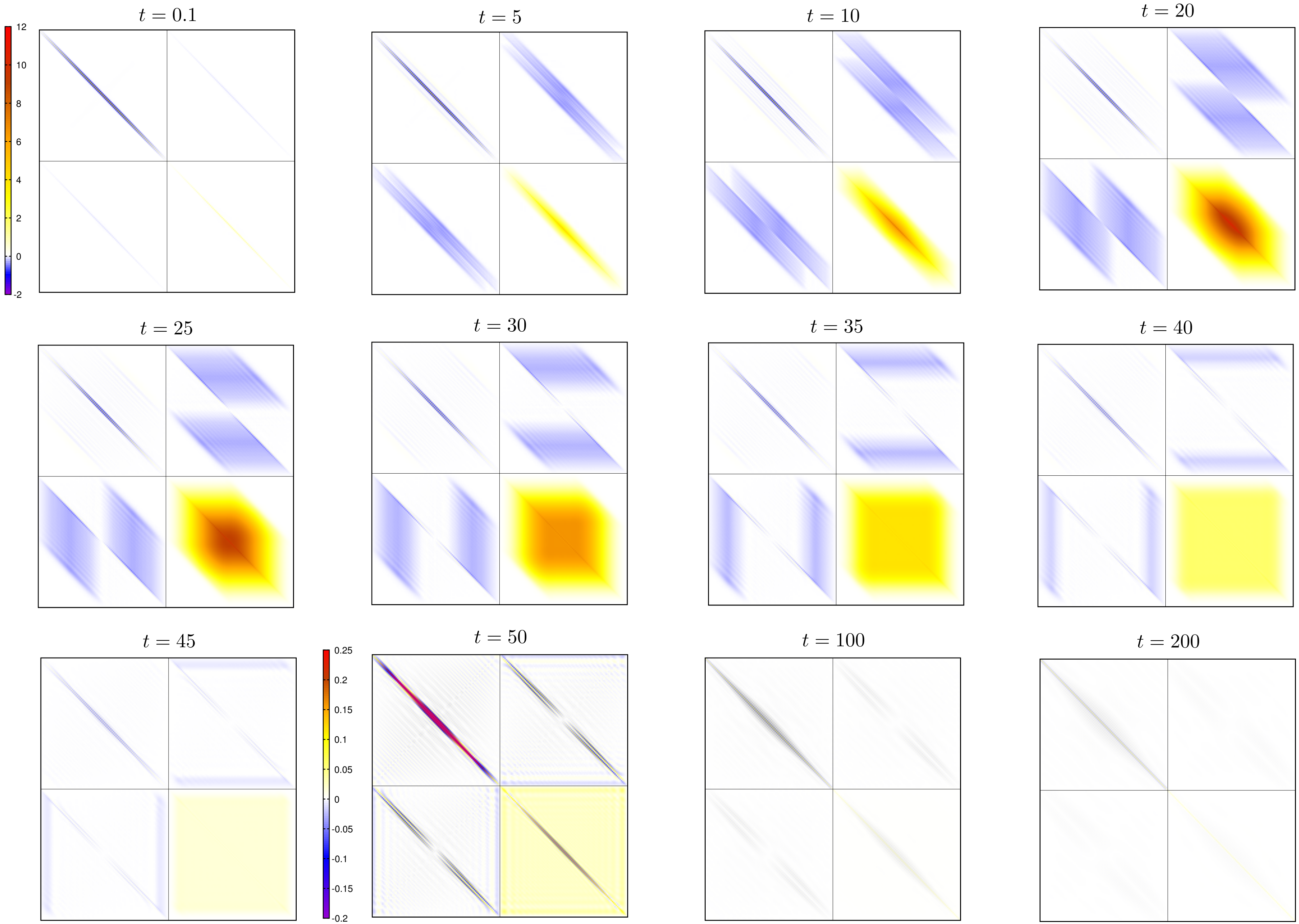
$$\widehat{H} = -\frac{1}{2}\sum_{n=-\infty}^{+\infty} \big(\hat{c}_n^\dagger\, \hat{c}_{n+1} + \hat{c}_{n+1}^\dagger\, \hat{c}_n\big)$$

- $C_A(t)$ for an interval is known analytically

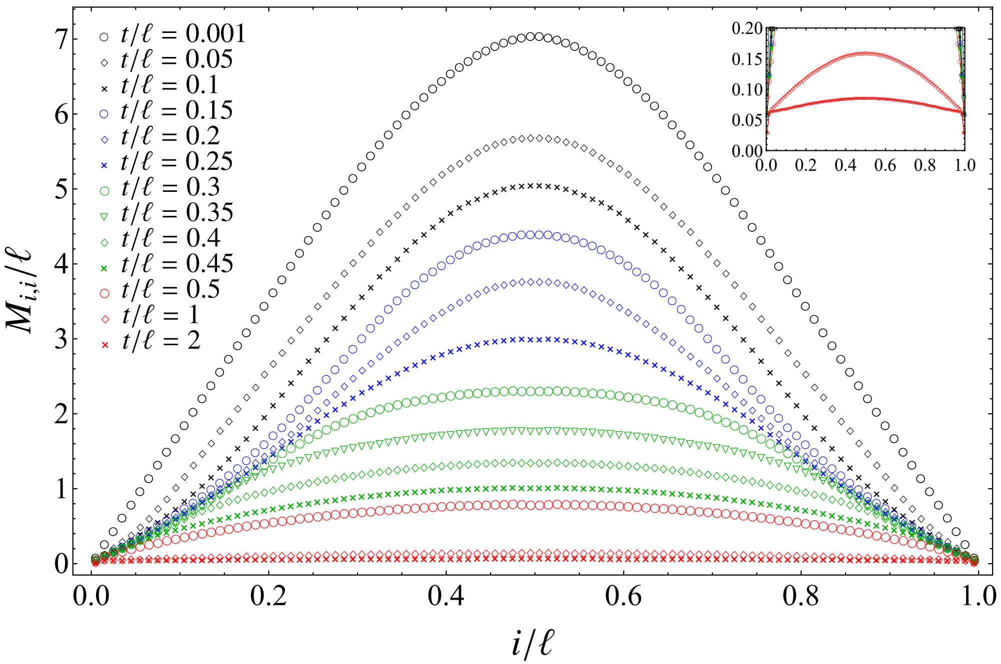
Evolution of the EH matrix in the HC

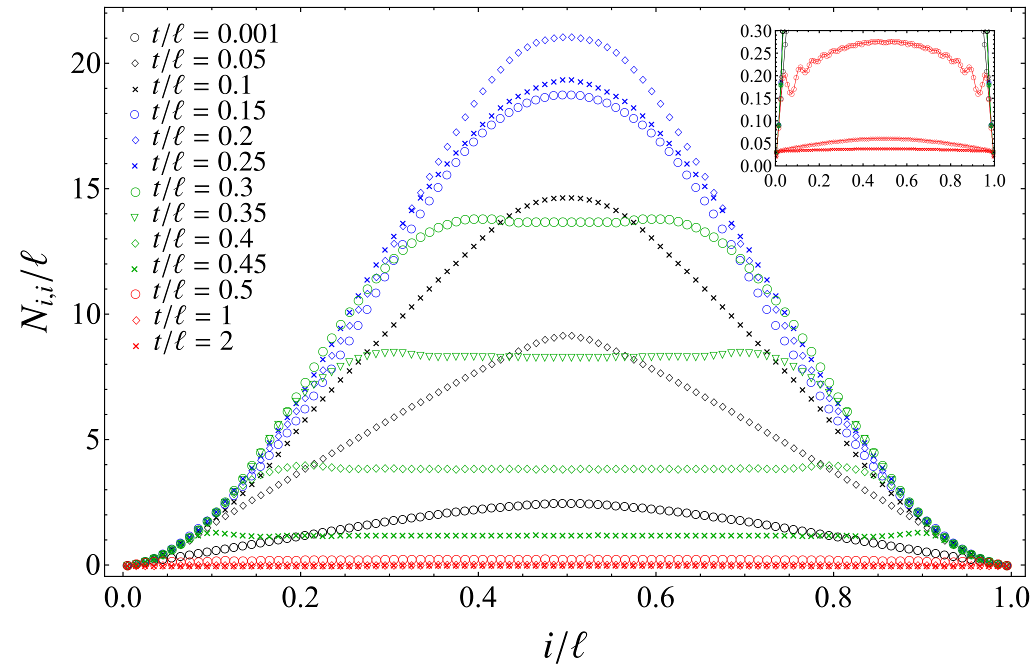
$\ell = 100 \quad \omega_0 = 1 \quad \omega = 0$

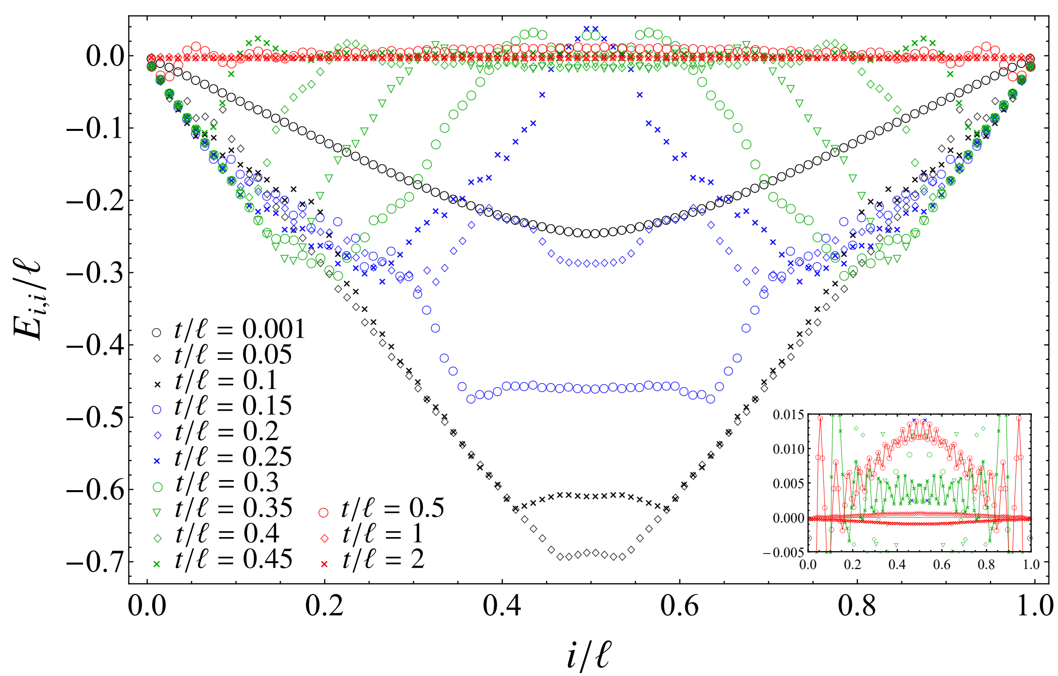
[Di Giulio, Arias, E.T., (2019)]

