UNIMODULAR GRAVITY
REDUX

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JHEP
From Pauli’s discussions with Enz and Thellung we know that Pauli estimated the influence of the zero-point energy of the radiation field – cut off at the classical electron radius – on the radius of the universe, and came to the conclusion that it “could not even reach to the moon”.

In units with $\hbar = c = 1$ the vacuum energy density of the radiation field is

$$< \rho >_{\text{vac}} = \frac{8\pi}{(2\pi)^3} \int_0^{\omega_{\text{max}}} \frac{\omega}{2} \omega^2 d\omega = \frac{1}{8\pi^2} \omega_{\text{max}}^4,$$

with

$$\omega_{\text{max}} = \frac{2\pi}{\lambda_{\text{max}}} = \frac{2\pi m_e}{\alpha}.$$

The corresponding radius of the Einstein universe in Eq.(2) would then be ($M_{\text{pl}} \equiv 1/\sqrt{G}$)

$$a = \frac{\alpha^2}{(2\pi)^{\frac{3}{2}}} \frac{M_{\text{pl}}}{m_e} \frac{1}{m_e} \sim 31 \text{km}.$$
comment of Henry Whitehead: 'It is the snobbishness of the young to suppose that a theorem is trivial because the proof is trivial.'
In Unimodular Gravity the full configuration space is restricted to unimodular metrics

\[ \hat{g} \equiv \text{det } \hat{g}_{\mu\nu} = -1 \]

Action principle

\[ S_{\text{in}} \equiv \int d^n x \left( -\frac{1}{2\kappa^2} R[\hat{g}] + \frac{1}{2} \hat{g}^{\mu\nu} \partial_{\mu}\psi \partial_{\nu}\psi - V(\psi) \right) \]

Potential energy (not only zero mode) naively decoupled from the gravitational field

How to integrate over unimodular metrics?
Van der Bij, van Dam and Ng.
TDiff has enough freedom to kill the three extra polarizations: $5-3=2$
Convenient new variables \( \hat{g}_{\mu\nu} \equiv g^{-\frac{1}{n}} g_{\mu\nu} \)

Their variation \( \delta g_{\alpha\beta} \) is unconstrained

Whereas \( \hat{g}^{\alpha\beta} \delta \hat{g}_{\alpha\beta} = 0 \)

In this variables there is (sort of spurious?) Weyl symmetry in addition to TDiff
Flat Space

\[ L \equiv \sum_{i=1}^{4} C_i \mathcal{O}^{(i)} \]

Most general l.c. of dimension 4 operators

\[ \mathcal{O}^{(1)} \equiv \frac{1}{4} \partial_m h_{\rho \sigma} \partial^\mu h^{\rho \sigma} \]

\[ \mathcal{O}^{(2)} \equiv -\frac{1}{2} \partial_\rho h_{\rho \sigma} \partial^\mu h^{\mu \sigma} \]

\[ \mathcal{O}^{(3)} \equiv \frac{1}{2} \partial_\mu h \partial^\lambda h^{\mu \lambda} \]

\[ \mathcal{O}^{(4)} \equiv -\frac{1}{4} \partial_\mu h \partial^\mu h \]

E.A., Blas, Garriga & Verdaguer, NPB
Two cases of enhanced symmetry
Both propagate spin 2 ONLY

Fierz-Pauli (LDiff)

\[ \delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \]

\[ \partial_\mu \xi^\mu = 0 \]

\[ C_2 = 1 \]

\[ C_3 = C_4 = 1 \]

(WTDiff)

\[ C_3 = \frac{2}{n} \]

\[ C_4 = \frac{n + 2}{n^2} \]
WTDiff is a truncation of Fierz-Pauli obtained by

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} - \frac{1}{n} h \eta_{\mu\nu} \]

But this is NOT A FIELD REDEFINITION, because it is not invertible.

WTDiff is the linear limit of Unimodular Gravity.
\[ S \equiv -M_P^{n-2} \int d^n x \left( R[\hat{g}] + L_{\text{matt}}[\psi_i, \hat{g}] \right) = \]
\[ = -M_P^{n-2} \int d^n x \left| g \right|^{\frac{1}{n}} \left( R + \frac{(n-1)(n-2)}{4n^2} \nabla_\mu g \nabla^\mu g - \frac{1}{n} \nabla^\nu g \nabla_\nu g \right) + L_{\text{matt}}[\psi_i, \left| g \right|^{-\frac{1}{n}} g_{\mu\nu}] \]

EM reminiscent of Einstein’s 1919 traceless Theory (E.A. JHEP)
UG is NOT equivalent to GR in the gauge $g=-1$

(Although classically it is quite similar)
(work in progress)
The Bianchi identities bring the trace back into the game

\[ \frac{n - 2}{2n} \nabla_\mu R = -\frac{1}{n} \nabla_\mu T \]

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - C g_{\mu\nu} = T_{\mu\nu} \]

The constant piece of the potential, $V_0$, does not source the c.c.
Quantum corrections

Does the cc get generated by quantum corrections?

\[ g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \quad |\bar{g}| = 1 \]

Gauge symmetry: \( T_{\text{diff}}(M) \& W_{\text{Weyl}}(M) \)

