Squashed fuzzy extra dimensions in $\mathcal{N} = 4$ SYM with large rank

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Motivation

- mechanism in large-$N$ Yang-Mills with adjoint scalars: dynamical generation of extra dimensions $\mathbb{R}^4 \times K_N$ (Higgs mechanism!)
  
  Madore, Myers, Arkani-Hamed et al, HS-Zoupanos-Chatzistavrakidis, Aschieri, Manousselis, O’Connor et al, Polchinski-Strassler, Andrews-Dorey,...

- aspects of string theory (intersecting branes, KK modes ...) realized in 4-D gauge theory

  not holographic, “direct”

- here: $\mathcal{N} = 4$ SYM, deformed by cubic potential
Motivation

**outline**

- $SU(N)$ $\mathcal{N} = 4$ SYM with cubic (flux) terms
- new vacuum solutions with $SU(3)$ structure
- geometry: squashed (fuzzy) coadjoint orbits of $SU(3)$
  - zero modes & chiral fermions
- stacks of branes,
  approaching physically interesting configurations
\[ \mathcal{N} = 4 \text{ SYM and squashed fuzzy branes} \]

starting point: \( SU(N) \mathcal{N} = 4 \text{ SUSY} \)

\[
S = \int d^4x \frac{1}{4g^2} tr \left( - F_{\mu\nu} F^{\mu\nu} - 2D^\mu \Phi^a D_\mu \Phi_a + [\Phi^a, \Phi^b][\Phi_a, \Phi_b] \right) \\
+ tr \left( \bar{\Psi} \gamma^\mu (i\partial_\mu + [A_\mu, .]) \Psi + \bar{\Psi} \Gamma^a[\Phi_a, \Psi] \right)
\]

- \( N = 1 \text{ SYM in 10 D dimensionally reduced to 4D} \)
- 6 scalar fields \( \Phi^a \), global \( SO(6)_R \)
- \( D_\mu \Phi^a = (\partial_\mu + i[A_\mu, .]) \Phi^a \)
- \( (\gamma^\mu, \Gamma^a) = \Gamma^A \ldots 10D \text{ Clifford generators, } \Psi \rightarrow 4 \text{ Weyl fermions} \)
- most symmetric 4D gauge theory, UV finite

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Squashed fuzzy extra dimensions in, \( \mathcal{N} = 4 \text{ SYM with large rank} \)
\[ \mathcal{N} = 4 \text{ SYM beautiful but too ”round“ for physics} \]

more structure:

- spontaneous symmetry breaking (SSB): still no ”interesting“ (chiral) low-energy physics
- add soft susy breaking terms to potential


scalar potential: \( V[\Phi] = (V_4[\Phi] + V_{\text{soft}}[\Phi]) \),

\[
\begin{align*}
V_4[\Phi] &= -\frac{1}{4} \text{tr}[\Phi_a, \Phi_b][\Phi^a, \Phi^b], \\
V_{\text{soft}}[\Phi] &= \text{tr}(im f_{abc} \Phi^a \Phi^b \Phi^c + M_{ab}^{2} \Phi^a \Phi^b)
\end{align*}
\]

\( f_{abc} \) ... tot. antisymmm., soft SUSY breaking

- known: \( f_{abc} \sim \epsilon_{abc} \) : \( \rightarrow \) fuzzy sphere solutions \( \Phi^a \sim J^a \)
- new: \( f_{abc} \sim \) truncated \( su(3) \) structure constants (roots) \( \rightarrow \) squashed coadjoint \( SU(3) \) orbits (4D, 6D)

