Heterotic Superpotentials and Moduli

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based on work in collaboration with Edward Hardy and Xenia de la Ossa,

June 9th 2015, IFT, Madrid
Introduction
Overview
General String Compactifications
Why Heterotic?
Superpotential
The Infinitesimal Moduli Space
Conclusions
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- String Compactifications.
- Gukov-Vafa-Witten superpotential and supersymmetry conditions.
- First order deformations, holomorphic structures and moduli.
- Conclusions and outlook..
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First need to find massless spectrum, i.e. infinitesimal moduli!
Why Heterotic?
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Need a nicer description to deal with moduli [Anderson et al 10, Anderson et al 14, de la Ossa EES 14, Garcia-Fernandez et al 15].
Superpotential and Supersymmetry
The Superpotential

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and where

\[ \omega^A_{CS} = \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) . \]
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Ignore D-terms and conformally balanced condition for this talk, and assume stable bundles.
The Infinitesimal Moduli Space

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Superpotential

The Infinitesimal Moduli Space

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\[
\delta_{12}W|_{\delta W=0} = \int_X \frac{\alpha'}{2} \left( \text{tr} \delta_1 A \wedge \delta_2 (F \wedge \Omega) - \text{tr} \delta_1 \Theta \wedge \delta_2 (R \wedge \Omega) \right)
+ \int_X d\tau_1 \wedge \delta_2 \Omega + \int_X \delta_2 (H + i d\omega) \wedge \delta_1 \Omega
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Complex Structure Moduli

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Note: Deformations \( \delta_2 \nabla = \kappa_2 \) non-physical. Can be thought of as infinitesimal field redefinitions preserving Strominger system [de la Ossa EES 14].
It follows that $\Delta_2$ is in the kernel of [Anderson et al 10]

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$\overline{\partial}_1$ defines an Atiyah algebroid [Atiyah 57]

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$$T\mathcal{M}_1 = H^{(0,1)}(Q_1)$$

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Conditions from the Anomaly
We also have the terms

\[ \int_X \delta_2 (H + i\omega) \wedge \delta_1 \Omega + \int_X (H + i\omega) \wedge \delta_2 \delta_1 \Omega \in \delta_{12} W |_{\delta W = 0} \]
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Algebra: ⇒ arrive at the following conditions
\[
\bar{\partial} \tau_2^{(0,2)} = 0
\]
\[
2\Delta^a_2 \wedge i\bar{\partial} [a \omega_b] \bar{c} \bar{d} z^{b\bar{c}} - \frac{\alpha'}{2} (\text{tr} \alpha_2 \wedge F - \text{tr} \kappa_2 \wedge R)
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= \partial \tau_2^{(0,2)} + \bar{\partial} \tau_2^{(1,1)}.
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Technicality: Assume \( H^{(0,1)}(X) = 0 \) ⇒ \( \partial \tau_2^{(0,2)} \) is \( \bar{\partial} \)-exact.
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Algebra: \(\Rightarrow\) arrive at the following conditions
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\[ 2\Delta^a_2 \wedge i\partial_{[a\omega_b]c} dz^{b\overline{c}} - \frac{\alpha'}{2} (\text{tr} \, \alpha_2 \wedge F - \text{tr} \, \kappa_2 \wedge R) \]
\[ = \partial \tau_2^{(0,2)} + \overline{\partial} \tau_2^{(1,1)}. \]

Technicality: Assume \( H^{(0,1)}(X) = 0 \Rightarrow \partial \tau_2^{(0,2)} \) is \( \overline{\partial} \)-exact.

It follows that \( x = (\Delta, \alpha, \kappa) \in H^{(0,1)}(Q_1) \) is in the kernel of
\[ \mathcal{H} : \quad H^{(0,1)}(Q_1) \rightarrow H^{(0,2)}(T^* X). \]
Holomorphic Double Extension

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The map $\mathcal{H}$ defines the holomorphic double extension

$$0 \to T^*X \to \mathcal{Q}_2 \to \mathcal{Q}_1 \to 0,$$

with corresponding holomorphic structure

$$\overline{\partial}_2 = \overline{\partial}_1 + \mathcal{H}, \quad \text{Heterotic Bianchi Identity} \iff \overline{\partial}_2^2 = 0.$$
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Note: $Q_2$ as a holomorphic bundle is \textit{self-dual}. 
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Infinitesimal moduli \cite{Anderson et al 14, de la Ossa EES 14}
\[ T \mathcal{M}_2 = H^{(0,1)}(Q_2) = H^{(0,1)}(T^* X) \oplus \ker(\mathcal{H}). \]
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Infinitesimal moduli [Anderson et al 14, de la Ossa ESS 14]

$$TM_2 = H^{(0,1)}(Q_2) = H^{(0,1)}(T^* X) \oplus \ker(\mathcal{H}).$$

Get same kernel structure.
Conclusions
Conclusions and Outlook

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Conclusions and Outlook

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- From the heterotic superpotential, we derived the massless moduli space, and saw that it agrees with the 10d computation of [Anderson et al 14, de la Ossa EES 14] for the infinitesimal moduli space of solutions to the Strominger system.
Conclusions:

- Heterotic string is a nice playground for phenomenology, but the moduli problem is hard.
- From the heterotic superpotential, we derived the massless moduli space, and saw that it agrees with the 10d computation of [Anderson et al 14, de la Ossa EES 14] for the infinitesimal moduli space of solutions to the Strominger system.
- We note that the heterotic anomaly condition may lead to lifting extra moduli, even in Calabi-Yau compactifications.
Outlook, and work in progress:
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- So far mostly a mathematical investigation into the structure of $\partial^2$. Interesting to look for more phenomenological examples.
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Thank you for your attention!