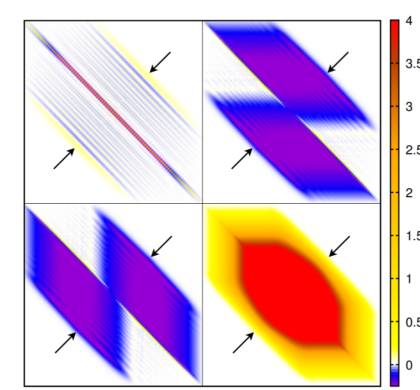


EH matrix in HC: main diagonals of the blocks

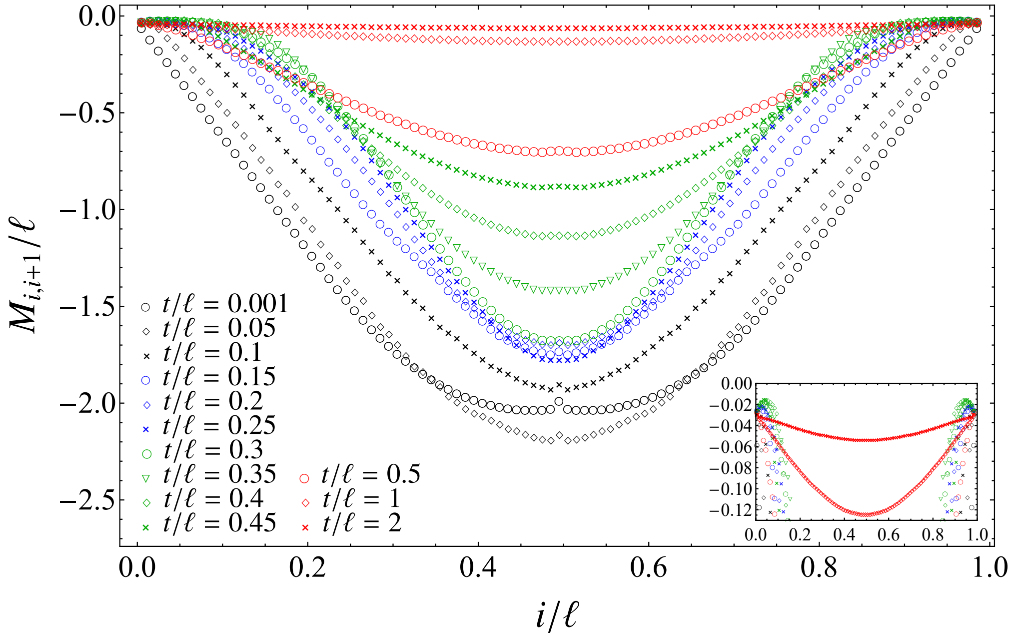


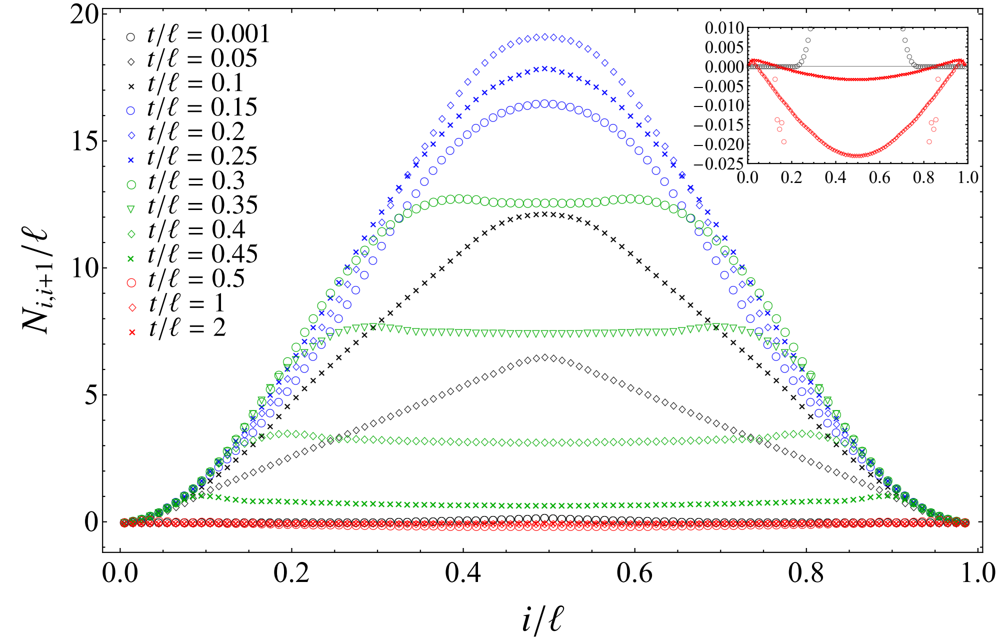


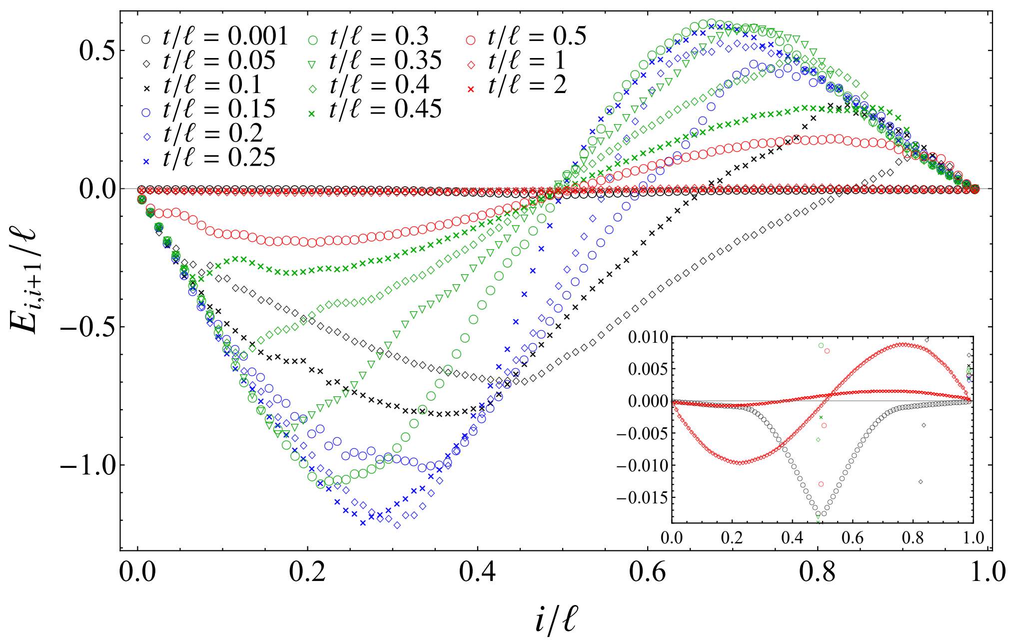


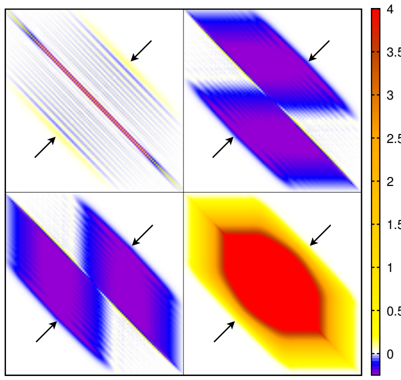


EH matrix in HC: first diagonals of the blocks

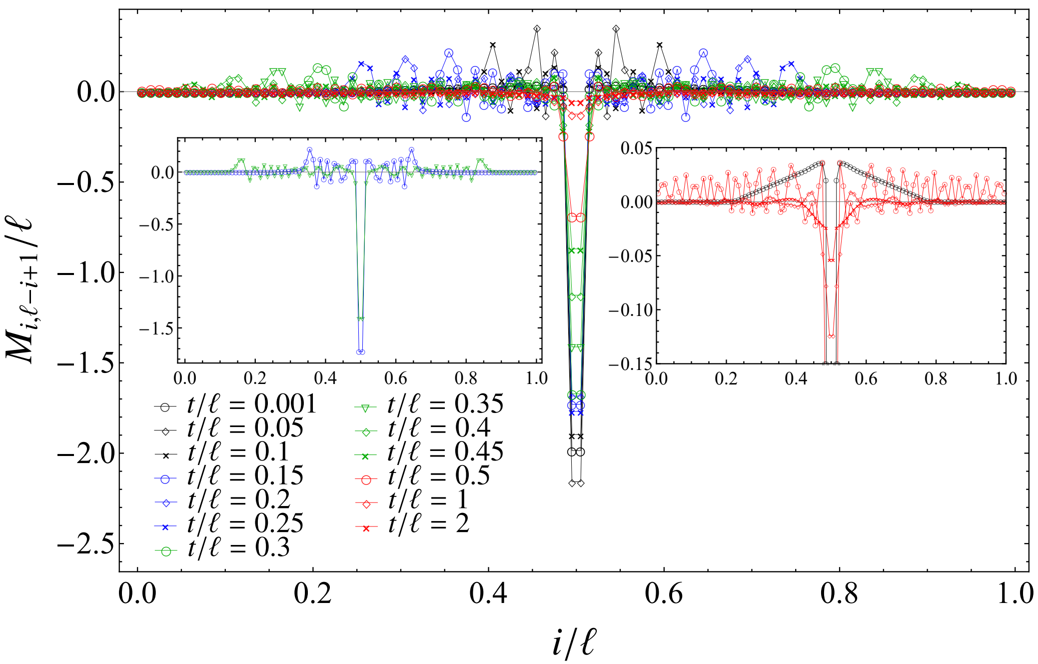


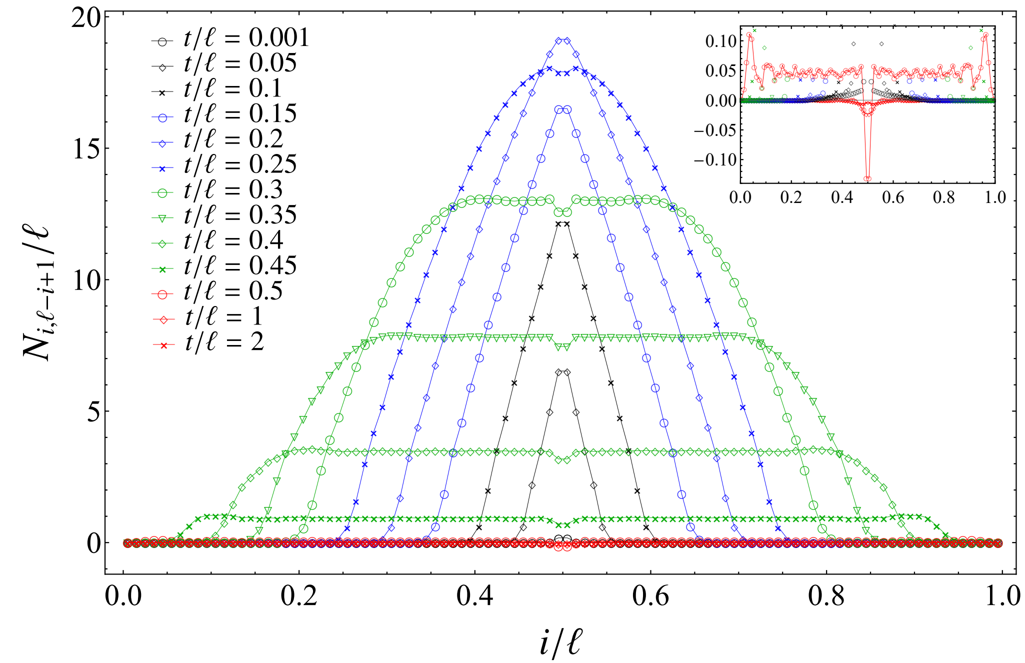


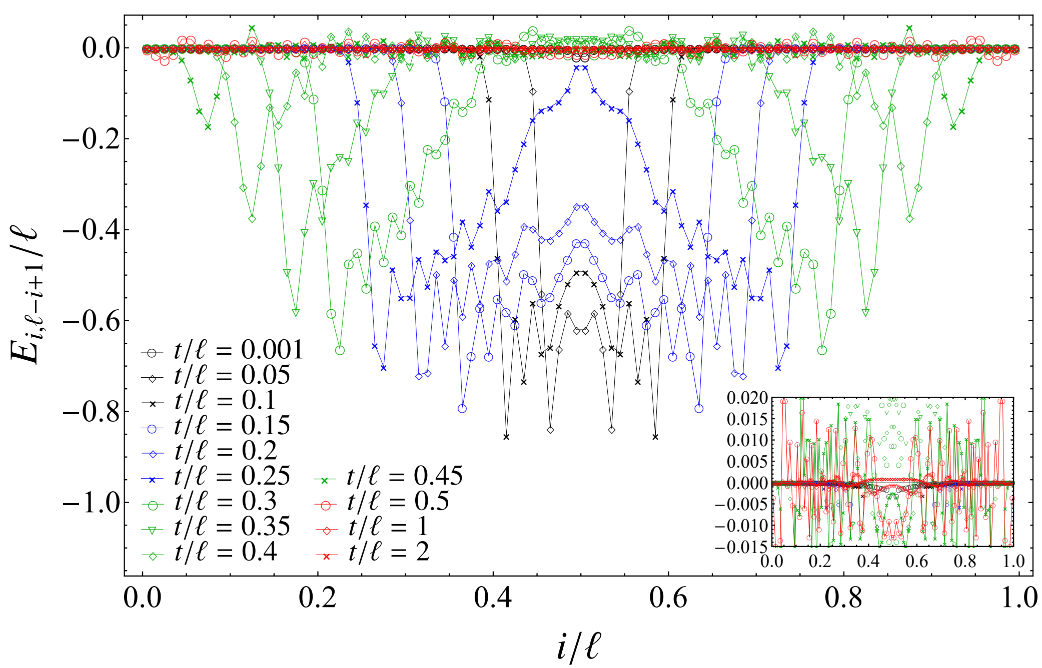


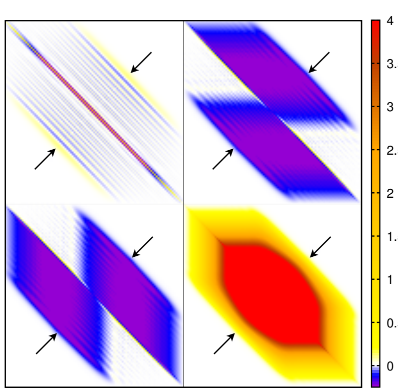


EH matrix in HC: antidiagonals of the blocks

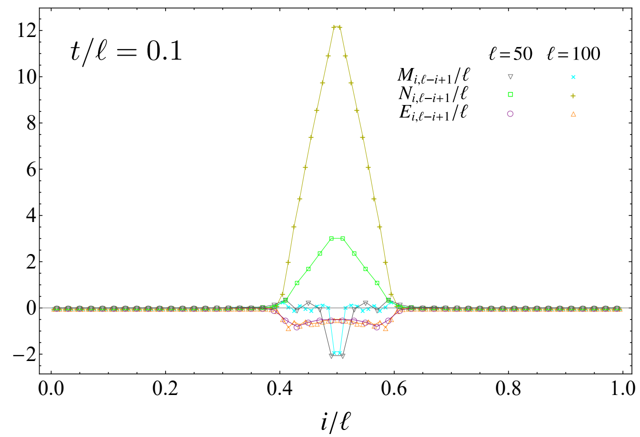


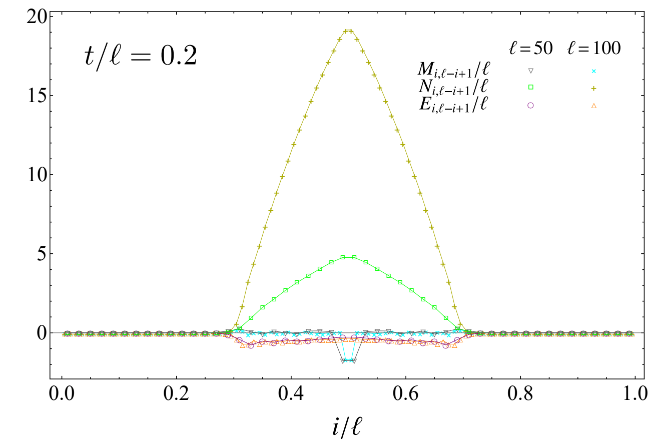


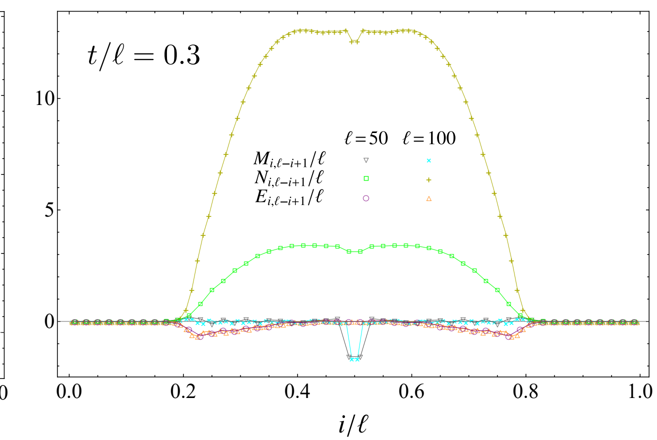


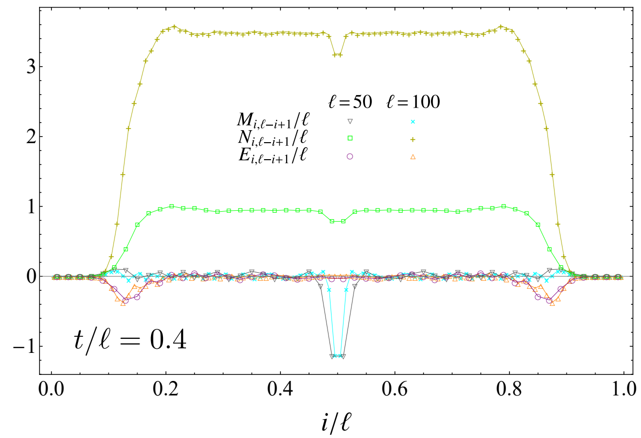


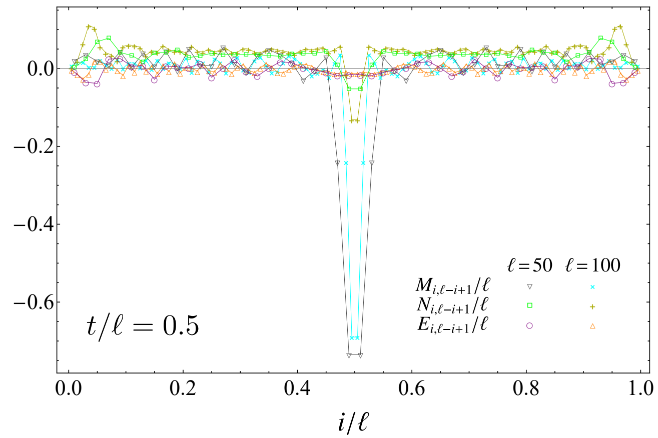
EH matrix in HC: antidiagonals of the blocks