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Motivation

Fluctuation modes

fuzzy coadjoint orbits

Fermions

stacks of branes

complex dimensionless fields $\Phi_a = mY_a$,

$Y_1^\pm = Y_4 \pm i Y_5 \equiv Y_{\pm \alpha_1}$,

$Y_2^\pm = Y_6 \mp i Y_7 \equiv Y_{\pm \alpha_2}$,

$Y_3^\pm = Y_1 \mp i Y_2 \equiv Y_{\pm \alpha_3}$

let $f_{abc} = c_{abc}\big|_{a,b,c\neq 3,8}$ ... structure constants of $su(3)$ without Cartan

observe

$\text{Tr}(if_{abc} Y^a Y^b Y^c) \sim \text{Tr}(\varepsilon_{ijk} Y_i^+ Y_j^+ Y_k^+ + h.c.)$

breaks $SO(6)_R$ to $SU(3)_R$

eom:

$0 = (\Box_4 + m^2(\Box_Y + 4M_i^2)) Y_i^+ + 4m^2 \varepsilon_{ijk} Y_j^- Y_k^-$

$\Box_Y \equiv [Y^a, [Y_a, \cdot]]$

squashed $SU(3)$ brane solutions

$Y_{\pm \alpha_i} = r_i \pi(T_{\pm \alpha_i})$

... root generators of $su(3)$, any rep.
(soft) SUSY breaking:

action is \textit{almost} $\mathcal{N} = 1^*$ deformation of $\mathcal{N} = 4$ SYM:
consider superpotential

\[ W = \frac{\sqrt{2}}{g} \text{tr}([\Phi^+_1, \Phi^+_2] \Phi^-_3 - m \Phi^-_3 \Phi^-_3) \quad \ldots \mathcal{N} = 1^* \]

(declare $\Phi^+_1 \Phi^+_2, \Phi^-_3$ as holomorphic coords)

$\rightarrow$ effective potential

\[ V(\Phi) = -\frac{1}{4g^2} \text{tr}[\Phi^\alpha, \Phi^\beta][\Phi^\alpha, \Phi^\beta] + 4 \frac{1}{g^2} \text{tr}( - m[\Phi^+_1, \Phi^+_2] \Phi^+_3 - m[\Phi^-_2, \Phi^-_1] \Phi^-_3 + 2m^2 \Phi^+_3 \Phi^-_3 ). \]

$\equiv$ present potential,
however, mass $M_3^2 = 2m^2$ too large for squashed $SU(3)$ brane solutions
Motivation

Symmetries:

solutions $Y_{\pm \alpha_i} = T_{\pm \alpha_i}$ break $SU(3)_R \rightarrow U(1) \times U(1) \cong$ gauge trafo

residual symmetry: $(U(1) \times U(1))_K$ generated by

$$K_i := 2\tau_i - [H_{\alpha_i}, \cdot],$$
$$\tau_i \phi_\alpha = \frac{1}{2} (\alpha_i, \alpha) \phi_\alpha,$$

generates $U(1)_i \subset U(3)_R$

$\Rightarrow$ 8-2 = 6 Goldstone bosons from $SU(3)/U(1) \times U(1)$

note:

the $Y_{i}^{\pm} \sim T_{\pm \alpha_i}$ generate $SU(3)_Y \subset SU(N)$, is not part of the $SO(6)_R$ symmetry

transform as \[\begin{cases}
(3) \text{ under } SU(3)_R \\
(8) \text{ under } SU(3)_Y
\end{cases}\]

$Y_{i}^{\pm} \sim \pi_{\mu}(T_{\pm \alpha_i})$ breaks $SU(N)$ gauge symmetry completely (Higgs)

$\rightarrow$ massive KK modes!

