Are there late-time solutions to the H_0 and σ_8 tensions?

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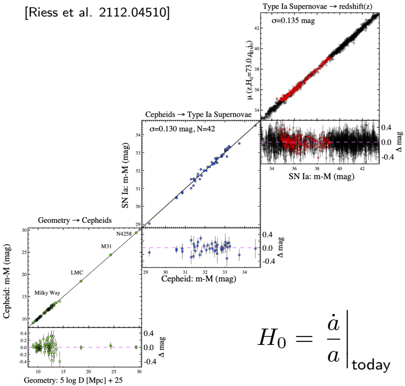
L. Heisenberg, HVR, J. Zosso, 2201.11623
L. Heisenberg, HVR, J. Zosso, 2202.01202

ETH zürich

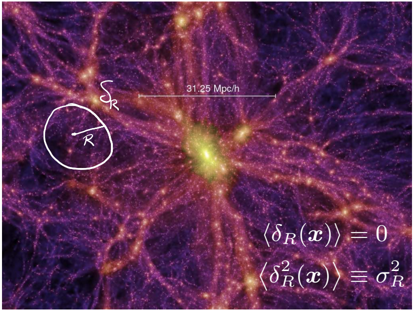
H_0 and σ_8

[Riess et al. 2112.04510]



$$H_0 = \frac{\dot{a}}{a}\Big|_{\text{today}}$$

[Millennium Simulation]



Clustering amplitude

$$\sigma_8 \equiv \sigma_R, \quad R = 8 \text{ Mpc}\, h^{-1}$$

H_0 and σ_8

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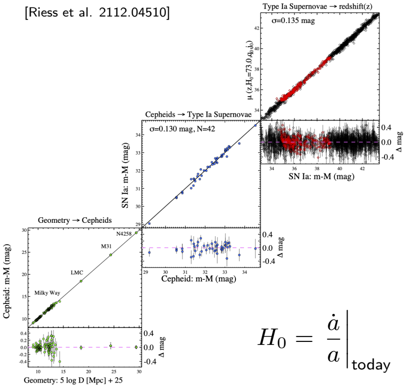

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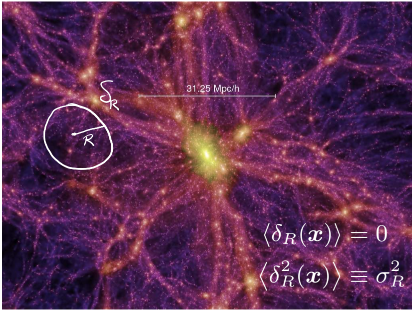

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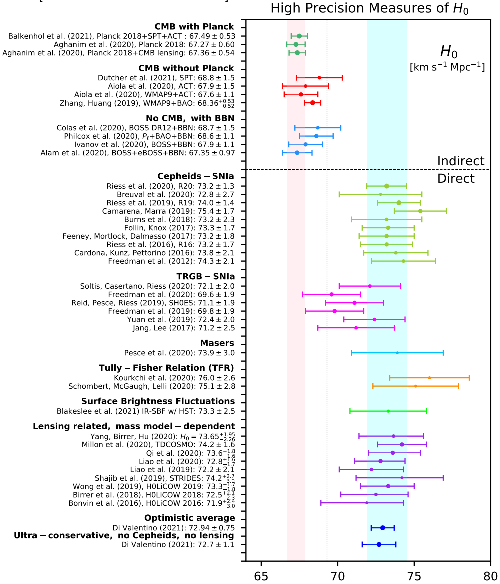
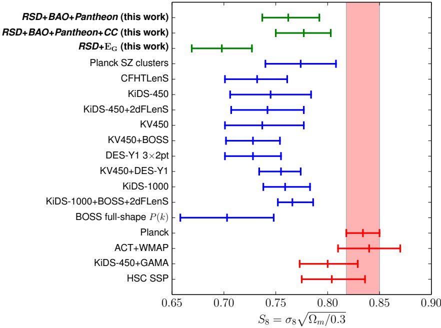
Both can be inferred from the CMB (after choosing a cosmoloy!!)