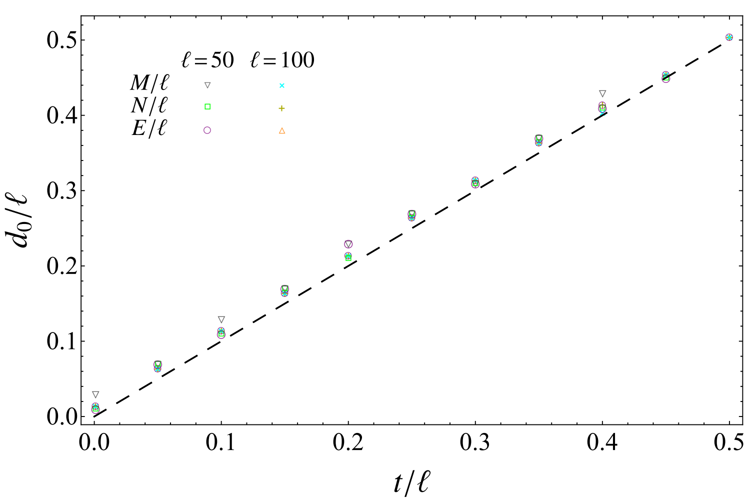


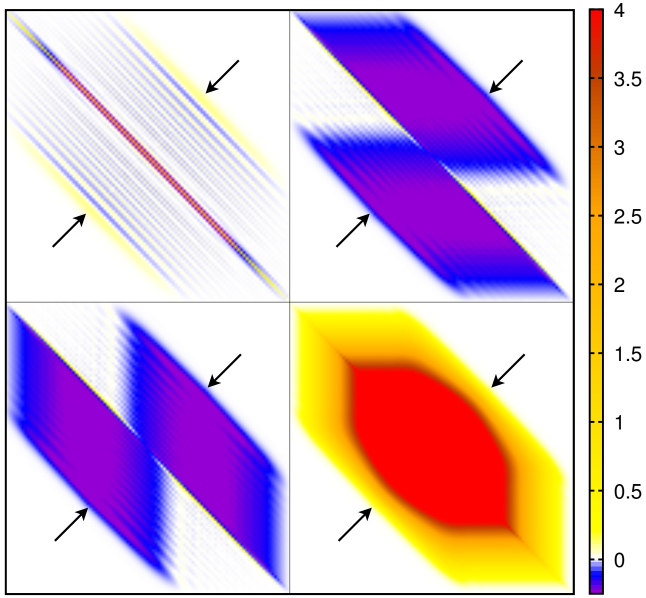




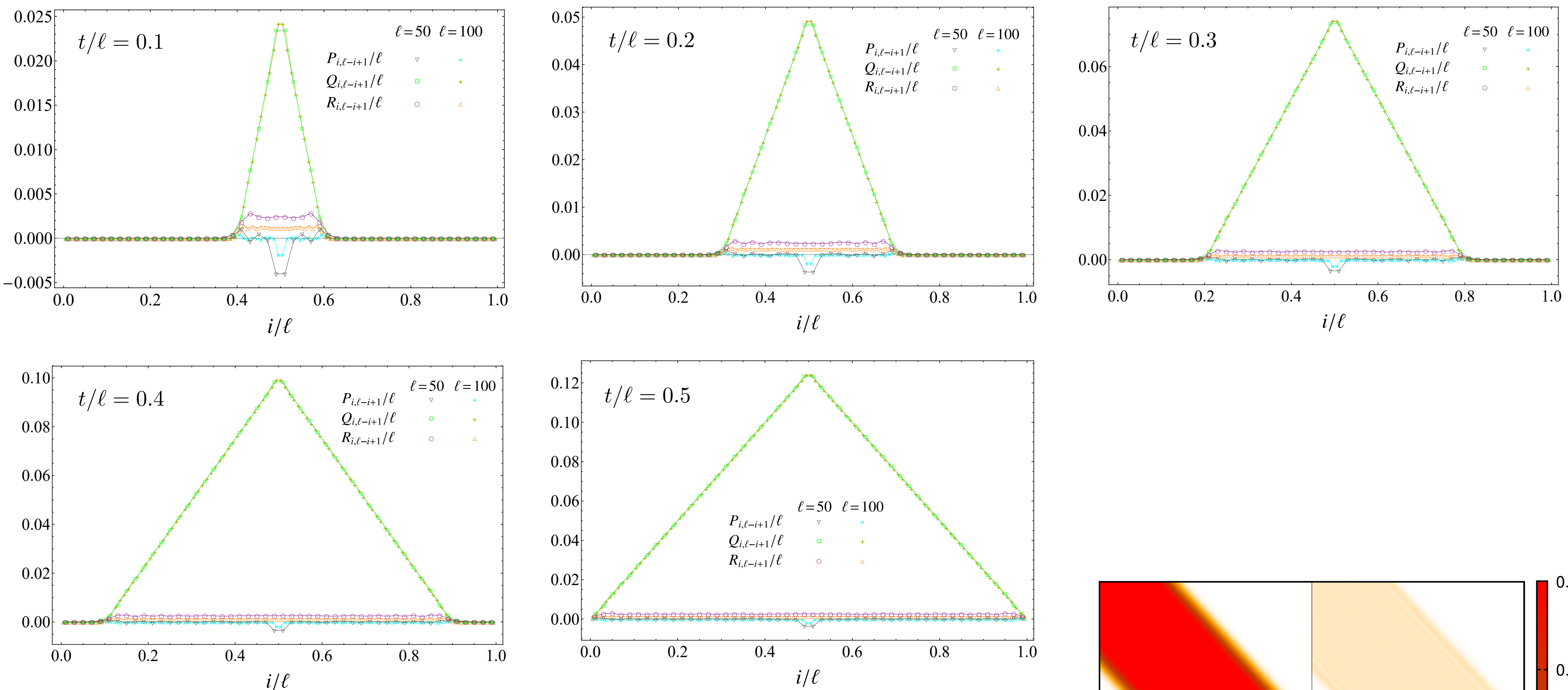


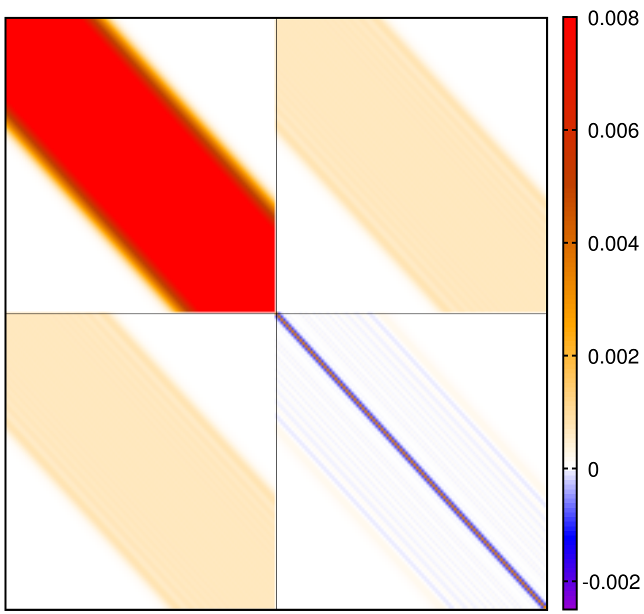






Reduced Covariance Matrix: antidiagonals of the blocks

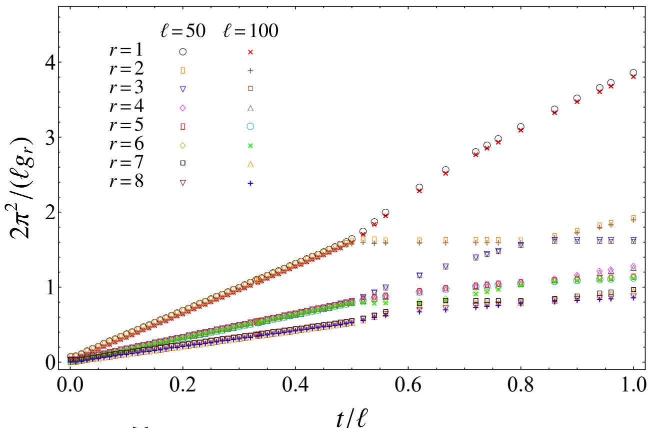


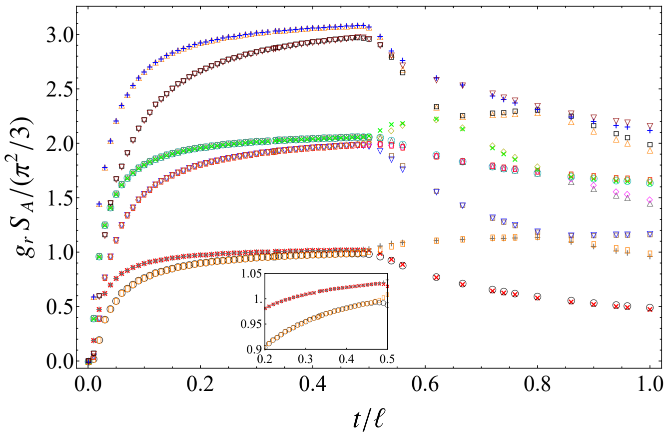


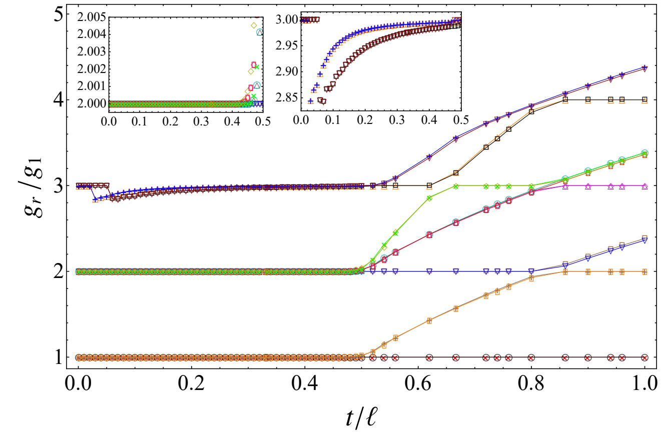
- The blocks of $\gamma_A(t)$ are constant along the diagonals

Gaps in the entanglement spectrum in the HC

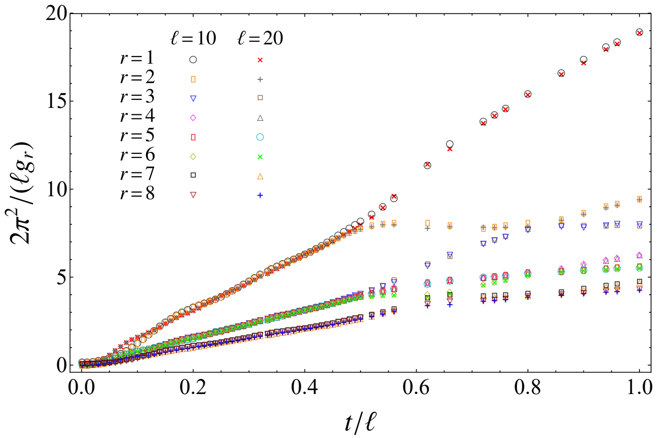
$\omega_0 = 1$

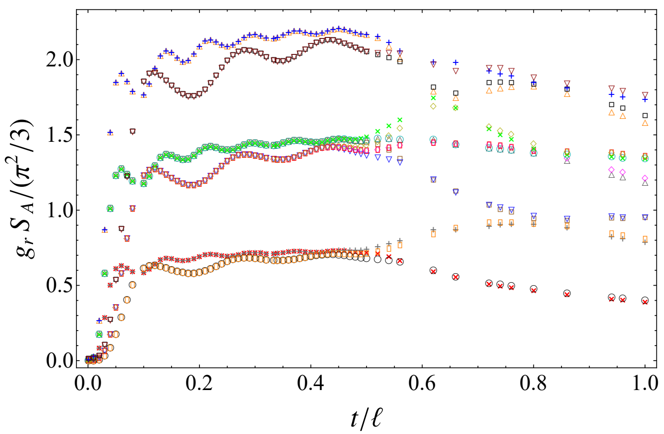


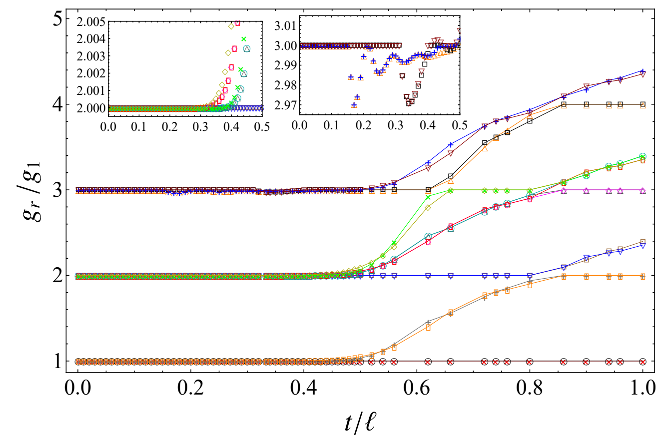




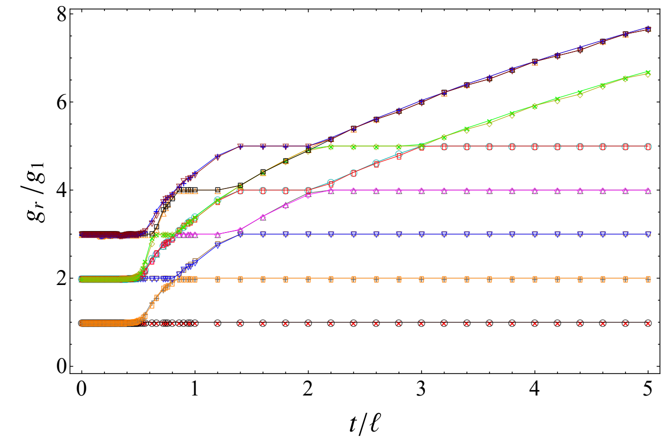
$\omega_0 = 5$







- Linear growths of $1/g_r$ for $t/\ell < 1/2$ that depend on the CFT spectrum



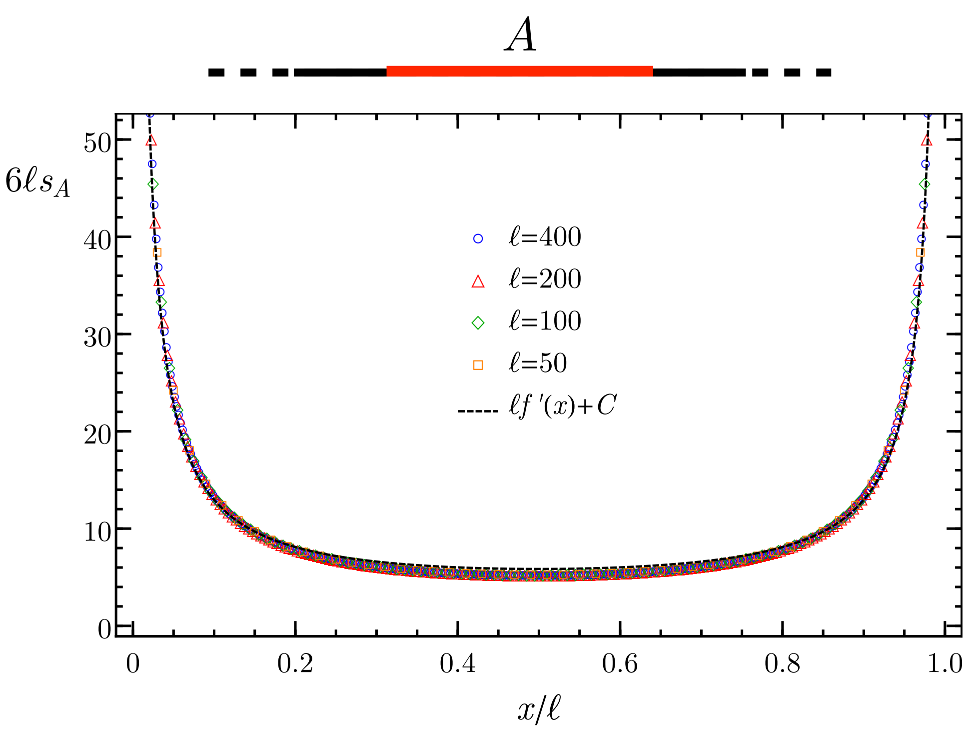
Contour for EE in the HC

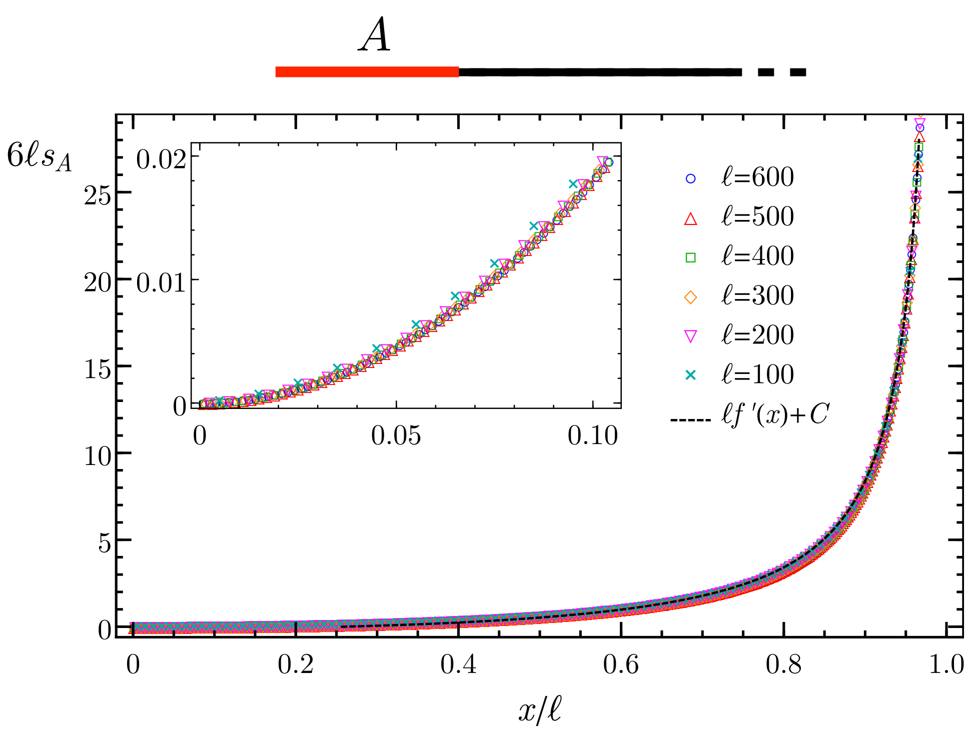
- Spatial structure of entanglement [Botero, Reznik, (2004)] [Chen, Vidal, (2014)]

$$S_A = \sum_{i \in A} s_A(i) \qquad s_A(i) \geqslant 0$$

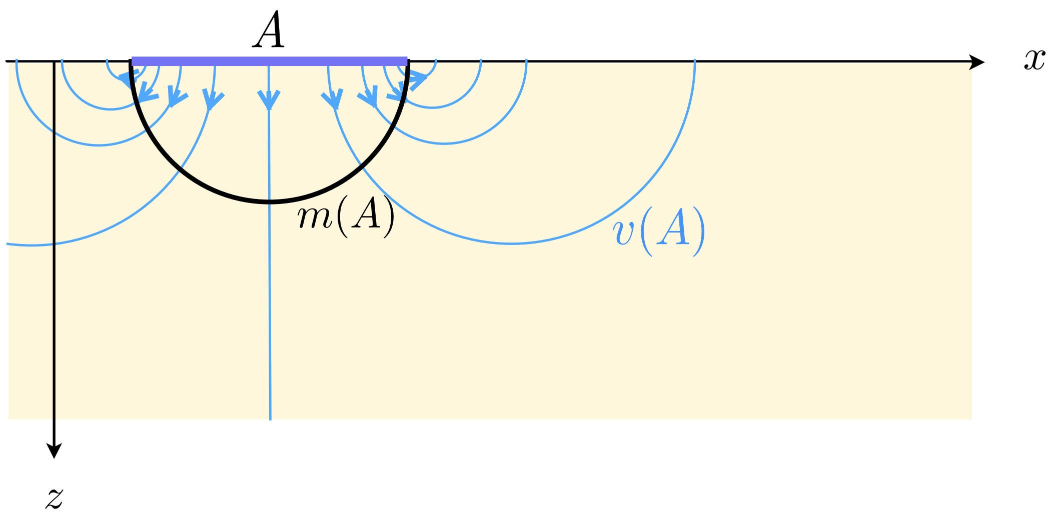
  - Other constraints are imposed on the contour function
  - A list of properties that provides $s_A(i)$ uniquely is not known

- In HC we use $s_A(i)$ constructed in [Coser, De Nobili, E.T., (2017)] that has been compared to the weight function in the EH from CFT





Contours for the holographic entanglement entropy?