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Squashed fuzzy extra dimensions in $\mathcal{N} = 4$ SYM with large rank
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Squashed fuzzy extra dimensions in $\mathcal{N} = 4$ SYM with large rank
all fluctuations (scalar, gauge fields) take values in

\[ u(N) = \text{End}(\mathcal{H}_\mu) \cong \mathcal{H}_\mu \otimes \mathcal{H}_\mu^* = \bigoplus \Lambda \mathcal{H}_\Lambda \]

\[ \cong \text{functions on squashed } C_N[\mu] \ldots \text{KK modes} ! \]

\[ \rightarrow \quad 8- \text{ or } 10\text{-dim gauge theory, below } \Lambda_{UV} \]

expand fields into \( SU(3)_Y \) harmonics

gauge fields:

\[ A_\mu(x) = \sum A^{(M,\Lambda)}_\mu(x) \hat{Y}_M^\Lambda \]

scalar fields:

\[ \Phi_a = mY^a + \varphi^a(x) \]

\[ \varphi_a(x) = \sum \varphi^{(M,\Lambda)}_a(x) \hat{Y}_M^\Lambda \]

fermions ... (similar)
massive gauge bosons

**example:** gauge field modes $A_\mu(x)$

kinetic term $tr(D_\mu Y_a D^\mu Y^a)$ gives mass term,

$$\int tr(D_\mu Y_a)^\dagger D_\mu Y_a = \int tr(\partial_\mu Y_a^\dagger \partial_\mu Y_a + \sum_{\Lambda,M} m_{\Lambda,M}^2 A^{\dagger}_{\mu,(\Lambda,M)} A^{\mu}_{(\Lambda,M)}) + S_{int}$$

using $D_\mu \phi_a = (\partial_\mu + i[A_\mu,\cdot])\phi_a$ and decomposition

$$A_\mu(x) = \sum A_{\mu}^{(M,\Lambda)}(x) \hat{Y}_M^\Lambda \equiv A_\mu(x,y) \quad \text{on } \mathbb{R}^4 \times \mathbb{C} \mathcal{N}[\mu]$$

into eigenmodes of

$$\square_Y \hat{Y}_M^\Lambda = [Y_a, [Y_a, \hat{Y}_M^\Lambda]] = m_{(\Lambda,M)}^2 \hat{Y}_M^\Lambda$$

$$\Rightarrow \text{ tower of massive KK modes } A_{\mu,(\Lambda,M)}(x), \text{ mass }$$

$$m_{(\Lambda,M)}^2 \sim (\langle \Lambda + \rho, \Lambda + \rho \rangle - \langle \rho, \rho \rangle - \langle M, M \rangle) > 0$$

(no massless gauge modes)
internal geometry: squashed fuzzy coadjoint orbits

claim: $Y_a = \pi_\mu(T_a)$ ... quantized coadjoint $SU(3)$ orbit, projected along Cartan directions

$C[\mu] \hookrightarrow \mathbb{R}^8 \ni \mathbb{R}^6$

$(y^a)_{a=1,\ldots,8} \mapsto (y^a)_{a=1,2,4,5,6,7}$

4- or 6-dimensional variety, self-intersecting embedding in $\mathbb{R}^6$
multiple covering at origin, spanning all 6 directions

H.S., J. Zahn arxiv:1409.1440

e.g.: 3-dim. section of squashed $\mathbb{C}P^2$ through $y_2 = y_5 = y_7 = 0$:

triple self-intersection at origin
classical coadjoint orbits of $SU(3)$

$$C[\mu] = \{ p = g^{-1}\mu g; \; g \in SU(3) \} \cong SU(3)/\mathcal{K} \subset su(3)^* \cong \mathbb{R}^8$$

$\mathcal{K}$ ... stabilizer group of weight $\mu \in g^*$

- $\mu$ 3 EV $\rightarrow C[\mu] \cong SU(3)/U(1) \times U(1)$ ... 6-dim.
- $\mu$ 2 EV $\rightarrow C[\mu] \cong SU(3)/SU(2) \times U(1) \cong \mathbb{C}P^2$ ... 4-dim.

embedding of $C[\mu] \subset \mathbb{R}^8$:

$$p = y^a \lambda_a \in C[\mu], \quad a = 1, ..., 8$$

$$y^a : \; C[\mu] \hookrightarrow \mathbb{R}^8 \cong su(3)$$
quantized (fuzzy) coadjoint orbits:

$C[\mu]$ is symplectic (Kirillov-Kostant) $\rightarrow$ quantize it:

$\mu = (n_1, n_2)$ ... dominant integral weight

$\mathcal{H}_\mu$ ... corresp. highest weight irrep

fuzzy $C_N[\mu]$:

$Y^a = \pi_\mu(T^a)$

generates matrix algebra

$A_N = \text{End}(\mathcal{H}_\mu)$

can show (in general):

decomposition into harmonics of $SU(3)$ agrees with classical harmonic analysis,

$Pol(C[\mu]) \cong \bigoplus \Lambda m_\Lambda \mathcal{H}_\Lambda \cong \mathcal{H}_\mu \otimes \mathcal{H}_\mu^* = \text{End}(\mathcal{H}_\mu) = A_N$

up to cutoff $\Lambda_{\text{max}}$
Motivation

quantization map

\[ Q : \text{Pol}(C[\mu]) \rightarrow A_N \]
\[ Y^A_M \leftrightarrow \hat{Y}^A_M \]

respects $SU(3)$, truncated at cutoff $\Lambda_{\text{max}}$.

$Y^a = Q(y^a) = \pi_{\mu}(T^a)$ interpreted as quantized embedding functions

$Y^a \sim y^a : C[\mu] \hookrightarrow \mathbb{R}^8$.

commutation relations:

\[ [Y^a, Y^b] = i c_{c}^{ab} Y^c \]

... quantizes Poisson structure (Kirillov-Kostant)

\[ \{y^a, y^b\} = c_{c}^{ab} y^c \]

$Y^a = \pi_{\mu}(T^a), \quad a = 1, ..., 8$ ... quantized coadjoint orbit $C_N[\mu]$

$Y^a = \pi_{\mu}(T^a), \quad a \neq 3, 8$ ... squashed $C_N[\mu]$

= solution of (deformed) $\mathcal{N} = 4$ SU(N) SYM
computer measurement of $Y^a = \pi^\mu(T^a)$:

squashed $C_N[\mu]$ for $\mu = (20, 0)$.

L. Schneiderbauer, H.S. (unpublished)
scalar fluctuations: \( \phi_a = m Y_a + \varphi_a \) governed by

\[
V_2[\varphi] = \text{tr} \varphi^\alpha (\Box Y + 2 \hat{\mathcal{D}}_{\text{diag}} - 2 \hat{\mathcal{D}}_{\text{mix}}) \varphi^\alpha,
\]

\[
(\hat{\mathcal{D}}_{\text{mix}} \varphi)^+_i = -\varepsilon_{ikj} [Y_k^-, \varphi_j^-], \quad \tau \hat{\mathcal{D}}_{\text{mix}} = -\hat{\mathcal{D}}_{\text{mix}} \tau
\]

\[
(\hat{\mathcal{D}}_{\text{diag}} \varphi)_{\alpha} = [H_{\alpha}, \varphi_{\alpha}] \quad \text{(no sum)}
\]

can show:

- no negative modes (!!)
- \( \exists \) zero modes (= flat deformations), corresponding to

\[
\hat{\mathcal{D}}_{\text{mix}} \phi_\alpha^{(0)} = 0.
\]
regular zero modes: 6 zero modes for each $\mathcal{H}_\Lambda$