Cosmological tensions

[Di Valentino et al. 2103.01183]

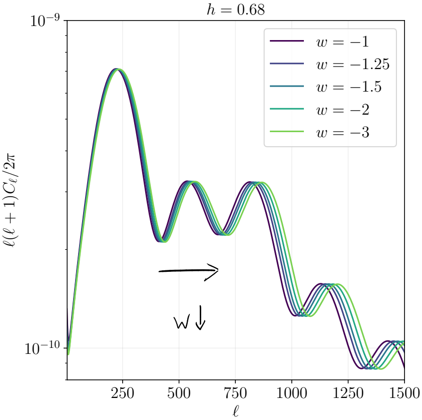


[Nunes, Vagnozzi, 2106.01208]



Early and late-time solutions to H_0

Toy model of late-time 'solution': phantom dark energy with $w = \text{const.} < -1$



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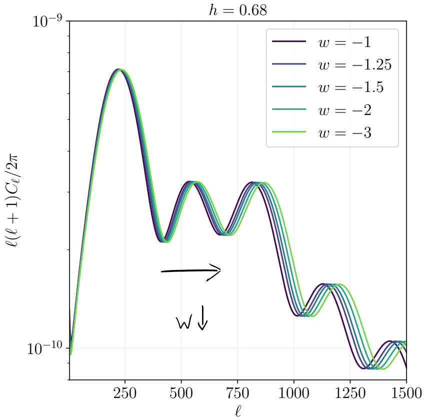

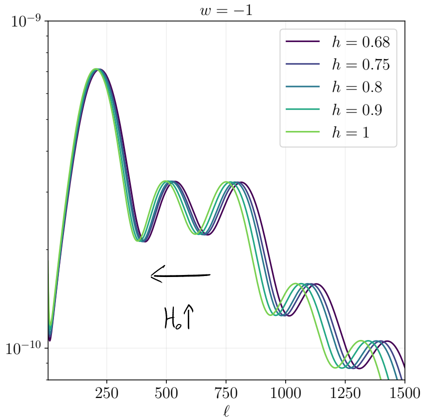

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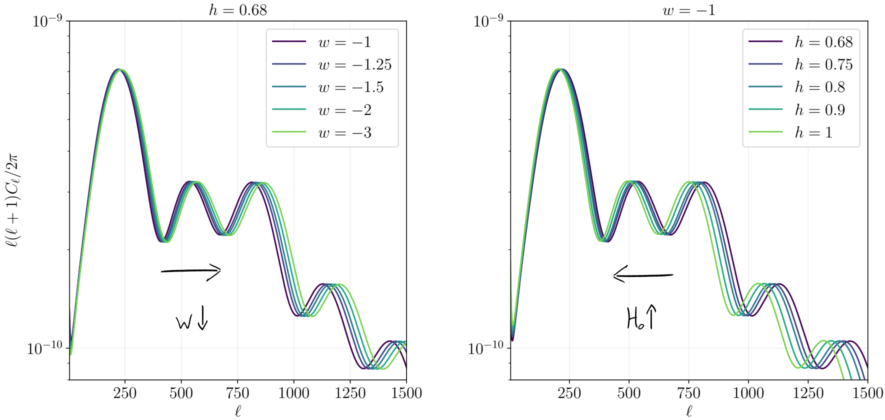
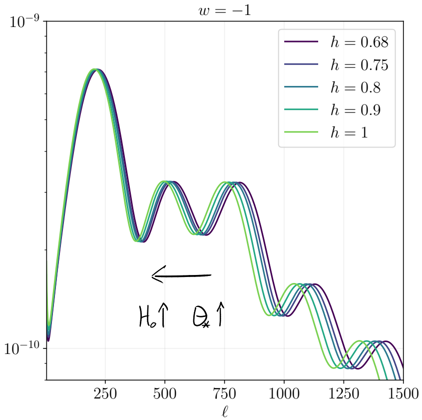


Why?