E.g.: AdS$_3$ $\quad \frac{1}{z^2}\left(-dt^2 + dz^2 + dx^2 \right)$

- The Ryu-Takayanagi prescription can be reformulated in terms of flows through A [Headrick, Freedman, (2016)] [Headrick, Hubeny, (2017)]

$$\begin{cases} \nabla_\mu v^\mu = 0 \\ |v| \leqslant \frac{1}{4G_{\rm N}} \end{cases} \xrightarrow{\text{max-cut min-flow theorem}} \boxed{S_A = \frac{\min\left[\text{area}(m(A))\right]}{4G_N} = \max \int_A v}$$

- A contour for the holographic entanglement entropy could be identified with a flow maximising the flux. [E.T., (2018)]

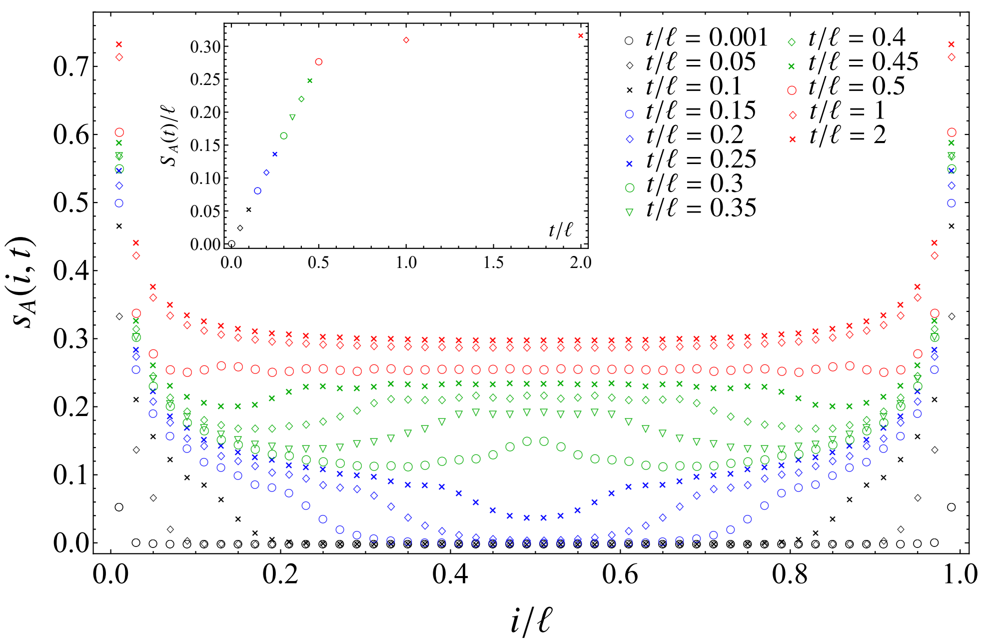
Contour for EE in the HC after a global quench

- Contour function after quantum quenches in fermionic chains
  [Chen, Vidal, (2014)]

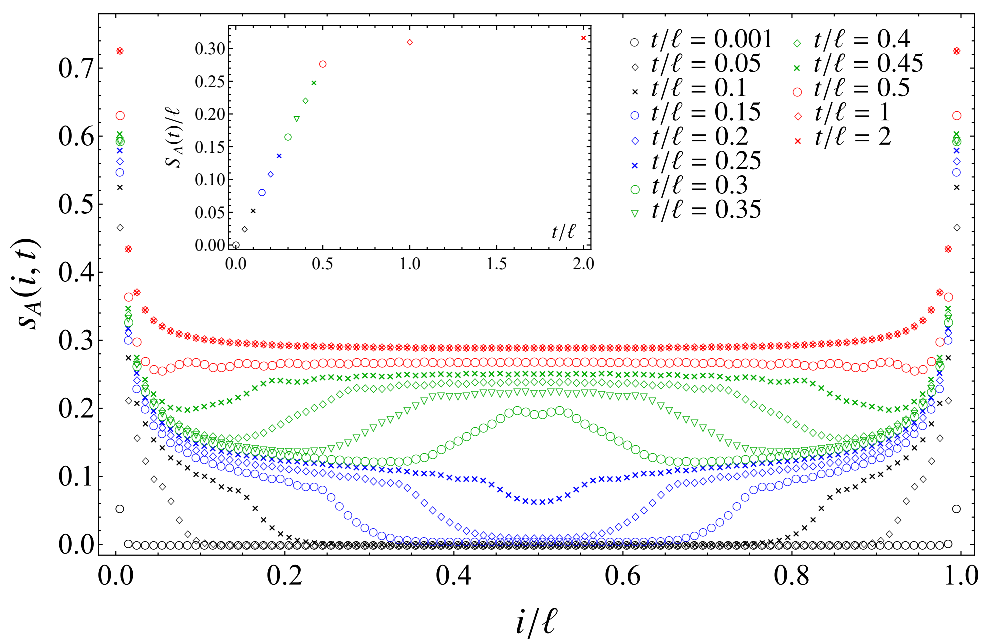
- Contour function after a local quench through holographic results
  [Kudler-Flam, MacCormack, Ryu, (2019)]

- Contour function in the harmonic chain after a quench $\omega_0 = 1 \to \omega = 0$
  [Di Giulio, Arias, E.T., (2019)]

$\ell = 50$

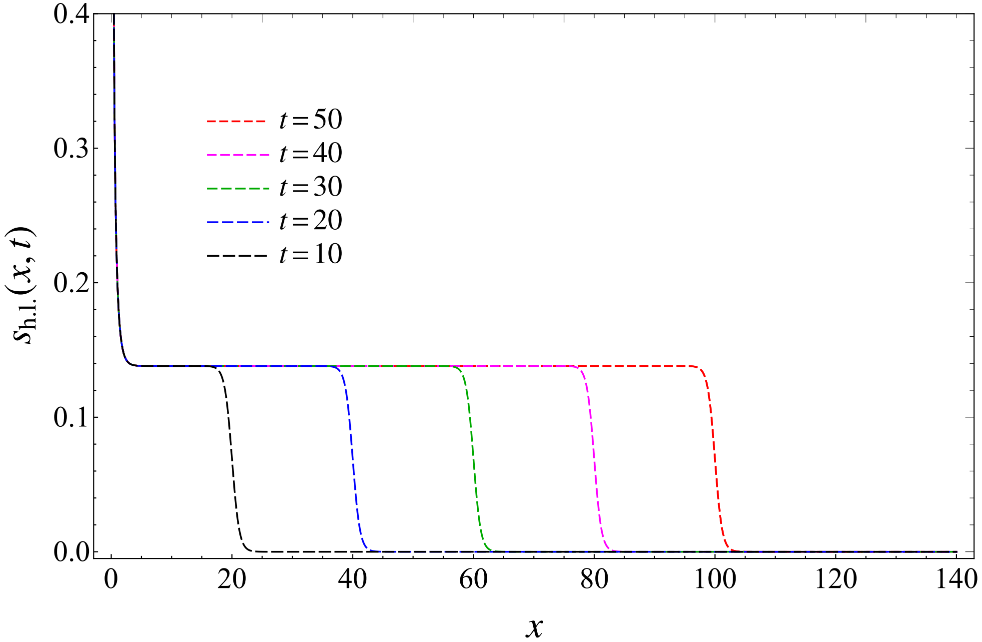


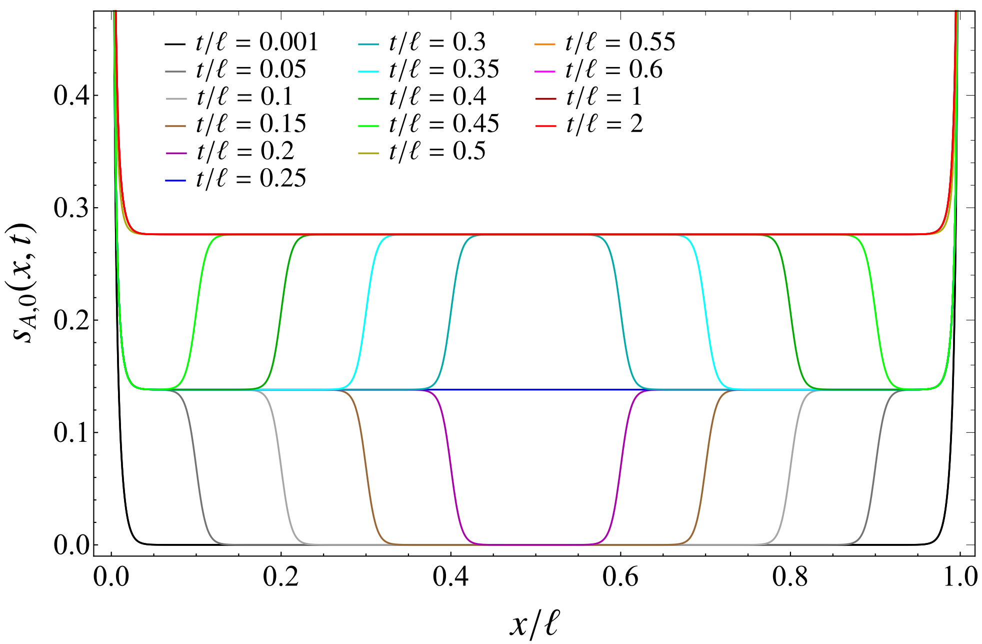
$\ell = 100$



- Same qualitative behaviour observed for fermions [Chen, Vidal, (2014)]

Insights from CFT: contour for the EE





- A is a half-line [Cardy, E.T., (2016)]

$$s_{\text{h.l.}}(x,t) = \frac{c}{6}\,\mathcal{F}_{\text{h.l.}}(x,t)$$

$$\mathcal{F}_{\text{h.l.}}(x,t) \equiv \frac{2\pi\,[\cosh(2\pi t/\tau_0)]^2 \coth(\pi x/\tau_0)}{\tau_0\,[\cosh(4\pi t/\tau_0) + \cosh(2\pi x/\tau_0)]}$$

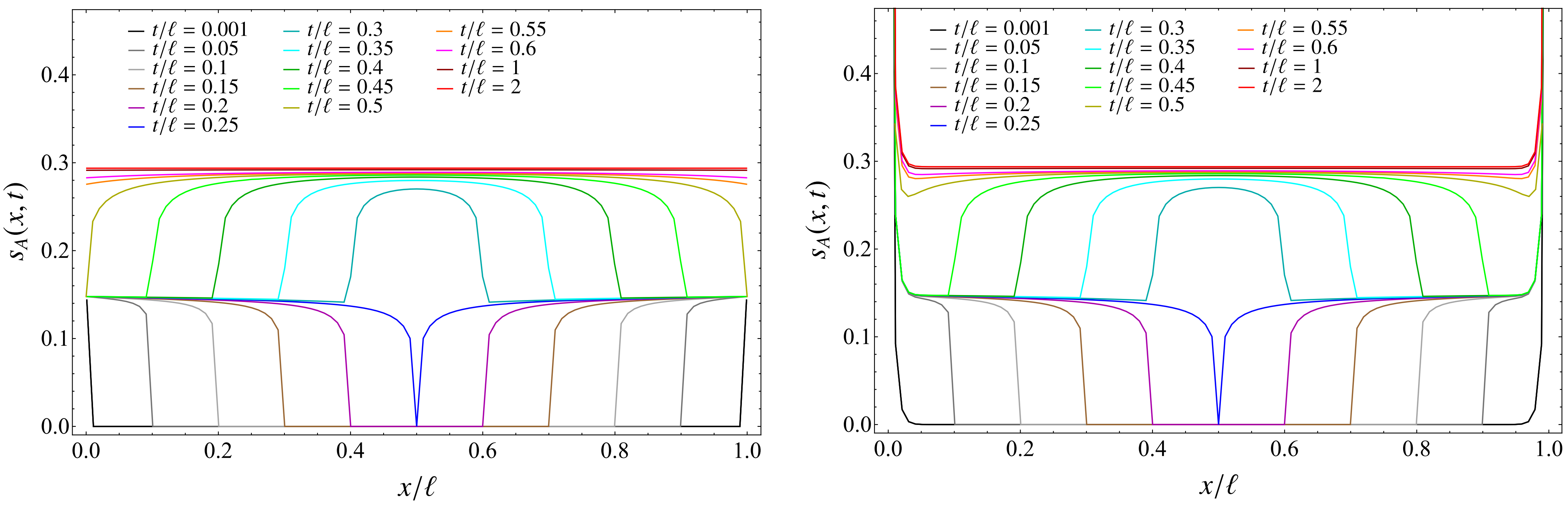
- Naive guess when A is an interval [Di Giulio, Arias, E.T., (2019)]

$$s_{A,0}(x,t) \equiv s_{\text{h.l.}}(x,t) + s_{\text{h.l.}}(\ell - x,t)$$

- Integrals of the contour function $(x_1,x_2) \subseteq A$

$$\mathcal{S}_{A,0}(x_1,x_2;t) \equiv \int_{x_1}^{x_2} s_{A,0}(x,t)\,dx$$

Contour for EE in the HC from quasi-particle picture



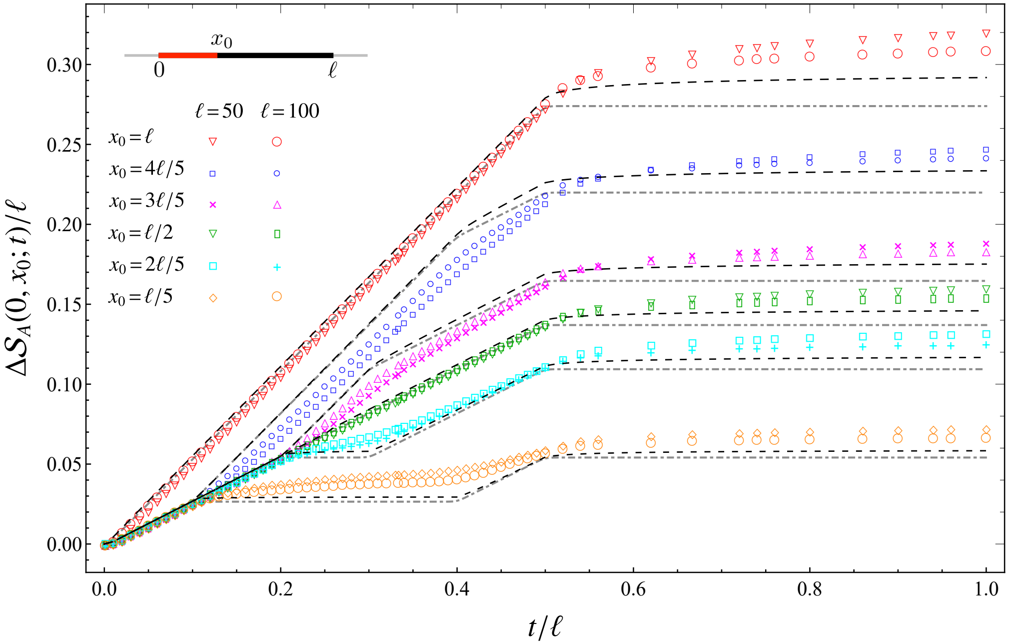
- By exploiting the quasi-particle picture of [Calabrese, Cardy, (2005)] we find [Di Giulio, Arias, E.T., (2019)]