Need of BRST gauge fixing

Nielsen-Kallosh ghosts (30)

\[ \nabla_\mu c^T \mu = 0. \]
The operator involving $h_{\mu\nu}, f$ and $c'$ is non-minimal. Usual heat kernel techniques do not work here. Its determinant has been computed using the (quite unwieldy) Barvinsky-Vilkovisky technique.
Background EM

\[ \bar{R}_{\mu\nu} = \frac{1}{n} \bar{R} \bar{g}_{\mu\nu} \]

\[ \bar{R} = \text{constant} \]

\[ \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} = \frac{1}{n} \bar{R}^2 = \text{constant} \]

\[ \hat{W}_4 = E_4 + 2 \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} - \frac{2}{3} \bar{R}^2 = E_4 + \text{constant} \]

Topological density modulo a dynamically irrelevant term
Divergences proportional to the EM can be eliminated by a one-loop field redefinition

\[ \delta S = \int \frac{\delta S}{\delta g_{\mu\nu}} (a \, R_{\mu\nu} + b \, R \, g_{\mu\nu}) \]

Point transformations leave formally invariant the path integral

Counterterms that vanish on-shell are gauge dependent (Kallosh)

Ergo, it is possible to find a gauge where the theory is finite, even off-shell
Consider a general gauge theory

Gauge fixing & ghost terms are together BRST exact

\[ Z[\bar{A}] \equiv \int D\bar{A} e^{-S_c[\bar{A}+A]+\int \frac{\delta S_c}{\delta \bar{A}} |_{\bar{A}} A-\xi \int L_{gf}} \]

Gauge fixing & ghost terms are together BRST exact

\[ L_{gf} = sF_{gf} \]

Let us examine the dependence of the partition function with the gauge fixing
\[
\frac{\partial Z}{\partial \xi} = \int \mathcal{D}A \left( \int L_{gf} \right) e^{-S(\bar{A}+A) + \frac{\partial S}{\partial A} |_{\bar{A}} + \xi \int L_{gf}} = \\
= \int \mathcal{D}A \ s \left( \int F_{gf} \right) e^{-S(\bar{A}+A) + \frac{\partial S}{\partial A} |_{\bar{A}} + \xi \int L_{gf}} = \\
= \int \mathcal{D}A \left( \int F_{gf} \right) s \left( S(\bar{A}+A) - \frac{\partial S}{\partial A} |_{\bar{A}} \right) e^{-S(\bar{A}+A) + \frac{\partial S}{\partial A} |_{\bar{A}} + \xi \int L_{gf}}
\]

The classical action is gauge invariant, then BRST invariant

\[s S_c = 0\]

On shell

\[\frac{\delta S_c}{\delta A} \bigg|_{\bar{A}} = 0\]

\[\frac{\partial Z}{\partial \xi} = 0\]
The one loop quantum correction reads

\[ S_\infty = \frac{1}{16\pi^2(n - 4)} \int d^4x \left( \frac{119}{90} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{83}{120} R^2 \right) = \]

\[ = \frac{1}{16\pi^2(n - 4)} \int d^4x \left( \frac{119}{90} E_4 + \text{constant} \right) \quad (25) \]

This term is non-dynamical; in GR this is not the case (Christensen-Duff)

\[ S_{GR} = \frac{1}{16\pi^2(n - 4)} \int \sqrt{|g|} d^4x \left( -\frac{1142}{135} \Lambda + \frac{53}{45} W_4 \right) \quad (26) \]
This result is not a consequence of $\bar{g} = -1$.

Rather, it is Weyl symmetry that forbids dimension zero operators.

It is possible to work out GR in the gauge $g=-1$. CC is still generated there; there is no symmetry in GR to prevent it from appearing.
How certain are we of the belief that the effective string gravitational equations are Diff invariant and not only TDiff invariant?
Impossible to tell the difference from any first quantized approach...
Session Two slides
In order not to clutter formulas

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + \bar{g}^{\frac{1}{n}} h_{\mu\nu}, \]

In terms of a unimodular background

\[ \bar{g}_{\mu\nu} = \bar{g}^{\frac{1}{n}} \tilde{g}_{\mu\nu}. \]

All covariant derivatives etc are defined wrt the background

\[ S_{UG}[g_{\mu\nu} = \bar{g}_{\mu\nu} + \bar{g}^{\frac{1}{n}} h_{\mu\nu}] = S_{UG}[g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}], \]

In order not to clutter formulas

\[ \tilde{g}_{\mu\nu} = g_{\mu\nu} \]
TDiff ghosts are transverse \( \nabla_\mu c^T_\mu = 0 \)

**Quadratic piece of the classical action**

\[
\begin{align*}
L_2 &= \frac{1}{4} h^{\mu\nu} \nabla^2 h_{\mu\nu} - \frac{n + 2}{4n^2} h \nabla^2 h + \frac{1}{2} \nabla_\mu h^{\mu\alpha} \nabla_\nu h_\alpha^\nu - \frac{1}{n} \nabla_\mu h \nabla_\nu h^{\mu\nu} + \\
&+ \frac{1}{2} h^{\alpha\beta} h_\mu^\beta \bar{R}_{\mu\alpha} - \frac{1}{n} h h^{\mu\nu} \bar{R}_{\mu\nu} - \frac{1}{2} h^{\mu\nu} h^{\alpha\beta} \bar{R}_{\mu\nu\alpha\beta} - \frac{1}{2n} h^{\mu\nu} h_{\mu\nu} \bar{R} + \frac{1}{2n^2} h^2 \bar{R}.
\end{align*}
\]