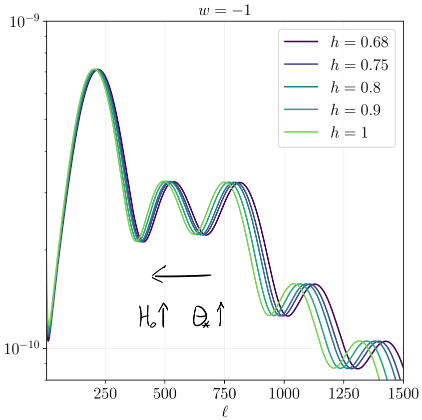
Early and late-time solutions to H_0



Overall position of the peaks: acoustic scale

$$\theta_* = \frac{r_s(z_*)}{d_A(z_*)}$$

Early and late-time solutions to H_0



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$$\theta_* = \frac{r_\mathrm{s}(z_*)}{d_A(z_*)}$$

$$r_\mathrm{s}(z_*) = \int_{z_*}^{\infty} \frac{\mathrm{d}z}{H} c_\mathrm{s} \quad \rightarrow \quad \text{Early } H(z)$$

$$d_A(z_*) = \int_0^{z_*} \frac{\mathrm{d}z}{H} \quad \rightarrow \quad \text{Late } H(z)$$

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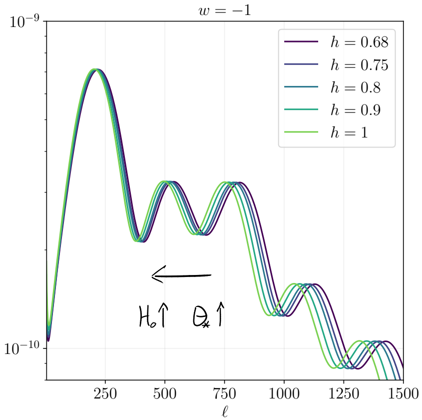

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$$d_A(z_*) = \int_{0}^{z_*} \frac{dz}{H} \quad \rightarrow \quad \text{Late } H(z)$$

To infer a larger H_0 from the CMB, we need models such that $\theta_* \downarrow$ (fixed H_0)

- Early-time solutions: $r_s(z_*) \downarrow$
- Late-time solutions: $d_A(z_*) \uparrow$

Both usually worsen the σ_8 tension!

ΛCDM cosmology:

$$H^2_{\Lambda\text{CDM}}(z) = H_0^2\Big(\Omega_m(z) + \Omega_r(z) + \Omega_\Lambda\Big)$$

Motivation for our work

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Alternative cosmology (late-time deformation of ΛCDM):

$$H(z) = H_{\Lambda\text{CDM}}(z) + \delta H(z)\ , \qquad \frac{\delta H(z)}{H(z)} \ll 1 \qquad \delta H(z > 100) = 0$$

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- Model-independent approach: we **do not** parameterize $\delta H(z)$
- Fully analytical results
- Generalization: we will also include (some) additional effects on the perturbations ($G_{\text{eff}} = G_N + \delta G(z)$)

Effects of late-time ΛCDM extensions

ΛCDM

$$\begin{cases} (H_0, \Omega_m) \\ H = H_{\Lambda\text{CDM}}(z) \\ G_{\text{eff}} = G = \text{const.} \end{cases}$$

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After we compare with observations, the preferred values will be different from ΛCDM

$$(H_0 + \Delta H_0,\ \Omega_m + \Delta\Omega_m,\ \dots \mid \sigma_8 + \Delta\sigma_8,\ \theta_* + \Delta\theta_*,\ \dots)$$

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These changes are connected to the properties of the new model

$$\frac{\Delta H_0}{H_0}, \frac{\Delta\sigma_8}{\sigma_8}, \ldots \quad \leftrightarrow \quad \frac{\delta H(z)}{H(z)}, \frac{\delta G(z)}{G}$$

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$$\frac{\Delta H_0}{H_0},\ \frac{\Delta\sigma_8}{\sigma_8},\ \ldots \quad \leftrightarrow \quad \frac{\delta H(z)}{H(z)},\frac{\delta G(z)}{G}$$