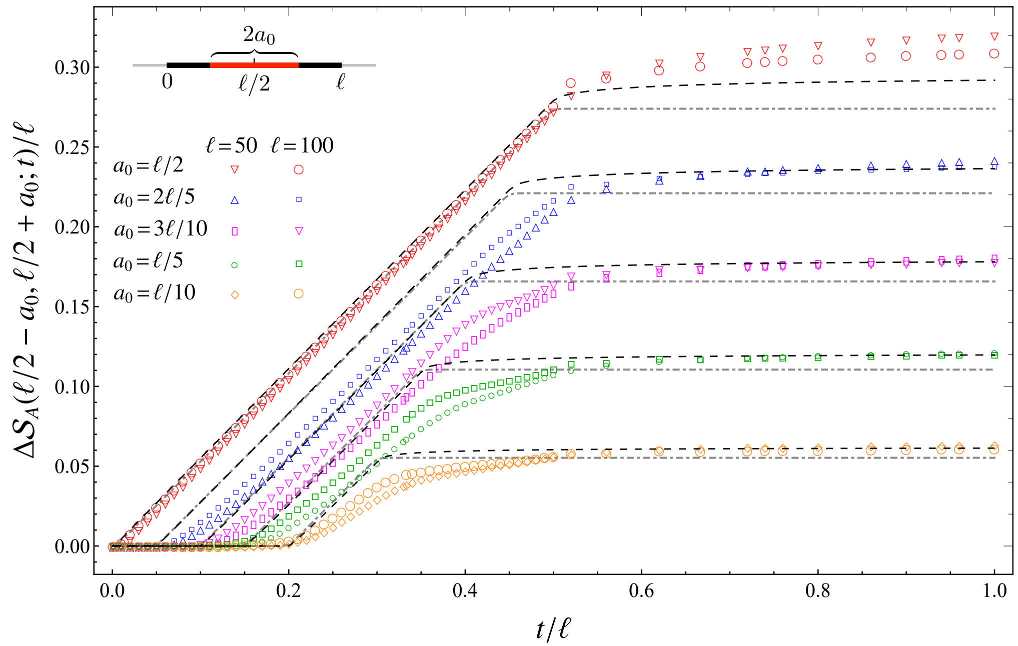
$$s_A(x,t) = \frac{1}{2}\left[\int_{x<2|v_p|t<\ell} \tilde{s}(p)\,dp + \int_{\ell - x<2|v_p|t<\ell} \tilde{s}(p)\,dp\right] + \int_{2|v_p|t>\ell} \tilde{s}(p)\,dp + f_0(x)$$

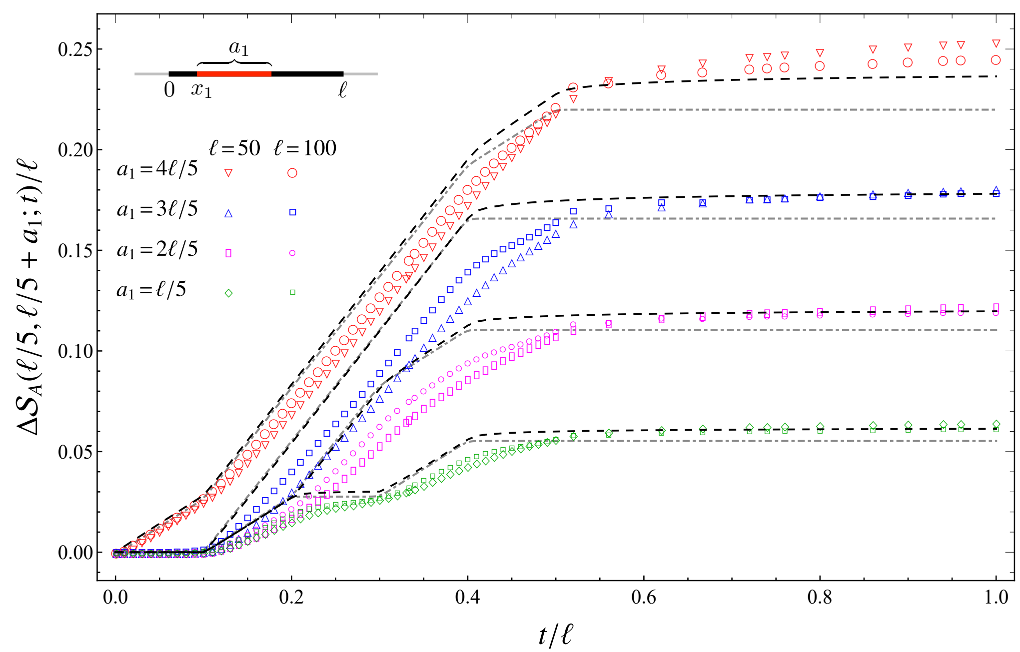
For the harmonic chain we used $\tilde{s}(p)$ obtained in [Alba, Calabrese, (2018)]

- Also $\mathcal{S}_A(x_1, x_2)$ can be written in a similar form

Integrals of the contour for EE in the HC







EH matrix & contour for the EE in the FFC

- The EH matrix T can be written in terms of the correlation matrix C_A restricted to the interval [Peschel, (2003)]

$$\widehat{K}_A = \sum_{i,j=1}^{\ell} T_{i,j}\, \hat{c}_i^\dagger\, \hat{c}_j$$

$$T^{\mathrm{t}} = \log\big(C_A(t)^{-1} - \mathbf{1}\big)$$

$$T_{i,j} = \sum_{k=1}^{\ell} \eta_k\, \widetilde{U}_{k,i}^*\, \widetilde{U}_{k,j}$$

We consider the global quench of [Eisler, Peschel, (2007)]

- The matrix T is complex after the quench

- The contour function $s_A(i)$ depends also on $\widetilde{U}_{k,i}$ [Chen, Vidal, (2014)]

$$S_A = \sum_{k=1}^{\ell} s(\zeta_k) \qquad \eta_k = \log(1/\zeta_k - 1)$$

$$\begin{cases} p_k(i) = \big|\widetilde{U}_{k,i}\big|^2 \\ \displaystyle\sum_{i=1}^{\ell} p_k(i) = 1 \end{cases}$$

$$S_A = \sum_{i\,\in\, A} s_A(i)$$

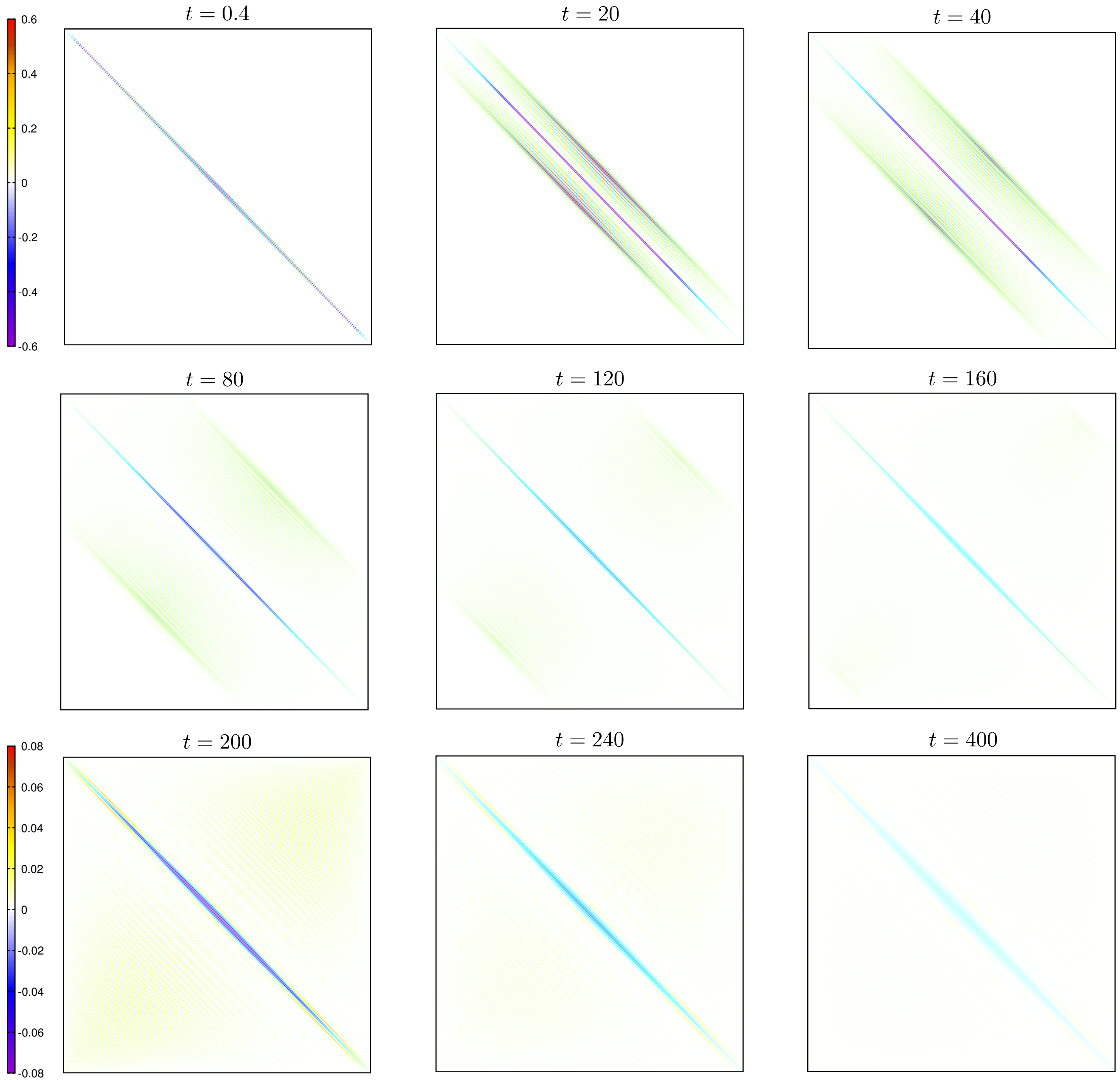
$$s_A(i) = \sum_{k=1}^{\ell} p_k(i)\, s(\zeta_k)$$

$$\mathcal{S}_A(i_1, i_2) = \sum_{i=i_1}^{i_2} s_A^{(n)}(i) \qquad i_1, i_2 \in A$$

Evolution of the EH matrix (real part) in the FFC

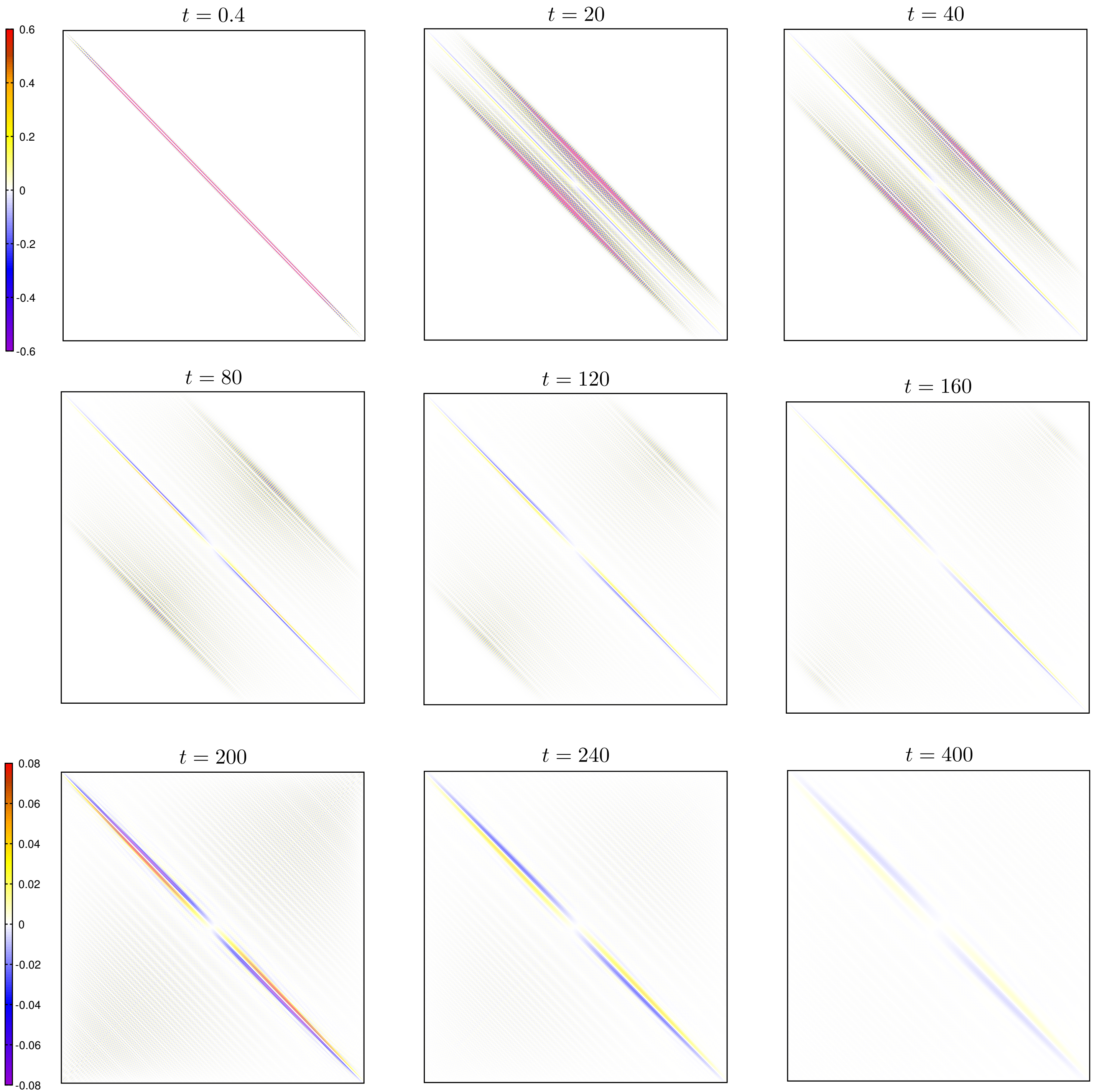
$\ell = 400$

[Di Giulio, Arias, E.T., (2019)]