Non-unimodular background can be easily gotten by going to the Jordan frame
In order to get a nilpotent BRST operator

\[ s = s_D + s_W \]

It is necessary to define

\[ [\mathcal{D}_c^T \mu], \]

**Better use unconstrained fields**

\[ c^T_\mu = \Theta_{\mu\nu} c^\nu = (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu - R_{\mu\nu}) c^\nu = (Q_{\mu\nu} - \nabla_\mu \nabla_\nu) c^\nu \]

Swapping transversality with gauge symmetry

\[ c_\nu \rightarrow \nabla_\nu f \]

New ghosts needed in order to get

\[ s_D^2 = s_W^2 \]
\[ \{s_D, s_W\} = 0 \]
Batalin-Vilkovisky

(Grassmann number, Ghost number)

\[ h_{\mu\nu}, c_{\mu}^{(1,1)}, b_{\mu}^{(1,-1)}, f_{\mu}^{(0,0)}, \phi^{(0,2)} \]
\[ \pi^{(1,-1)}, \pi^{(1,1)}, \bar{c}^{(0,-2)}, c^{(0,0)} \]
\[ c^{(1,1)}, b^{(1,-1)}, f^{(0,0)} \]
\[
\n\nabla_\mu (c^\rho T c^{T\mu}) = 0, \quad \nabla_\mu \left[ (Q^{-1})^\mu_\nu \left( c^\rho T \nabla_\rho c^{T\nu} \right) \right] = 0.
\]

Results as above follow:

\[
s_D c^{T\mu} = c^{T\rho} \nabla_\rho c^{T\nu},
\]

\[
S_{\text{gauge-focusing}} = \int d^n x \sqrt{|g|} s(X_{TD} + X_W).
\]

\[\text{X}=(1,-1)\]

Free invertible kinetic terms

Quadratic in the quantum fields
<table>
<thead>
<tr>
<th>field</th>
<th>$s_D$</th>
<th>$s_W$</th>
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<td>$g_{\mu\nu}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$h_{\mu\nu}$</td>
<td>$\nabla_\mu c^T_\nu + \nabla_\nu c^T_\mu + c^\rho T \nabla_\rho h_{\mu\nu} + \nabla_\mu c^\rho T h_{\rho\nu} + \nabla_\nu c^\rho T h_{\rho\mu}$</td>
<td>$2c^{(1,1)} (g_{\mu\nu} + h_{\mu\nu})$</td>
</tr>
<tr>
<td>$c^{(1,1)}_{\mu}$</td>
<td>$(Q^{-1})^\mu_\nu (c^\rho T \nabla_\rho c^{T\nu}) + \nabla^\mu \phi^{(0,2)}$</td>
<td>0</td>
</tr>
<tr>
<td>$\phi^{(0,2)}$</td>
<td>0</td>
<td>0</td>
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<tr>
<td>$b^{(1,-1)}_\mu$</td>
<td>$f^{(0,0)}_\mu$</td>
<td>0</td>
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<td>$c^{(0,-2)}$</td>
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<td>$c^T_\rho \nabla_\rho f^{(0,0)}$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1. BRST transformations of the fields involved in the path integral.
The TDiff Sector

\[ X_{TD} = b^{(1,-1)}_{\mu} \left[ F^{\mu} + \rho_1 f^{(0,0)}_{\mu} \right] + \bar{c}^{(0,-2)}_{(1,1)} \left[ F^{\mu}_{2} c_{\mu} + \rho_2 \pi^{(1,1)} \right] + \\
+ c^{(0,0)}_{(1,-1)} \left[ F^{\mu}_{1} b^{(1,-1)}_{\mu} + \rho_3 \pi^{(1,-1)} \right] \]

Contains the graviton and is akin to the usual Faddeev-Popov gauge fixing

\[
\int d^n x \sqrt{|g|} \ sX_{TD} = \int d^n x \ f^{(0,0)}_{\mu} \left( F^{\mu} + \rho_1 f^{(0,0)}_{\mu} \right) - b^{(1,-1)}_{\mu} sF^{\mu} + \\
+ \pi^{(1,-1)} \left( F^{\mu}_{2} c^{(1,1)}_{\mu} + \rho_2 \pi^{(1,1)} \right) + \bar{c}^{(0,-2)} F^{\mu}_{2} \nabla_{\mu} \phi^{(0,2)} + \\
+ \pi^{(1,1)} \left( F^{\mu}_{1} b^{(1,-1)}_{\mu} + \rho_3 \pi^{(1,-1)} \right) + c^{(0,0)} F^{\mu}_{1} f^{(0,0)}_{\mu} 
\]
Completing the square

\[
f^{(0,0)}_\mu \left( F_\mu + \rho_1 f_\mu^{(0,0)} \right) + f^{(0,0)}_\mu \bar{F}_1^{\mu} c^{(0,0)}
\]