(we will fix $\omega_m = \Omega_m h^2$ here, more details in the paper)

Some variations

The Hubble parameter is

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Variation at fixed h
Effect of, e.g., dark energy, alternative dark matter

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$$\frac{\Delta\chi(z)}{\chi(z)} = I_\chi(z)\frac{\delta h}{h} + \int_0^\infty \frac{\mathrm{d}x_z}{1 + x_z} R_\chi(x_z, z)\frac{\delta H(x_z)}{H(x_z)}$$

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We can relate δh and $\delta H(z)$ by fixing $\Delta\theta_* = 0$ (CMB angular scale)

Fixing the angular scale of the CMB ($\Delta\theta_* = 0$), we can relate δh and $\delta H(z)$

$$\frac{\delta h}{h} = \int_0^\infty \frac{\mathrm{d}z}{1+z}\mathcal{R}_h(z)\frac{\delta H(z)}{H(z)}$$

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Response functions

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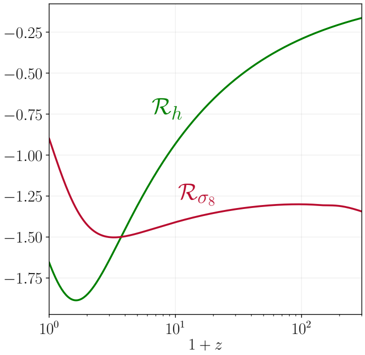

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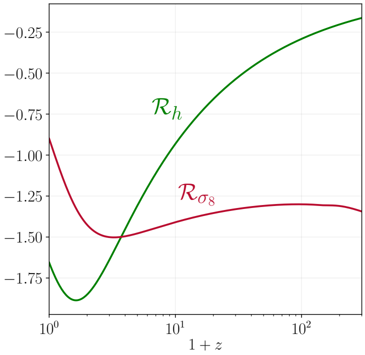
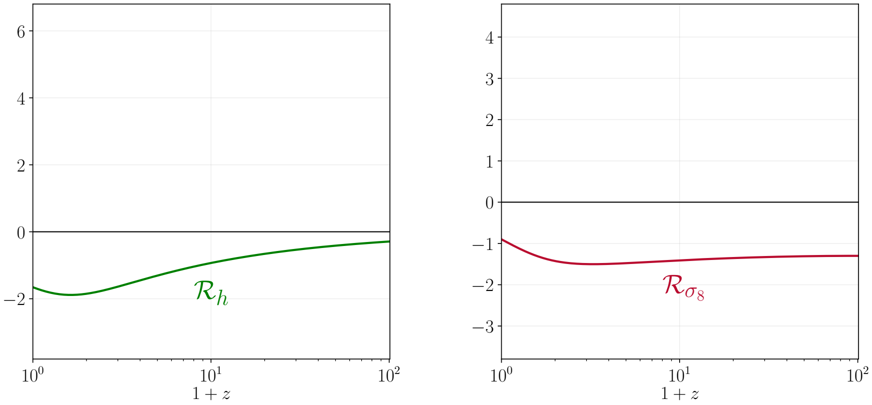


(Both $\mathcal{R}_h$ and $\mathcal{R}_{\sigma_8}$ are fully analytical!)


6
4
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$1+z$
$\mathcal{R}_h$
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10^0
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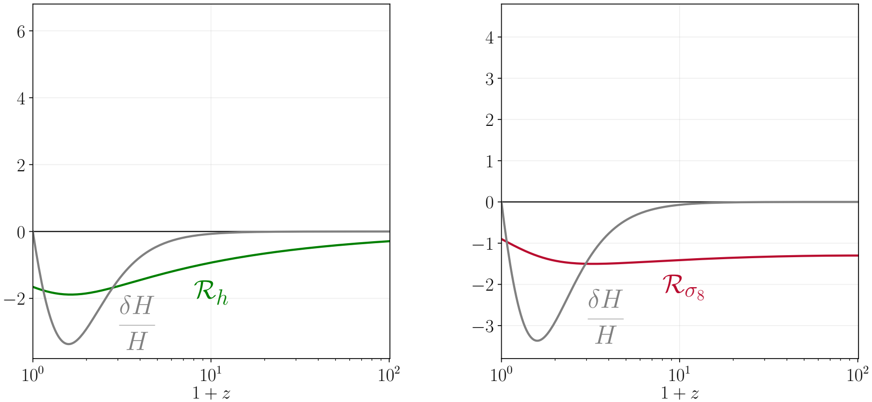
$\mathcal{R}_{\sigma_8}$