(corresponding to extremal weights $\Lambda'$ in $\mathcal{H}_\Lambda \otimes (8)$)

with distinct $U(1)_{K_i}$ charges

geometric interpretation:

string between coincident sheets of $C[\mu]$ at/near origin

e.g. $\varphi_\Lambda = |\Omega_\mu\rangle \langle \mu|$, $|\mu\rangle$ ... coherent states

exceptional zero modes:

in particular: 6 Goldstone bosons

(verified numerically)
Dirac operator on squashed $\mathcal{C}_N[\mu]$:

$$\mathcal{D}_{(6)}\psi = 2 \sum_{i=1}^{3} \left( \gamma_i [Y_i^+, \cdot] + \gamma_i^\dagger [Y_i^-, \cdot] \right)$$

spinorial ladder operators $\{\gamma_i, \gamma_j^\dagger\} = \delta_{ij}$ ... $SO(6)$ Clifford algebra

zero modes:

$$\mathcal{D}_{(6)}\psi_\Lambda = 0$$

- in one-to-one correspondence to regular zero modes
  (fermionic strings connecting branes!)
- but not corresponding to exceptional zero modes
- have distinct chirality determined by gauge charge (weight $\Lambda'$)
Motivation Fluctuation modes fuzzy coadjoint orbits Fermions stacks of branes

**chiral fermions**

\[ \Gamma^{(6)} \psi_{w^\Lambda} = (-1)^{|w|} \psi_{w^\Lambda}. \]

\[ \Psi = \psi_+ \otimes \chi_+ + \psi_- \otimes \chi_- , \]

\[ \mathcal{D}_{(4)} \psi_\pm = 0 , \quad \gamma_5 \psi_\pm = \pm \psi_\pm , \quad \psi_\pm^C = \psi_{\mp}. \]

3 generations of Weyl/ Majorana spinors on \( \mathbb{R}^4 \) (Weyl rotations \( \frac{2\pi}{3} \)) distinguished by \( U(1)_{K_i} \) charges

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Squashed fuzzy extra dimensions in \( \mathcal{N} = 4 \) SYM with large rank
towards interesting physics:

**Stacks** of squashed $D_i = C[ζ_i]$ branes (=reducible rep. of $su(3)$)

\[
\gamma^a = \left( \begin{array}{cc} \gamma^a_{\mu_1} & \gamma^a_{\mu_2} \end{array} \right)
\]

off-diagonal fermions

\[
\psi = \left( \begin{array}{cc} 0 & \psi_{12} \\ \psi_{21} & 0 \end{array} \right)
\]

transform in bi-fundamental $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ of $U(N_1) \times U(N_2)$

→ zero-modes as above

e.g. **branes linked to 3 point-branes**: $\mathcal{H}_2 \cong \mathbb{C}^3$ → “quarks“

\[
\psi_{12} \in \mathcal{H}_\mu \otimes \mathbb{C}^3
\]

→ 3+3 chiral zero-modes attached to extremal weights

get **quiver gauge theory** corresponding to stack of branes, zero modes connecting branes = chiral superfields (+ exceptionals)
A standard-model-like brane configuration

- 3 generations ($\mathbb{Z}_3$ Weyl rotations!), chiralities distinguished by (massive) gauge modes

- can realize all standard model fermions (+ more ...)
  e.g: quarks link $C[\mu]$ with 3 coincident point branes ($\rightarrow$ color)

same solutions in IKKT matrix model, $\rightarrow$ dynamical (NC) space-time, (emergent) gravity?
off-diagonal fermionic zero modes $\rightarrow$ (extended) SSM matter:

$$
\Psi = \begin{pmatrix}
*_{2} & \tilde{H}_d & \tilde{H}_u & l_L & Q_L \\
* & e' & e_R & \nu_R & u_R \\
* & * & \nu_R & u & u' \\
*_{3} & & & & \\
\end{pmatrix}.
$$

- correct quantum numbers, no exotics ($u' \sim u_R$, $e' \sim e_R$)
- $\exists$ cubic potential for bosonic ("Higgs") modes $\phi_u, \phi_d$
  $\rightarrow$ further SSB expected, complicated
- appropriate symmetry-breaking pattern possible, leaving $SU(3) \times U(1)_Q$ in light sector (assuming suitable Higgs)
- $\exists$ mirror fermions (opposite chirality), distinguished coupling to (mirror) Higgs, $\rightarrow$ may have (much?) higher masses