Completely the square

\[
\rho_1 \left( f^{(0,0)}_\mu + \frac{1}{2\rho_1} \left( F_\mu + \bar{F}_1^{\mu} c^{(0,0)} \right) \right)^2 - \frac{1}{4\rho_1} \left( F_\mu + \bar{F}_1^{\mu} c^{(0,0)} \right)^2
\]

\[
S_{gf} = -\frac{1}{4\rho_1} \int d^nx \sqrt{|g|} \left( F_\mu + \bar{F}_1^{\mu} c^{(0,0)} \right)^2
\]

(This is essentially Faddeev-Popov)

Consider now the fermionic terms

\[
\pi^{(1,-1)} \left( F_2^{\mu} c^{(1,1)}_\mu + \rho_2 \pi^{(1,1)} \right) + \pi^{(1,1)} \left( F_1^{\mu} b^{(1,-1)}_\mu + \rho_3 \pi^{(1,-1)} \right) =
\]

\[
= \left( \pi^{(1,-1)} - F_1^{\mu} b^{(1,-1)}_\mu (\rho_2 - \rho_3)^{-1} \right) (\rho_2 - \rho_3) \left( \pi^{(1,1)} + (\rho_2 - \rho_3)^{-1} F_2^{\mu} c^{(1,1)}_\mu \right)
\]

\[
+ F_1^{\mu} b^{(1,-1)}_\mu (\rho_2 - \rho_3)^{-1} F_2^{\mu} c^{(1,1)}_\mu
\]
The knack for choosing $F_\mu$

- Cancel nonminimial operators for graviton fluctuations
- Weyl invariant so that the gauge fixings decouple

\[ F_\mu = \nabla^\nu h_{\mu\nu} - \frac{1}{n} \nabla_\mu h \]
In terms of unconstrained fields

\[ sF_\mu = (g_{\mu\alpha} \Box + R_{\mu\alpha}) (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu - R_{\mu\nu}) c^{\nu(1,1)} = \]
\[ = \Box^2 c^{(1,1)}_\mu - \nabla_\mu \Box \nabla_\nu c^{\nu(1,1)} - 2R_{\mu\rho} \nabla^\rho \nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} - \]
\[ - 2\nabla_\sigma R_{\mu\rho} \nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho} R^{\rho\nu} c^{(1,1)}_\nu \]

\[ S_{bc} = \int d^n x \, b^{(1,-1)}_\mu \left( \Box^2 c^{(1,1)}_\mu - \nabla_\mu \Box \nabla_\nu c^{\nu(1,1)} - 2R_{\mu\rho} \nabla^\rho \nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} - \right. \]
\[ -2\nabla_\sigma R_{\mu\rho} \nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho} R^{\rho\nu} c^{(1,1)}_\nu \right) \]

Choose

\[ F_1^{\mu} b^{(1,-1)}_\mu = \nabla^\alpha b^{(1,-1)}_\alpha \]
\[ F_2^{\mu} c^{(1,1)}_\mu = \nabla^\mu c^{(1,1)}_\mu \]
\[ (\rho_2 - \rho_3)^{-1} = -\Box \]

Then

\[ F_1^{\mu} b^{(1,-1)}_\mu (\rho_2 - \rho_3)^{-1} F_2^{\mu} c^{(1,1)}_\mu = - \left( \nabla^\nu b^{(1,-1)}_\nu \right) \Box \nabla^\mu c^{(1,1)}_\mu = b^{(1,-1)}_\nu \nabla^\nu \Box \nabla^\mu c^{(1,1)}_\mu \]
\[ S_{bc} + S_{gf}^{bc} = \int d^n x \sqrt{|g|} b^{(1,-1)} \left( \Box^2 c^{(1,1)}_{\mu} - 2 R_{\mu\rho} \nabla^\rho \nabla_\nu c^{\nu(1,1)}_{\mu} - R_{\mu\rho} c^{\rho(1,1)}_{\nu} - 2 \nabla_\sigma R_{\mu\rho} \nabla^\sigma c^{(1,1)}_{\rho\nu} - R_{\mu\rho} R^{\rho\nu} c^{(1,1)}_{\nu} \right) \]

\[ S_{\pi} = \int d^n x \, \pi^{(1,-1)} \Box^{-1} \pi^{(1,1)} \]

\[ S_{\psi c'} = -\int d^n x \, \frac{1}{4\rho_1} \left[ F^\mu_1 \psi^{(0,0)} \tilde{F}^\mu_1 c'^{(0,0)} + 2 F_\mu \tilde{F}^\mu_1 c'^{(0,0)} \right] = \]

\[ = -\int d^n x \, \frac{1}{4\rho_1} \left[ g^{2-n} \nabla_\mu c'^{(0,0)} \nabla^\mu c^{(0,0)} + 2 g^{2-n} F_\mu \nabla^\mu c^{(0,0)} \right] \]

\[ S_{\tilde{c} \phi} = \int d^n x \, \tilde{c}^{(0,-2)} \Box \phi^{(0,2)} \]
Summarizing

\[ S^{TDiff}_{BRST} = \int d^nx b^\mu \left( \Box^2 c^{(1,1)}_{\mu} - 2R_{\mu\rho} \nabla^\rho \nabla_{\nu} c^{(1,1)}_{\nu\mu} - \Box R_{\mu\rho} c^{(1,1)}_{\rho} \right) - 2\nabla_\sigma R_{\mu\rho} \nabla^\sigma c^{(1,1)}_{\rho} - R_{\mu\rho} R^{\rho\nu} c^{(1,1)}_{\nu\rho} \right) + \bar{c}^{(0,-2)} \Box \phi^{(0,2)} + \pi^{(1,-1)} \Box -1 \pi^{(1,1)} - \frac{1}{4\rho_1} \left( F_\mu F^\mu + g \frac{2-n}{2n} \nabla_{\mu} c^{(0,0)} \nabla^{\mu} c^{(0,0)} + 2g \frac{2-n}{4n} F_\mu \nabla^{\mu} c^{(0,0)} \right) = \]