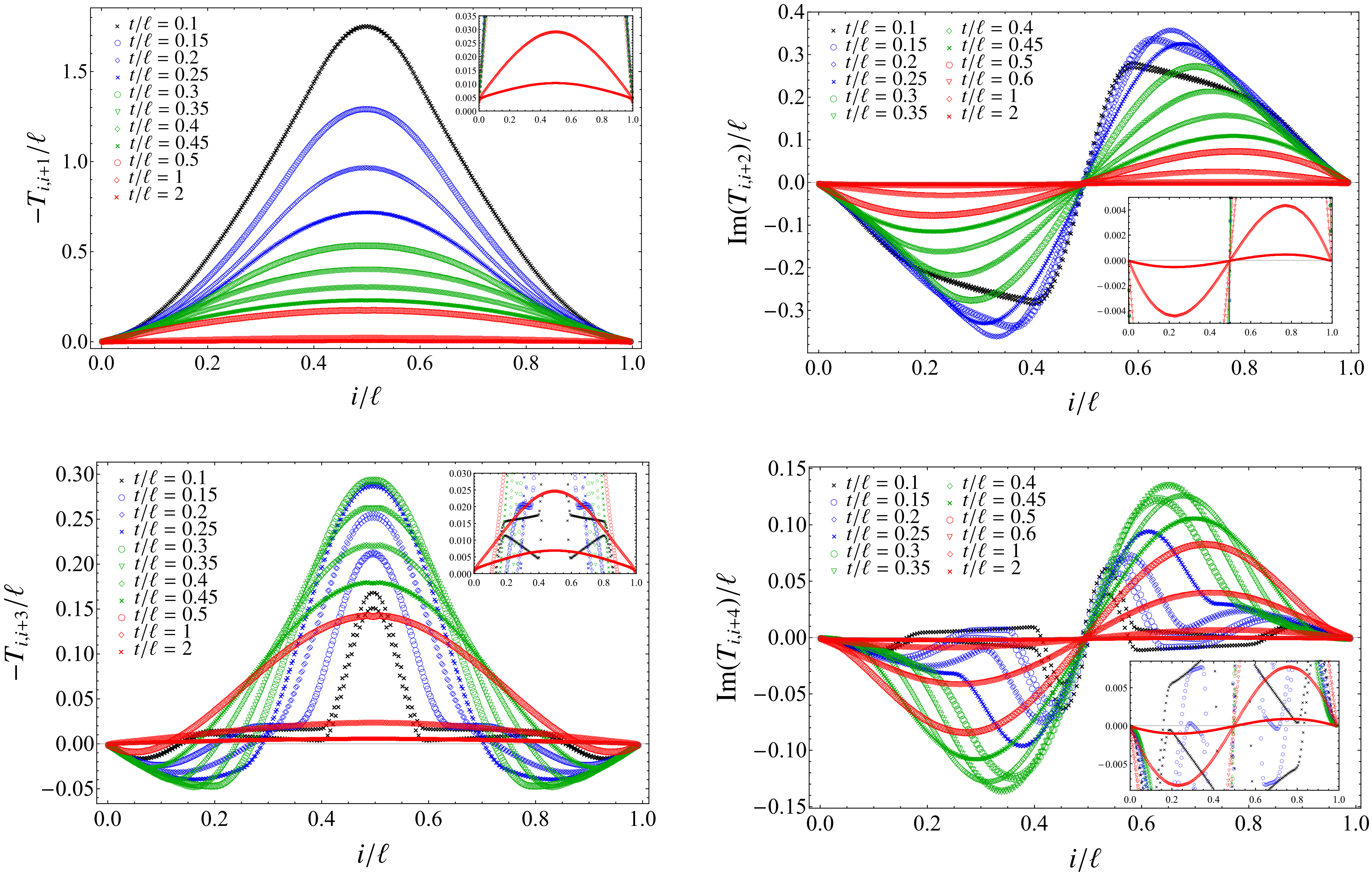


Evolution of the EH matrix (imaginary part) in the FFC

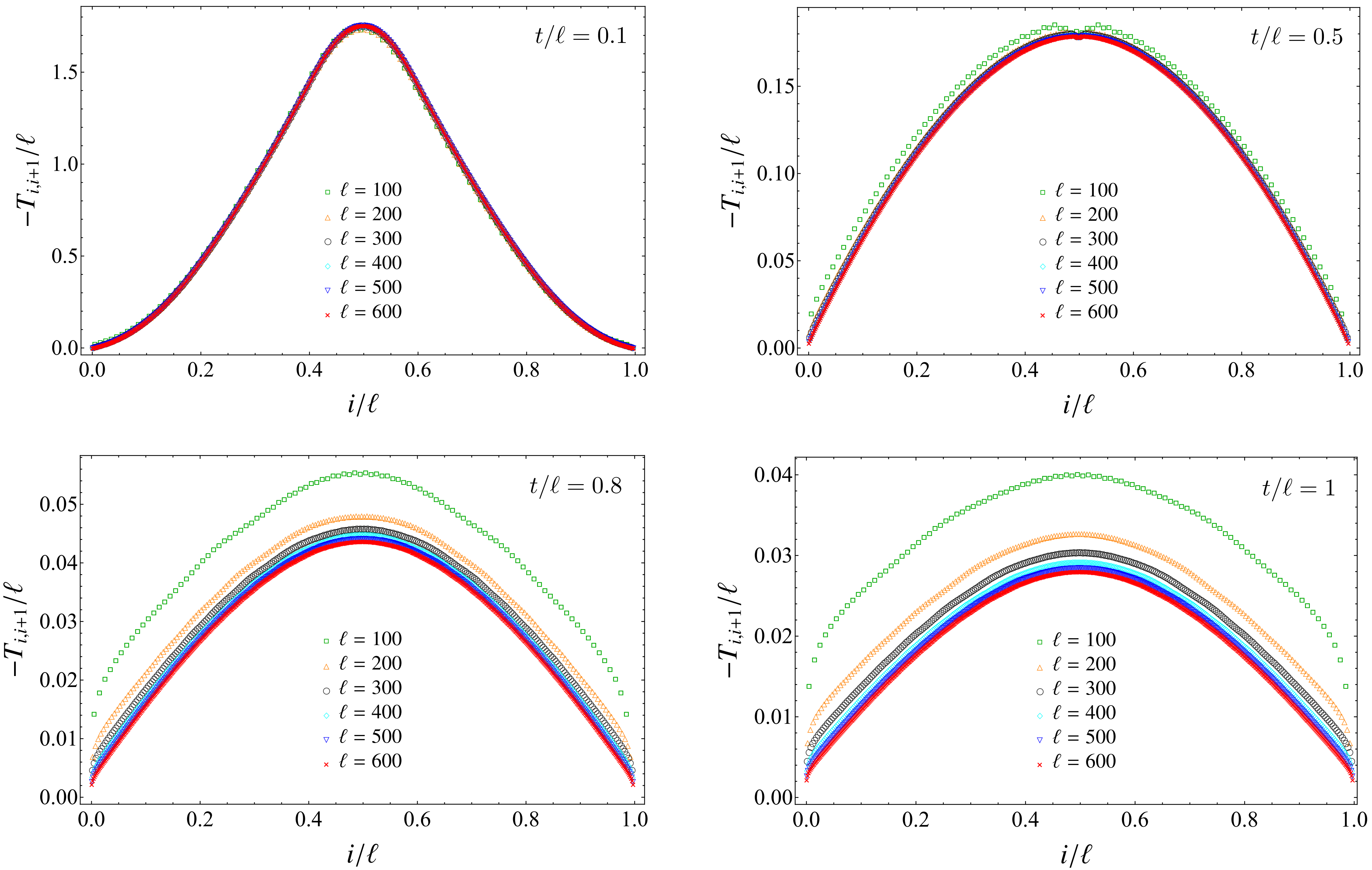
$\ell = 400$



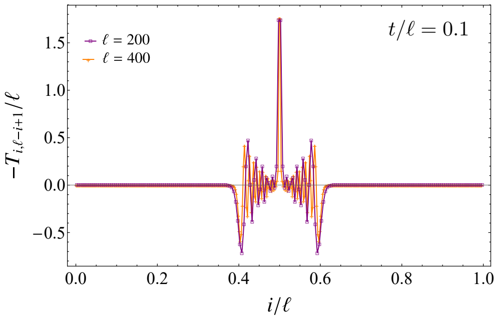
EH matrix in FFC: first diagonals

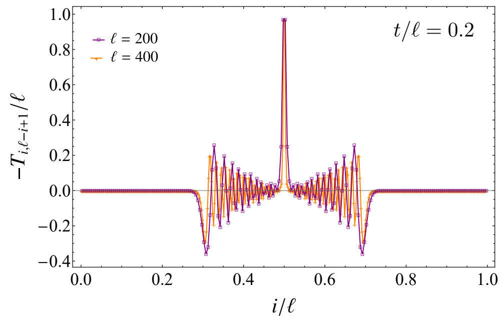


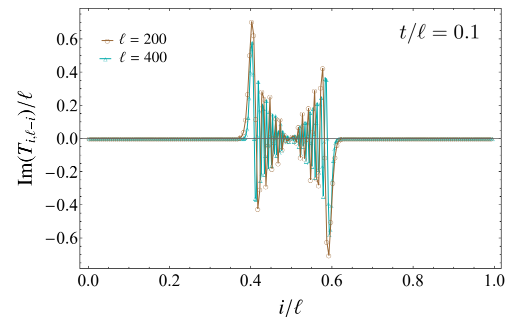
EH matrix in FFC: first diagonal

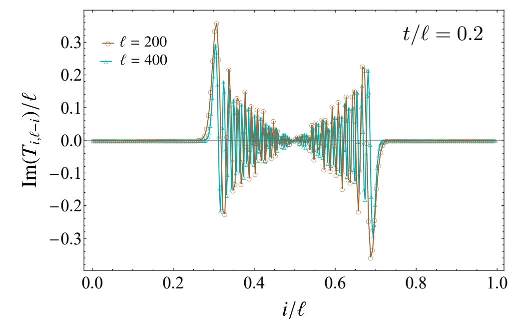


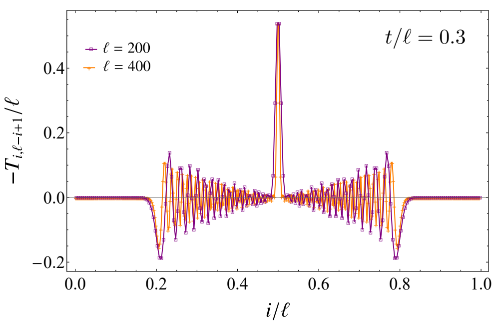
EH matrix in FFC: antidiagonals

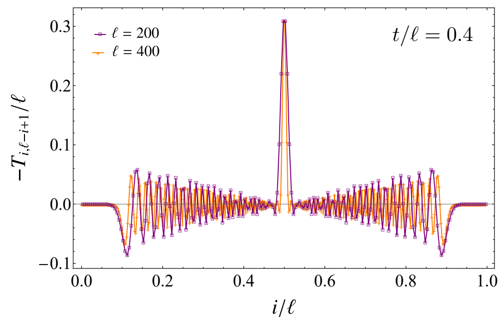


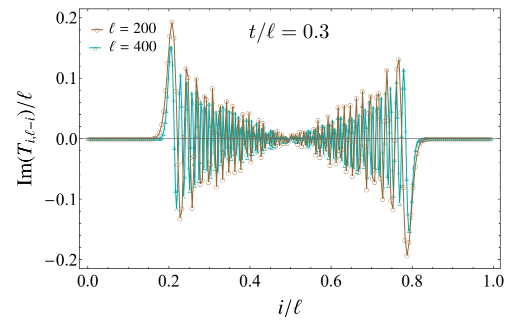


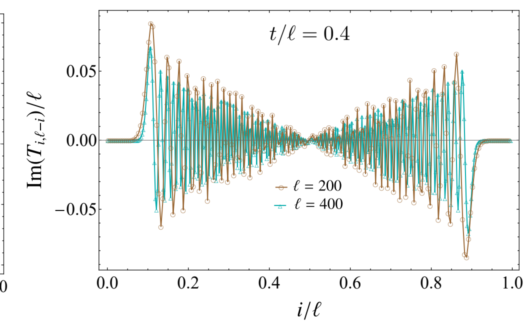


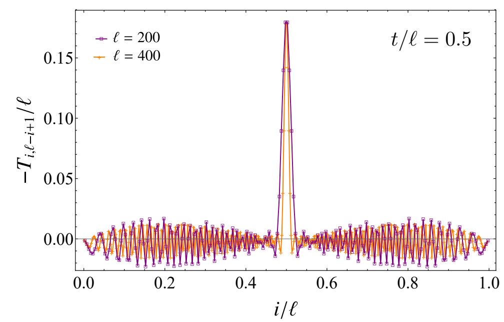


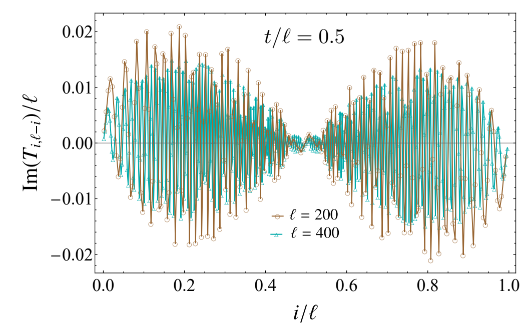


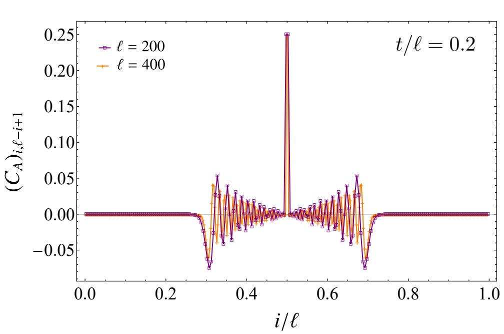