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Squashed fuzzy extra dimensions in $\mathcal{N} = 4$ SYM with large rank
$C_N[\mu]$ branes can be stabilized by rotation in pure $\mathcal{N} = 4$ SYM (no flux!) \cite{steinacker14}

ansatz:

$$Y_i^\pm = r_i e^{\pm i\omega_i x} \pi_\mu(T_i^\pm)$$

$$\omega_i = \omega_{i,\mu} \quad \text{... 3 time-like vectors ("frequencies")}$$

zero mode sector has Lorentz-invariant kinematics
∃ rich class of background solutions of $SU(N) \mathcal{N} = 4$ SYM

... self-intersecting $SU(3)$ branes $\mathbb{R}^4 \times \mathbb{C}[\mu]$

interesting low-energy physics:
space-filling $\rightarrow$ chiral fermions, scalars & gauge fields

can be surprisingly similar to standard model in broken phase (?!)

open issues:
generalizations, further SSB
relation with in string theory?
strong coupling?
gravity in matrix model version?
coherent states on $\mathbb{C}[^\mu]$: let $p \in \mathbb{C}[^\mu]$ ... north pole

$$SU(3) \rightarrow \mathbb{C}[^\mu]$$

$$g \mapsto g \triangleright p$$

stabilizer $\mathcal{K} \subset SU(3) \Rightarrow \mathbb{C}[^\mu] \cong SU(3)/\mathcal{K}$ ... coadjoint orbit

let $|^\mu\rangle \in \mathcal{H}_{^\mu}$ ... highest weight vector in $\mathcal{H}_{^\mu}$

$$\langle ^\mu|\vec{y}|^\mu\rangle = \vec{p} \quad \text{...localized at north pole}$$

def. $|\psi_g\rangle := g \triangleright |^\mu\rangle$

projector $\Pi_p = |\psi_g\rangle\langle\psi_g| \in End(\mathcal{H}_{^\mu})$ ... independent of $\mathcal{K} \subset SU(3)$

$$\mathbb{C}[^\mu] \cong SU(3)/\mathcal{K} \rightarrow End(\mathcal{H}_{^\mu})$$

$$p \mapsto \Pi_p := |\psi_g\rangle\langle\psi_g| =: \delta_N(y - p)$$

$$\langle p|\vec{y}|p\rangle = tr(\Pi_p\vec{y}) = \vec{p} \quad \text{... sweeps out } \mathbb{C}[^\mu]$$
example: fuzzy $\mathbb{C}P^2_N$ (Grosse & Strohmaier, Balachandran et al, ...)
arises for $\mu = N\Lambda_1$ or $\mu = N\Lambda_2$, stabilizer $K = SU(2) \times U(1)$
satisfy the relations

\[
\begin{align*}
[Y^a, Y^b] &= \frac{i}{2} \Lambda_N f_{abc} Y^c \\
\delta_{ab} Y^a Y^b &= R^2, \\
d_{abc} Y^a Y^b &= R \left( \frac{2N/3+1}{\sqrt{\frac{1}{3}N^2+N}} \right) Y^c.
\end{align*}
\]

harmonic decomposition

\[
\mathcal{A} = \text{End}(\mathcal{H}_\mu) = \bigoplus_{n=0}^{N} \mathcal{H}_{(n,n)}.
\]
squashed $\mathbb{CP}^2_N$:

projection $\Pi \equiv$ take away $H_3, H_8$

- $\langle \mu | \vec{y} | \mu \rangle = \vec{0}$, projected to 0 by $\Pi$
- acting with $SU(3)$ on $|\mu\rangle$: stabilizer $SU(2)_3$
  $\Rightarrow$ 4D orbit along 4567 plane
- analogous for 3 Weyl images of $\mu$

- three 4D sheets intersecting at origin
- extremal weight states $|\mu\rangle =$ coherent states on different sheets at origin