\[ = S_{bc} + S_{gf}^{bc} + S_{\bar{c}\phi} + S_{\pi} + S_{hc'} \]
The Weyl sector

\[ X_W = \nabla_\mu b^{(-1)} \nabla^\mu \left( f^{(0,0)} - \alpha g(h) \right) \]

\[ S_{BRST}^{Weyl} = \int d^n x \left[ \nabla_\mu f^{(0,0)} \nabla^\mu \left( f^{(0,0)} - \alpha g(h) \right) - \alpha \nabla_\mu b^{(-1)} \nabla^\mu \left( s g(h) \right) \right] \]

simplest choice

\[ g(h) = h \]

\[ S_{Weyl} = \int d^n x \nabla_\mu f^{(0,0)} \nabla^\mu \left( f^{(0,0)} - \alpha h \right) - 2n\alpha \nabla_\mu b^{(-1)} \nabla^\mu c^{(1,1)} = \]

\[ = \int d^n x \left( -f^{(0,0)} \Box f^{(0,0)} + \frac{\alpha}{2} f^{(0,0)} \Box h + \frac{\alpha}{2} h \Box f^{(0,0)} \right) + 2n\alpha b^{(-1)} \Box c^{(1,1)} = S_W + S_{hf} \]

Keeping the free parameter alpha until we put the result on shell yields a powerful check of our computations.
One Loop Effective Action

\[ W_\infty = W_\infty^{(UG)} + W_\infty^{(bc)} + W_\infty^{(\pi)} + W_\infty^{(\bar{c}\bar{\phi})} + W_\infty^{(W)} \]

\[ S^{UG} = S_2 + S_{hc'} + S_{hf} \]

General one-loop structure

\[ S = \int d^n x \Psi^A F_{AB} \Psi^B \]

For example

\[ \Psi^A = \begin{pmatrix} b \\ c \end{pmatrix} \]

\[ F_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times n\alpha\Box \]
Minimal operators

\[ F_{AB} = \gamma_{AB} \Box^m + K_{AB} \]

The computation of their determinant is a standard Schwinger-DeWitt calculation

Non-minimal operators need some care

\[ \mathcal{L} = \frac{1}{4} h^\mu{}^\nu \Box h_{\mu \nu} - \frac{1}{4n} h \Box h + \frac{1}{2} h^{\alpha \beta} h^\mu{}_{\beta} R_{\mu \alpha} + \frac{1}{2} h^\mu{}^\nu h^{\alpha \beta} R_{\mu \alpha \nu \beta} - \frac{1}{n} h h^\mu{}^\nu R_{\mu \nu} - \frac{1}{2n} h^\mu{}^\nu h_{\mu \nu} R + \]
\[ + \left( -f \Box f + \frac{\alpha}{2} f \Box h + \frac{\alpha}{2} h \Box f \right) - \frac{1}{2} \left( \nabla_\mu c^{(0,0)} \nabla_c^{(0,0)} + 2 \left( \nabla_\nu h^\nu{}_{\mu} - \frac{1}{n} \nabla_\mu h \right) \nabla_\mu c^{(0,0)} \right) + \]
\[ + \frac{1}{2n^2} h^2 R \]

\[ \rho_1 = \frac{1}{2} \]
The nonminimal operator

\[ \Psi^A = \begin{pmatrix} h^{\mu\nu} \\ f \\ c' \end{pmatrix} \]

\[ \gamma_{AB} = \begin{pmatrix}
-\frac{1}{4} \left( \frac{1}{n} \mathcal{K}_{\mu\nu\rho\sigma}^{\alpha\beta} - \mathcal{P}_{\mu\nu\rho\sigma}^{\alpha\beta} \right) g_{\alpha\beta} & \frac{\alpha}{2} g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \\
\frac{\alpha}{2} g_{\rho\sigma} & -1 & 0 \\
-\frac{1}{2} g_{\rho\sigma} & 0 & \frac{1}{2}
\end{pmatrix} \]

\[ M_{AB} = \begin{pmatrix}
M_{hh} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \]

\[ J_{AB}^{\alpha\beta} = \begin{pmatrix}
0 & 0 & \frac{1}{4} \left( g_{\mu}^{\alpha} g_{\nu}^{\beta} + g_{\nu}^{\alpha} g_{\mu}^{\beta} \right) \\
0 & 0 & 0 \\
\frac{1}{4} \left( g_{\mu}^{\alpha} g_{\nu}^{\beta} + g_{\nu}^{\alpha} g_{\mu}^{\beta} \right) & 0 & 0
\end{pmatrix} \]