6
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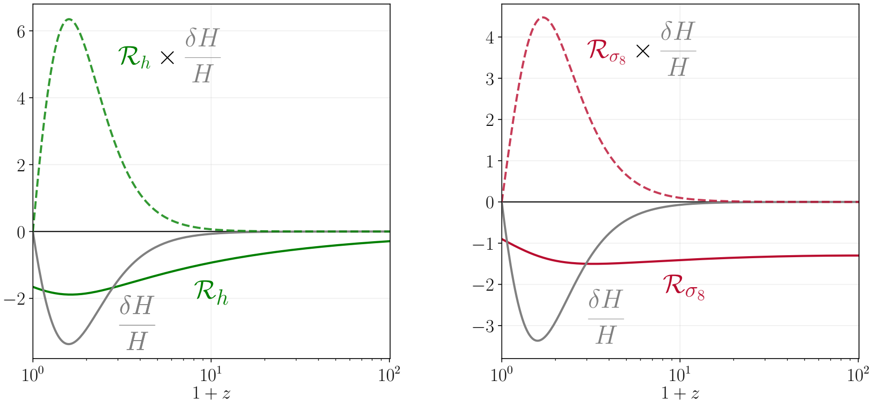
$\mathcal{R}_h \times \frac{\delta H}{H}$
$\mathcal{R}_h$
$\frac{\delta H}{H}$
$\mathcal{R}_{\sigma_8} \times \frac{\delta H}{H}$
$\mathcal{R}_{\sigma_8}$
$\frac{\delta H}{H}$
$1+z$
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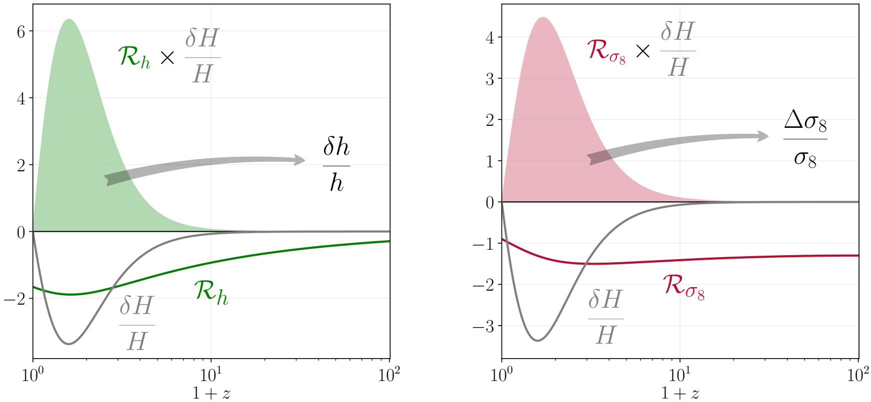