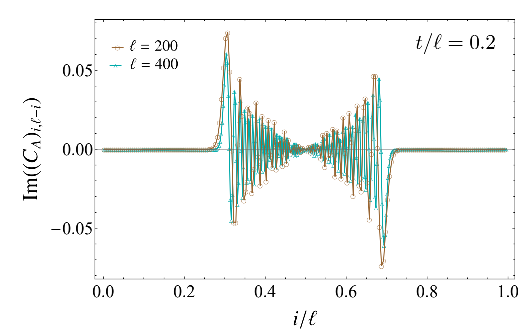






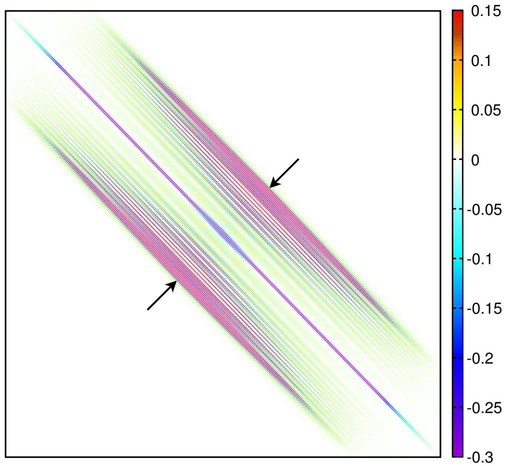




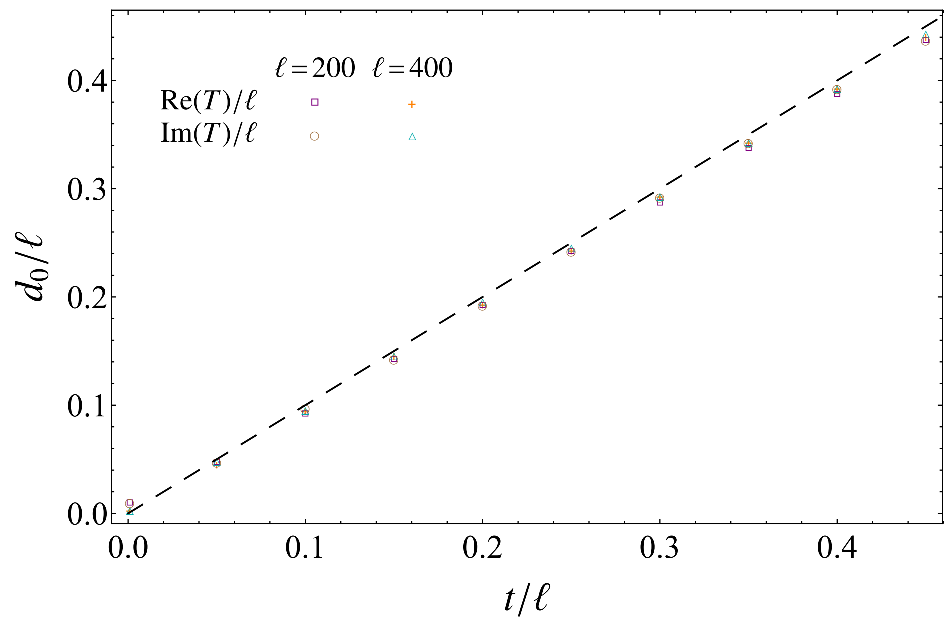


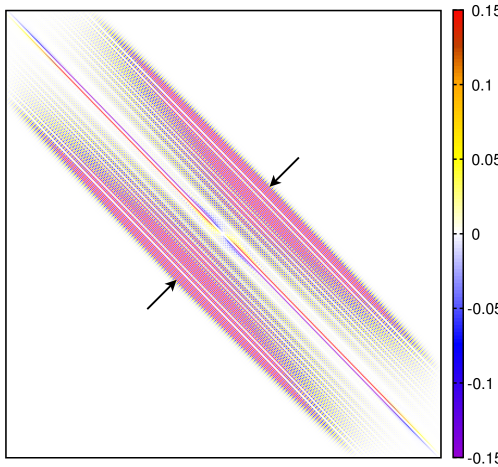
EH matrix in FFC: antidiagonals

- Linear growth of the antidiagonals of $\mathrm{Re}(T)$ and $\mathrm{Im}(T)$



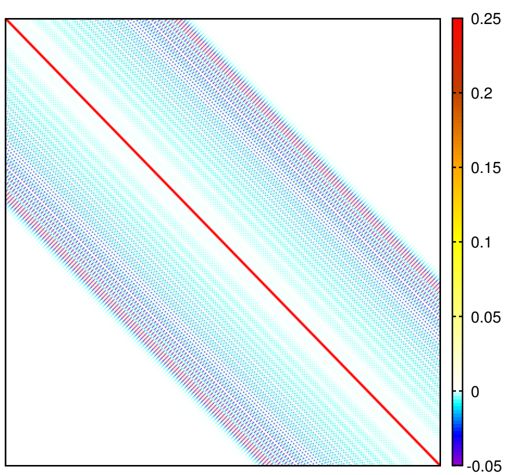
$\mathrm{Re}(T)$





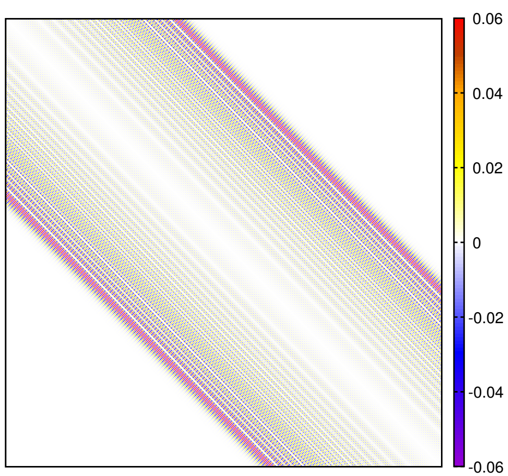
$\mathrm{Im}(T)$

- Same linear growth observed for antidiagonals of $C_A(t)$

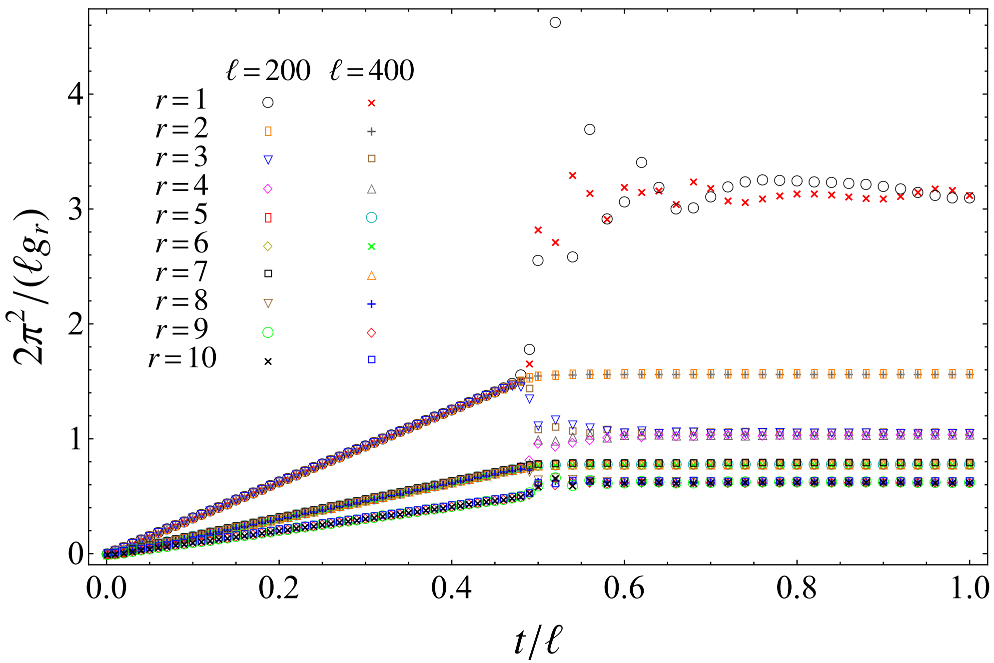


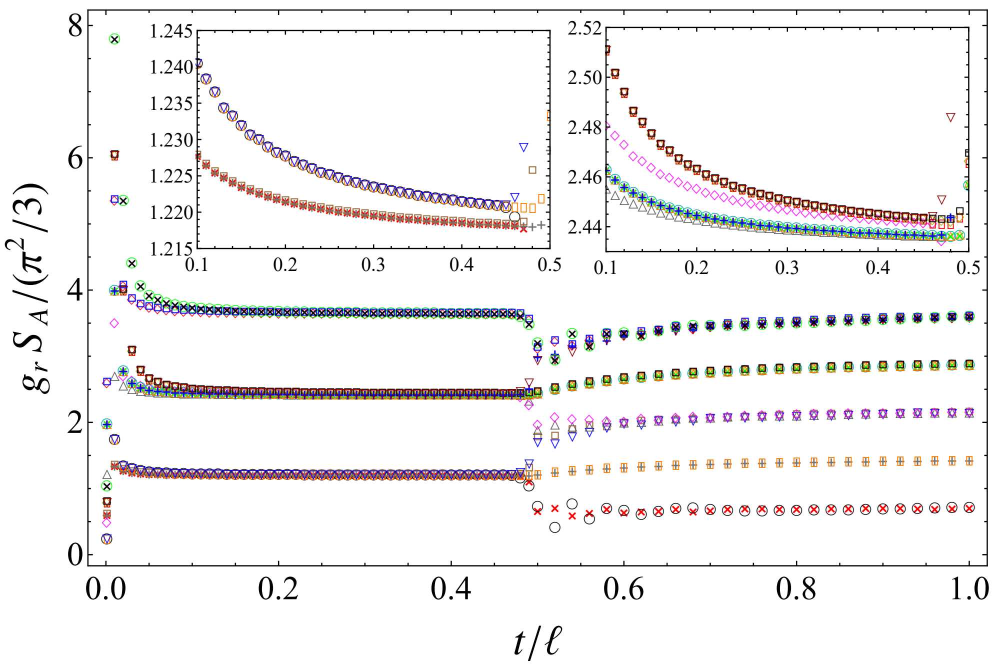
$\mathrm{Re}(C_A)$

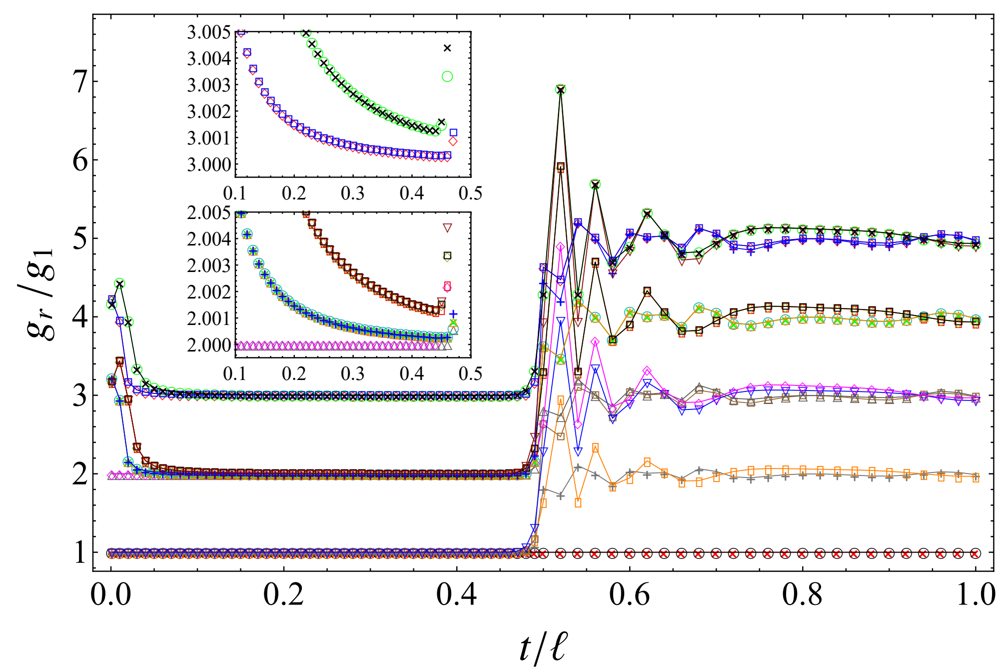
$\mathrm{Im}(C_A)$



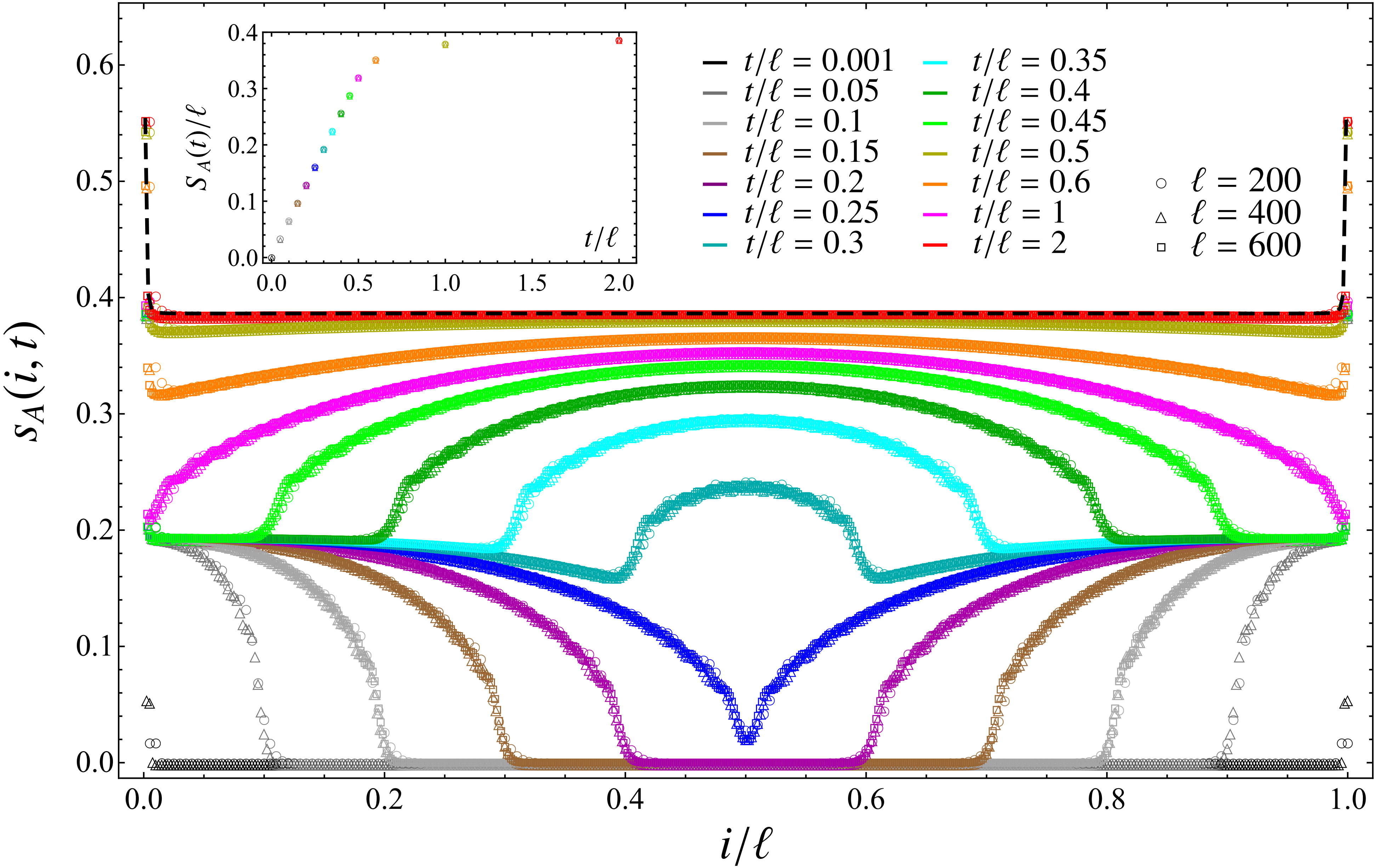
Gaps in the entanglement spectrum in the FFC