\[ M_{hh} = \left( \frac{1}{2} \mathcal{P}_{\mu\nu\rho\sigma}^{\alpha\beta} - \frac{1}{n} \mathcal{K}_{\mu\nu\rho\sigma}^{\alpha\beta} \right) R_{\alpha\beta} - \frac{1}{2n} \left( \mathcal{P}_{\mu\nu\rho\sigma}^{\alpha\beta} - \frac{1}{n} \mathcal{K}_{\mu\nu\rho\sigma}^{\alpha\beta} \right) \gamma_{\alpha\beta} R + \frac{1}{2} R_{(\mu\nu\rho\sigma)} \]

\[ \mathcal{P}_{\mu\nu\rho\sigma}^{\alpha\beta} = \frac{1}{4} \left( g_{\mu\nu} \delta_{\rho\sigma}^{(\alpha\beta)} + g_{\mu\sigma} \delta_{\nu\rho}^{(\alpha\beta)} + g_{\nu\rho} \delta_{\mu\sigma}^{(\alpha\beta)} + g_{\nu\sigma} \delta_{\mu\rho}^{(\alpha\beta)} \right) \]

\[ \mathcal{K}_{\mu\nu\rho\sigma}^{\alpha\beta} = \frac{1}{2} \left( g_{\mu\nu} \delta_{\rho\sigma}^{(\alpha\beta)} + g_{\rho\sigma} \delta_{\mu\nu}^{(\alpha\beta)} \right) \]
$D_{AB}(\nabla) = \gamma_{AB} \Box + J_{AB}^{\alpha \beta} \nabla_\alpha \nabla_\beta.$

$F_{AB}(\nabla|\lambda) = D_{AB}(\nabla|\lambda) + M_{AB}, \quad 0 \leq \lambda \leq 1$

$F_{AB}(\nabla|\lambda) = \gamma_{AB} \Box + \lambda J_{AB}^{\alpha \beta} \nabla_\alpha \nabla_\beta + M_{AB}$

$\hat{F}(\lambda) \hat{G}(\lambda) = \mathbb{I},$

**Schwinger**

$$W(\lambda) = W(0) - \frac{1}{2} \int_0^\lambda d\lambda' Tr \left[ \frac{d\hat{F}(\lambda)}{d\lambda'} \hat{G}(\lambda') \right]$$

$$\hat{D}(k) = D_B^A(k) = \gamma^{AC} D_{CB}(k)$$

$$\hat{D}^{-1}(k) = \frac{\hat{K}(k)}{(k^2)^m}$$

$$\hat{D}(k) \hat{K}(k) = (k^2)^m \mathbb{I}.$$
In our case the computation simplifies somewhat

\[
\hat{F}(\nabla)\hat{K}(\nabla) = \Box^m + \hat{M}(\nabla)
\]

\[
\hat{G} = -\hat{K} \sum_{p=0}^{4} (-1)^p \hat{M}_p \frac{\Box^m}{m(m+1)} + O(m^5)
\]

\[
\hat{G} = -\hat{K} \sum_{p=0}^{4} (-1)^p \hat{M}_p \frac{\Box^m}{m(m+1)} + O(m^5)
\]

\[
\hat{M}_0 = \mathbb{I}
\]

\[
\hat{M}_{p+1} = \hat{M} \hat{M}_p + [\Box^m, \hat{M}_p]
\]

\[
W(\lambda) = W(0) - \frac{1}{2} \int_0^\lambda d\lambda' Tr \left[ \hat{J}^{\alpha\beta} \nabla_\alpha \nabla_\beta \left\{ \hat{K} \left( -\frac{\mathbb{I}}{3} + \hat{M} \frac{\mathbb{I}}{6} - 3[\Box, \hat{M}] \frac{\mathbb{I}}{7} - \hat{M}^2 \frac{\mathbb{I}}{9} \right) \right\} \right]
\]
The minimal piece is a standard computation

\[ W(0) = \frac{1}{16\pi^2} \frac{1}{n-4} \int d^n x \left\{ \frac{16}{15} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \left( \frac{2}{8\alpha^2 - 1} - \frac{46}{15} \right) R_{\mu\nu} R^{\mu\nu} + \left( \frac{13}{24} + \frac{1}{2 - 16\alpha^2} \right) R^2 \right\} \]

(3.30)

The nonminimal part is reduced through the Barvinsky-Vilkovisky technique to the computation of functional traces

\[ Tr \left( O_{\nu_1 \nu_2 \ldots \nu_j} \nabla_{\mu_1} \nabla_{\mu_2} \ldots \nabla_{\mu_p} \frac{I}{n} \right) \]
Effective Action = sum of gauge piece plus BRST exact (ghosts)

\[ W_\infty = W_{\infty}^{UG} + W^{bc} + W^{\pi} + W^{\phi\phi} + W^{W} \]

\[ W_\infty = \frac{1}{16\pi^2} \frac{1}{n-4} \int d^n x \left( \frac{16}{15} R_{\mu \nu \alpha \beta} R^{\mu \nu \alpha \beta} + \left( \frac{1}{6\alpha^2} - \frac{46}{15} \right) R_{\mu \nu} R^{\mu \nu} + \left( \frac{1}{3} - \frac{1}{24\alpha^2} \right) R^2 \right) \]

Christensen & Duff

\[ W_{\infty}^{GR} = \frac{1}{16\pi^2 (n-4)} \int \sqrt{|g|} d^4 x \left( \frac{53}{45} W_4 - \frac{1142}{135} A^2 \right) \]
On-shell divergences

\[ R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} = 0 \]

\[ R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = E_4 \]

\[ R_{\mu\nu} R^{\mu\nu} = \frac{1}{4} R^2 \]

\[ R = \text{constant} \]

\[ W_{\text{on-shell}}^{\infty} = \frac{1}{16\pi^2} \frac{1}{n-4} \int d^n x \left( \frac{119}{90} E_4 - \frac{83}{120} R^2 \right) \]

\[ \frac{1}{M^{2\beta-n}} \int d^n x \ R^\beta \]
Conclusions

Unimodular gravity solves one aspect of the cosmological constant problem, namely, why the vacuum energy does not generate a huge value for it.