$\mathcal{R}_h \times \frac{\delta H}{H}$
$\frac{\delta h}{h}$
$\mathcal{R}_h$
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$\mathcal{R}_{\sigma_8} \times \frac{\delta H}{H}$
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-3
10^0
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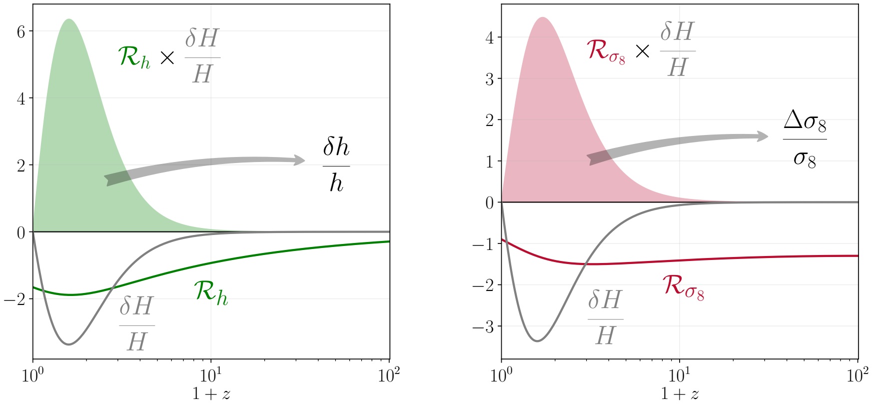
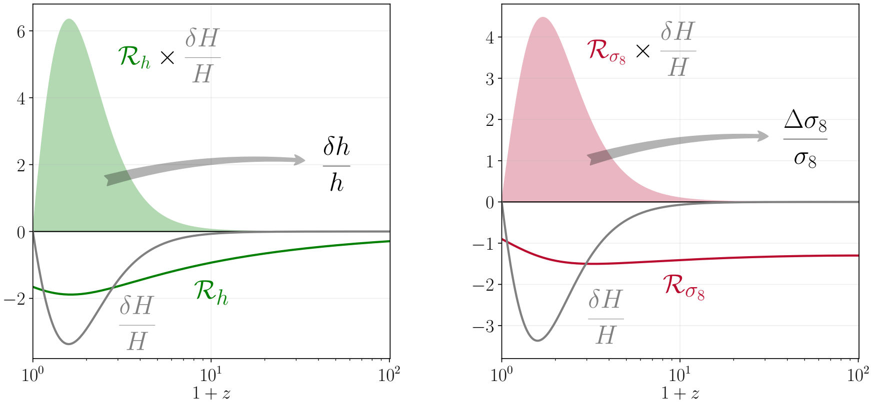


- Solving H_0 tension ($\delta h > 0$) $\Rightarrow\ \exists z \,|\, \delta H(z) < 0\ \Rightarrow\ \exists z \,|\, w(z) < -1$



- **I** Solving H_0 tension ($\delta h > 0$) $\Rightarrow$ $\exists z \,|\, \delta H(z) < 0$ $\Rightarrow$ $\exists z \,|\, w(z) < -1$
- **II** Solving both tensions ($\delta h > 0,\ \Delta\sigma_8 < 0$):

  a) If $G_{\text{eff}} = G \implies \delta H(z)$ changes sign $\implies w(z)$ crosses -1

  b) If $G_{\text{eff}} \neq G \implies$?

The matter density perturbation on small scales evolves as

$$\frac{\mathrm{d}^2\delta_m}{\mathrm{d}a^2} + \frac{\mathrm{d}\log\left(a^3 H\right)}{\mathrm{d}a}\frac{\mathrm{d}\delta_m}{\mathrm{d}a} - \frac{3\Omega_m H_0^2}{2a^5 H^2}\frac{G_{\mathsf{eff}}}{G}\delta_m = 0$$

where $G_{\mathsf{eff}} = G$ in ΛCDM.

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We will assume that the effects of the new model can be described with a scale-independent, small deviation $\delta G(z)$

$$G_{\text{eff}} = G + \delta G(z)$$

Then, only the growth factor D is modified

$$\delta_m(z,k) \propto \Big(D(z) + (\Delta D)|_{\delta G}\Big)T(k)$$

Following the same steps, we compute the response to $\delta G(z)$ so now the full variation is

$$\frac{\Delta\sigma_8}{\sigma_8} = \int_0^\infty \frac{\mathrm{d}z}{1+z}\mathcal{R}_{\sigma_8}\frac{\delta H}{H} + \int_0^\infty \frac{\mathrm{d}z}{1+z}\mathcal{G}_{\sigma_8}\frac{\delta G}{G}$$