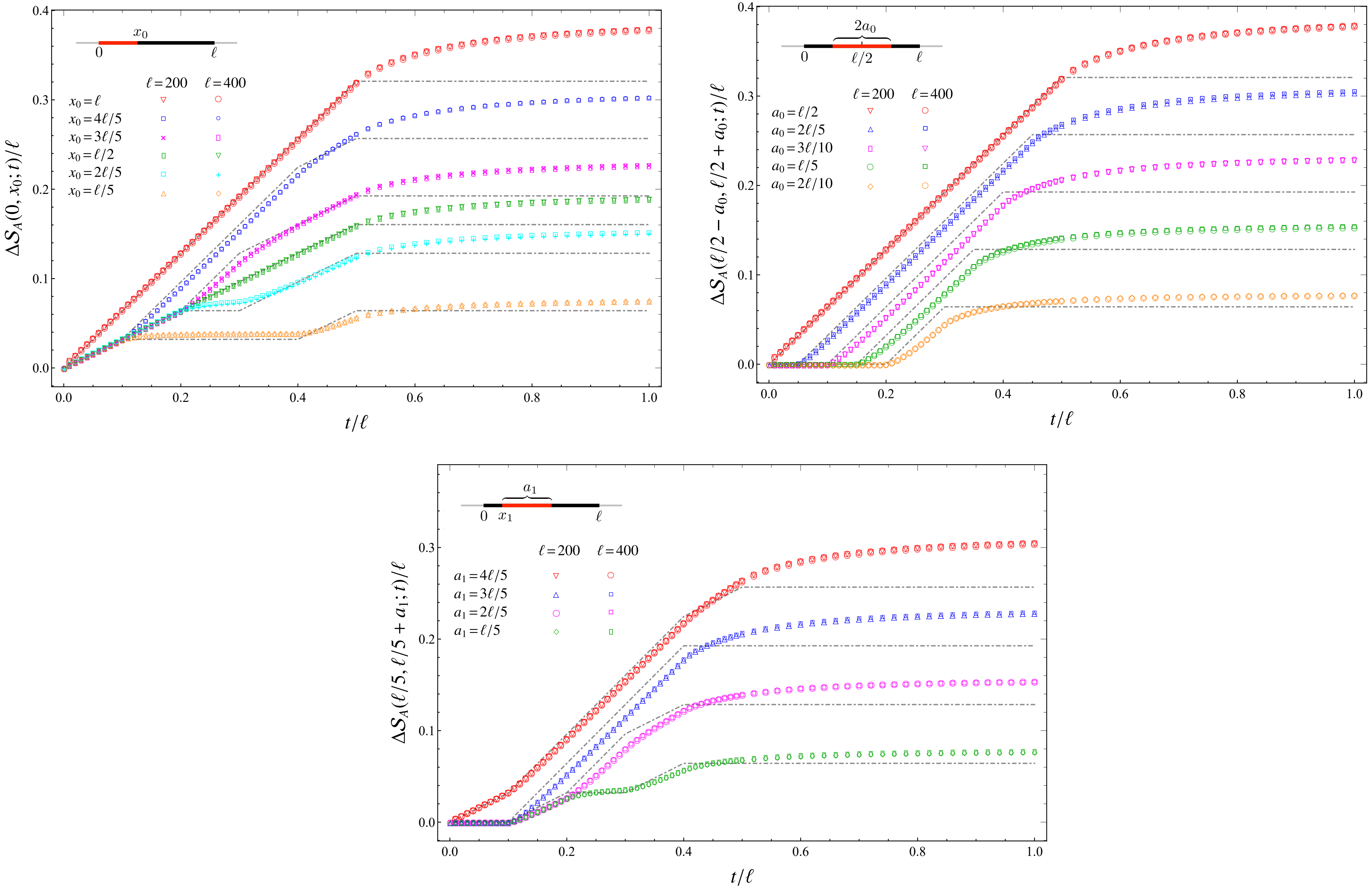




Contour for the EE in the FFC



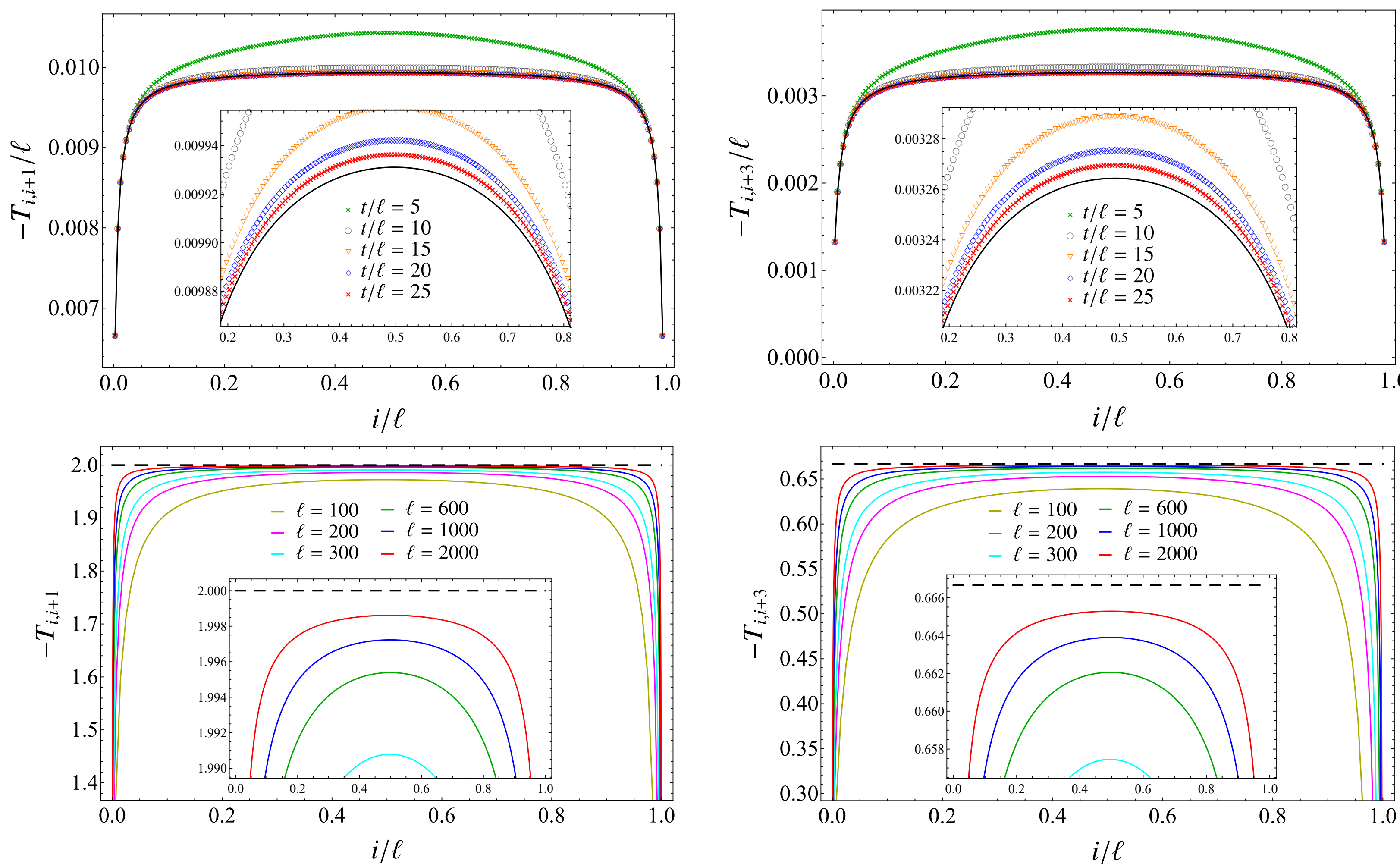
Integrals of the contour for EE in the HC



FFC: EH matrix in the long time regime

- The results of [Eisler, Peschel, (2007)] allow to write an analytic expression for the EH matrix T and the $s_A(i)$ as $t \to \infty$ [Di Giulio, Arias, E.T., (2019)]

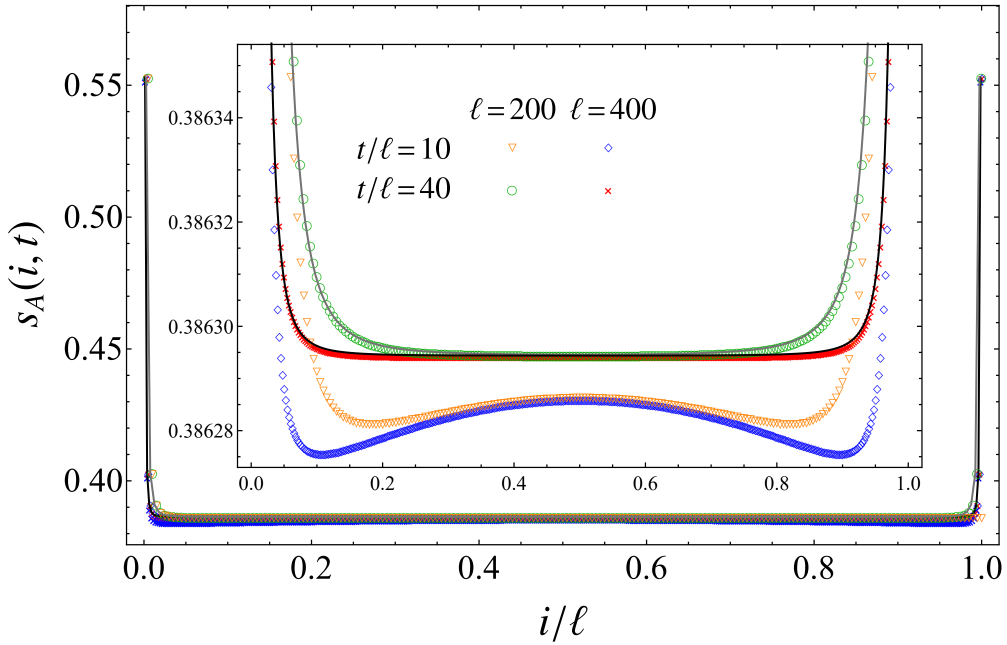
$$T_{i,j} = \frac{4}{\ell+1} \sum_{k=1}^{\ell} \log\big[\tan(\theta_k/2)\big] \, \sin(i\theta_k) \, \sin(j\theta_k) \qquad \theta_k \equiv \frac{\pi\, k}{\ell+1}$$

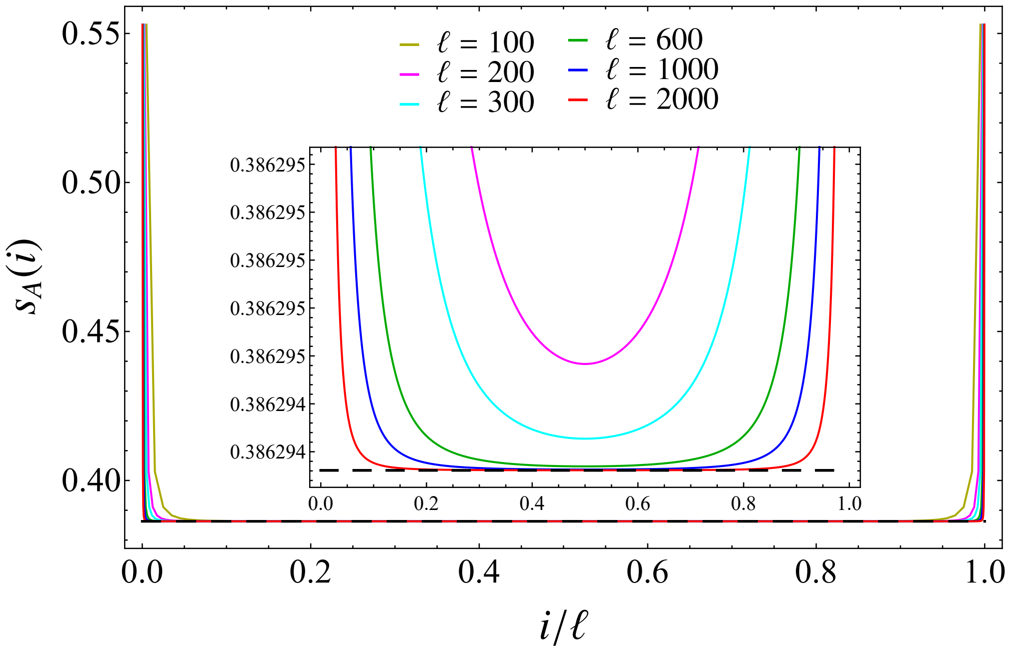


FCC: Contour for EE in the long time regime

- The contour function for EE as $t \to \infty$ becomes

$$s_A^{(n)}(i) = \sum_{k=1}^{\ell} s_n(\zeta_k)\, u_k(i)^2 = \frac{2}{\ell+1} \sum_{k=1}^{\ell} s_n(\zeta_k) \left[\sin(i\theta_k)\right]^2$$





Conclusions & some open issues

- Continuum limit of an Entanglement Hamiltonian
- Entanglement Hamiltonians of an interval & contour for EE after a global quench:
  - Harmonic chain: quench of the frequency parameter
  - Chain of free fermions: a quench of the couplings
- Analytic insights from CFT and quasi-particle picture

- Some open problems:
  - Interacting models
  - Other quenches
  - Higher dimensions
  - Other spatial configurations
  - Holography

Thank you!

Entanglement contours in the lattice: further properties

[Chen, Vidal, (2014)]

- Besides $S_A^{(n)} = \sum_{i \in A} s_A^{(n)}(i)$ and $s_A^{(n)}(i) \geqslant 0$, other properties can be imposed on the contour function.

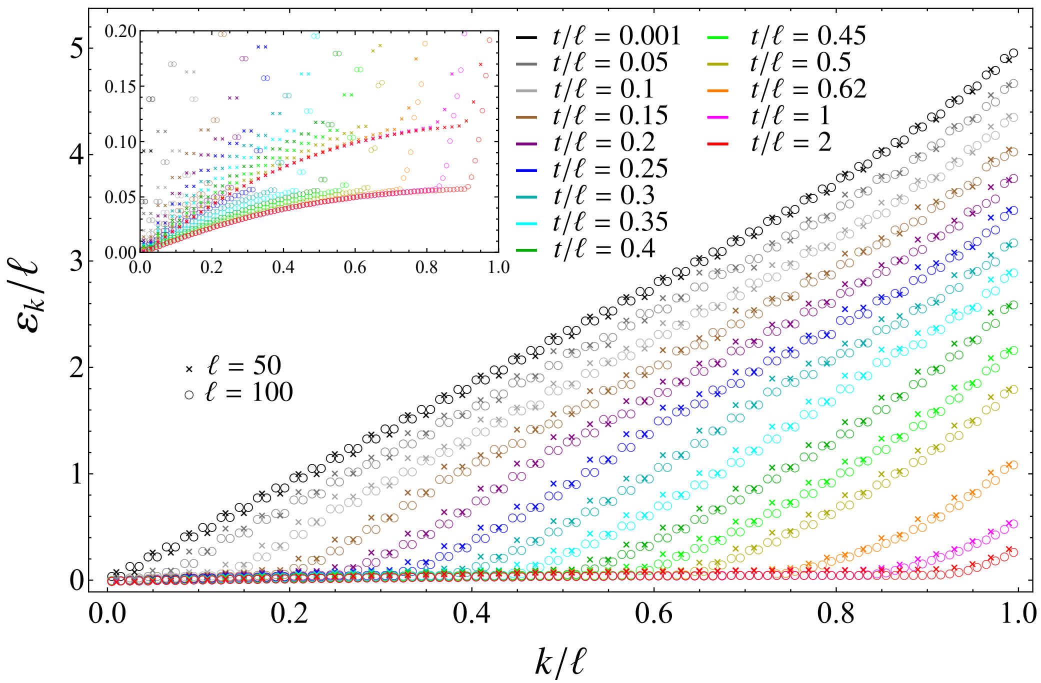
(a) *Spatial symmetry.* If ρ_A is invariant under a transformation relating the sites i and j in the subsystem A, then $s_A^{(n)}(i) = s_A^{(n)}(j)$.

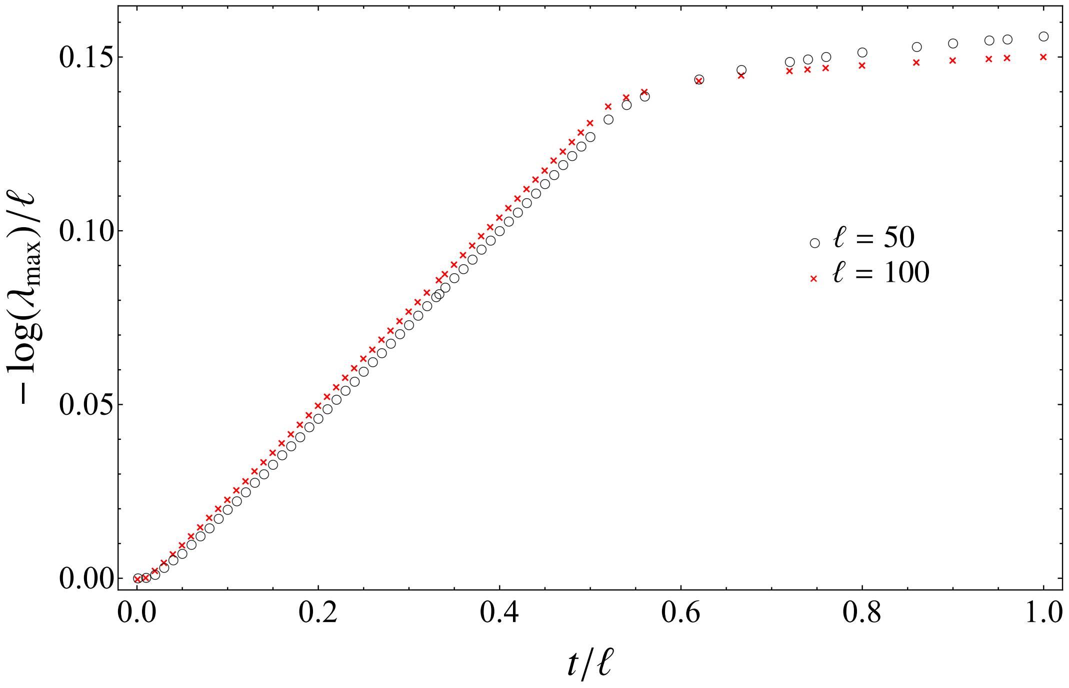
The contour $s_A^{(n)}(G)$ of a subregion $G \subseteq A$ is $s_A^{(n)}(G) \equiv \sum_{i \in G} s_A^{(n)}(i)$.

(b) *Invariance under local unitary transformations.* Given a system in the state ρ and a unitary transformation U_G acting non trivially only on $G \subseteq A$, denoting by ρ' the state of the system after such transformation, the same contour $s_A^{(n)}(G)$ should be found for ρ and ρ'.

(c) *A bound.* Given a system in the pure state $|\Psi\rangle$ and $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, let us assume that the further decompositions $\mathcal{H}_A = \mathcal{H}_{\Omega_A} \otimes \mathcal{H}_{\bar{\Omega}_A}$ and $\mathcal{H}_B = \mathcal{H}_{\Omega_B} \otimes \mathcal{H}_{\bar{\Omega}_B}$ lead to the factorisation of the state $|\Psi\rangle = |\Psi_{\Omega_A \Omega_B}\rangle \otimes |\Psi_{\bar{\Omega}_A \bar{\Omega}_B}\rangle$. Considering a subregion $G \subseteq A$ such that $\bigotimes_{i \in G} \mathcal{H}_i \subseteq \mathcal{H}_{\Omega_A}$, we must have $s_A^{(n)}(G) \leqslant S^{(n)}(\Omega_A)$, where $S^{(n)}(\Omega_A)$ are the entanglement entropies obtained by tracing over $\mathcal{H}_{\bar{\Omega}_A} \otimes \mathcal{H}_B$.

Entanglement spectrum in HC





Entanglement spectrum in the FFC