It is a Wilsonian solution to the problem of weighing vacuum energy (No need to modify UV behavior)
In spite of the fact that UG is quite close to GR, which can be worked out in the gauge $g=-1$, this mechanism does not work in GR because there is no Weyl symmetry there.

EM should be derived BEFORE gauge fixing!

$\sqrt{|g|} \text{ (Something)} = \sqrt{|g|} \ 8\pi G \text{ (Something Else)}$
Incidentally, the low energy limit of string theory could as well be UG instead of GR.

With on-shell amplitudes only it is not possible to discriminate the two.
Backup
Schwinger–DeWitt

\[ W = -\frac{1}{2} \log \det D \]

\[ K(s, f, D) = Tr \left( f e^{-sD} \right) \]

\[ \zeta(s, D) = \Gamma(s)^{-1} \int_0^\infty dt \ t^{s-1} \langle \psi_i | e^{-tD} | \psi_i \rangle = \int_0^\infty dt \ t^{s-1} K(t, D) \]

\[ \log (\det(D)) = -\int_0^\infty \frac{dt}{t} \langle \psi_i | e^{-tD} | \psi_i \rangle = -\int_0^\infty \frac{dt}{t} K(t, D) \]

\[ W = -\frac{1}{2} \int_0^\infty \frac{dt}{t} K(t, D) \]

Regularized Effective Action

\[ W_{reg} = -\frac{1}{2} \mu^{2s} \int_0^\infty \frac{dt}{t^{1-s}} K(t, D) = -\frac{1}{2} \mu^{2s} \Gamma(s) \zeta(s, D) \]
Short time expansion

\[ W_{\text{reg}} = -\frac{1}{2} \left( \frac{1}{s} - \gamma_E + \log (\mu^2) \right) \zeta(0, D) - \frac{1}{2} \zeta'(0, D) \]

\[ W_{\text{ren}} = -\frac{1}{2} \zeta'(0, D) - \frac{1}{2} \log (\mu^2) \zeta(0, D) \]

Zeta Function Out of Heat Kernel

\[ K(s, D) = \frac{1}{(4\pi s)^n/2} \sum_{i=0} a_i(D) \]

Pole Part of the Effective Action

\[ \zeta(0, D) = a_n(D) \]

\[ W_\infty = \frac{1}{(4\pi)^{n/2}} \frac{1}{n-4} a_n(D) \]
Laplace-type Operator

\[ D = -\gamma_{AB} \Box + N^\mu_{AB} \nabla_\mu + M_{AB} \]

\[ D = \nabla + \omega, \quad D = -\gamma_{AB} D^2 - E_{AB} \]

\[ \omega^A_{\mu B} = \frac{1}{2} \gamma^{AC} N_{\mu CB} \]

\[ E^A_B = \gamma^{AC} (-M_{CB} - \omega_{\mu CF} \omega^\mu_{BF} - \nabla_\mu \omega^\mu_{CB}) \]

\[ a_4 = \frac{1}{360} \int d^n x \sqrt{|g|} \ Tr(60 \Box E + 60 R E + 180 E^2 + 12 \Box R + 5 R^2 - 2 R_{\mu \nu} R^{\mu \nu} + 2 R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} + 30 \hat{R}_{\mu \nu} \hat{R}^{\mu \nu}) \]

(A.15)

\[ [D_\mu, D_\nu] V_A = \hat{R}_{\mu \nu A}^B V_B \]
Starting Point: The d’Alembert Operator

\[
[\nabla_\mu, \nabla_\nu] h^{\rho\sigma} = \mathcal{R}_\mu^{\rho\sigma} h^{\alpha\beta} \]

\[
\mathcal{R}_\mu^{\rho\sigma} = \frac{1}{2} \left( R_\alpha^{\rho\sigma} g^\beta_\mu + R_\beta^{\rho\sigma} g^\alpha_\mu + R_\mu^{\rho\sigma} g^\alpha_\beta + R_\nu^{\rho\sigma} g^\beta_\nu \right)
\]

\[
a_\Delta(\Box) = \frac{1}{360} \int d^n x \sqrt{|g|} \left( 12 \Box R + 5 R^2 - 2 R_{\mu\nu} R^{\mu\nu} + 2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right)
\]

\[D = -\Box\]
Quartic Operator

\[ D = \gamma_{AB} \Box^2 + \Omega^{\mu\nu\alpha}_{AB} \nabla_\mu \nabla_\nu \nabla_\alpha + J^{\mu\nu}_{AB} \nabla_\mu \nabla_\nu + H^\mu_{AB} \nabla_\mu + P_{AB} \]