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  $$\Longrightarrow \qquad \frac{\delta G(z)}{G} < \alpha(z)\frac{\delta H(z)}{H(z)} < 0 \quad \text{for some } z$$

  Again, $\alpha(z)$ is a known, analytical function

Summary of the results

General conditions for any late dark energy model to simultaneously solve the H_0 tension (i.e. $H_0 \uparrow$) and the σ_8 tension (i.e. $\sigma_8 \downarrow$):

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   $\Longrightarrow$ $\boxed{w(z) < -1 \text{ for some } z}$

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II. Without modifying the perturbations ($G_{\text{eff}} = G$):
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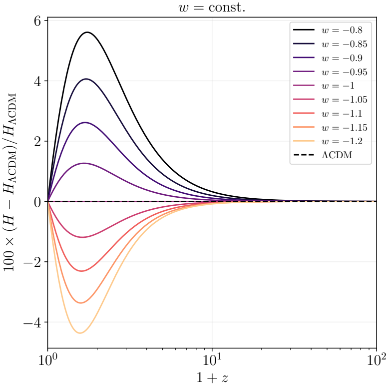
III. If $G_{\text{eff}} = G + \delta G(z)$ and $\delta H(z)$ does not change sign:

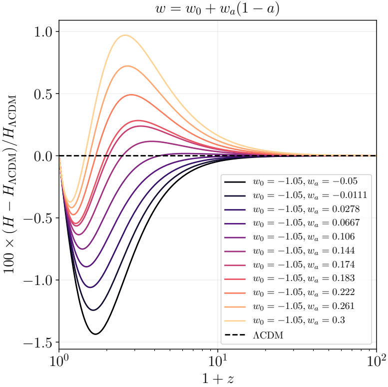
Solving H_0 and σ_8 tensions $\Longrightarrow$ $\boxed{\frac{\delta G(z)}{G} < \alpha(z)\frac{\delta H(z)}{H(z)} < 0 \text{ for some } z}$

Example 1: wCDM



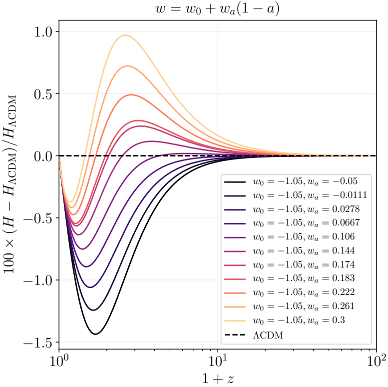
	$100 \times \delta h/h$		$100 \times \Delta\sigma_8/\sigma_8$	
w	class	Analytical	class	Analytical
-0.8	-8.57	-10.97	-8.98	-7.13
-0.85	-6.47	-7.76	-6.29	-5.32
-0.9	-4.35	-4.89	-3.93	-3.53
-0.95	-2.19	-2.32	-1.85	-1.76
-1.05	2.22	2.10	1.65	1.75
-1.1	4.47	4.01	3.12	3.48
-1.15	6.75	5.74	4.45	5.21
-1.2	9.07	7.33	5.66	6.93

Example 2: w_0w_aCDM



$w_0 = -1.05$	$100 \times \delta h/h$		$100 \times \Delta\sigma_8/\sigma_8$	
w_a	class	Analytical	class	Analytical
-0.05	2.81	2.61	2.24	2.08
-0.01	2.35	2.22	1.86	1.75
0.03	1.89	1.80	1.47	1.39
0.07	1.42	1.37	1.07	1.03
0.11	0.946	0.93	0.66	0.64
0.14	0.465	0.46	0.24	0.23
0.174	0.095	0.093	-0.089	-0.092
0.18	-0.022	-0.025	-0.19	-0.20
0.22	-0.52	-0.53	-0.64	-0.65
0.26	-1.02	-1.07	-1.09	-1.13
0.3	-1.54	-1.62	-1.56	-1.64

Example 2: $w_0 w_a$CDM



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Additional response functions