\[ W_\infty = \frac{1}{16\pi^2} \frac{1}{n-4} \int d^n x \sqrt{|g|} \text{Tr} \left( \frac{1}{90} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - \frac{1}{90} R_{\mu\nu} R^{\mu\nu} + \frac{1}{36} R^{2} \right) - \hat{P} + \frac{1}{6} \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} - \frac{1}{6} J^{(\mu\nu)} R_{\mu\nu} + \frac{1}{12} J^\mu R + \frac{1}{48} (J^\mu)^2 + \frac{1}{24} J_{(\mu\nu)} J^{(\mu\nu)} - \frac{1}{2} J_{[\mu\nu]} \hat{R}_{\mu\nu} \right) \]
The (b c) System

\[ \int d^n x \ b^\mu \left( \Box c^{(1,1)}_{\mu} - 2 R_{\mu \rho} \nabla^\rho \nabla_\nu c^{(1,1)}_{\nu} - \Box R_{\mu \rho} c^{(1,1)}_{\rho} - 2 \nabla_\sigma R_{\mu \rho} \nabla^\sigma c^{(1,1)}_{\rho} - R_{\mu \rho} R^{\rho \nu} c^{(1,1)}_{\nu} \right) \]

\[ J_{\alpha \beta}^{\mu \nu} = -2 R_{\alpha \rho}^{\mu} \delta_{\beta}^{\nu} \]

\[ H_{\alpha \beta}^{\mu} = -2 \nabla^{\mu} R_{\alpha \beta} \]

\[ P_{\alpha \beta} = -\Box R_{\alpha \beta} - R_{\alpha \rho} R_{\beta}^{\rho} \]

\[ [\nabla_\mu, \nabla_\nu] c^\alpha = \tilde{R}_{\mu \nu}^{\alpha \beta} c_\beta = R_{\mu \nu}^{\alpha \beta} c_\beta \]

\[ W_{\infty}^{bc} = \frac{1}{16 \pi^2 n} \frac{1}{n-4} \int d^n x \left( \frac{11}{45} R_{\mu \nu \alpha \beta} R_{\mu \nu \alpha \beta} - \frac{41}{45} R_{\mu \nu} R_{\mu \nu} - \frac{1}{18} R^2 \right) \]
The \((\pi, \pi')\) System

\[
S_\pi = \int d^n x \, \pi^{(1,-1)} \Box^{-1} \pi'^{(1,1)}
\]

\[
a_4(\Box) = -\frac{1}{360} \int d^n x \, \left( 12 \Box R + 5 R^2 - 2 R_{\mu\nu} R^{\mu\nu} + 2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right)
\]
The \((\bar{c}\phi)\) System

\[
\int d^n x \, c^{(0,-2)} \Box c^{(0,2)}
\]

\[
W_{\infty}^{\bar{c}\phi} = \frac{1}{16\pi^2} \frac{1}{n-4} \frac{1}{180} \int d^n x \left( 12 \Box R + 5 R^2 - 2 R_{\mu\nu} R^{\mu\nu} + 2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right)
\]
The (w) System

\[2n\alpha \int d^n x \ b\Box c\]

\[W^W_\infty = -\frac{1}{16\pi^2} \frac{1}{n} \frac{1}{4180} \int d^n x \ (12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})\]
Functional Traces

\[ (F(\nabla))^{-n} = \frac{1}{(n-1)!} \left[ \left( \frac{d}{dm^2} \right)^{n-1} G(m^2) \right]_{m^2=0} \]

\[ G(m^2) = \int_0^\infty \exp(-sm^2) \exp(-sF(\nabla)) \]

\[ \exp(-s\hat{F}(\nabla))\delta(x, x') = \frac{1}{(4\pi)^{n/2}} \frac{D^{1/2}(x, x')}{s^{n/2}} \exp \left( -\frac{\sigma(x,x')}{2s} \right) \hat{\Omega}(s|x, x') \]

\[ \hat{\Omega}(s|x, x') = \sum_{n=0}^\infty s^n \hat{a}_n(x, x') \]
\[ D(x, x') = \left| \det \left( \frac{\partial \sigma}{\partial \omega^\mu \partial \omega'^\nu} \right) \right| \]
\[ D(x, x') = g^{1/2}(x) g^{1/2}(x') \Delta(x, x') \]

\[ \frac{\Gamma}{\Box^n} = \frac{1}{(n-1)!} \int_0^\infty ds \ s^{n-1} \exp(-s \Box) \]

\[ \int_0^\infty \frac{ds}{s^{n/2+k}}, \text{ with } k = -1, 0, 1 \]
\[ \nabla_\mu \nabla_\nu \square = \frac{\sqrt{g}}{8(n-4)\pi^2} \frac{1}{2} \left\{ -g_{\mu\nu} \left( \frac{1}{180} R_{\alpha\beta\lambda\sigma} R^{\alpha\beta\lambda\sigma} - \frac{1}{180} R_{\alpha\beta} R^{\alpha\beta} + \frac{1}{72} R^2 + \frac{1}{30} \Box R \right) \right\} + \frac{1}{45} R^{\alpha\beta} R_{\alpha\mu\beta\nu} + \right. \\
+ \frac{1}{45} R_{\alpha\beta\lambda\mu} R^{\alpha\beta}_{\lambda\mu} - \frac{2}{45} R_{\mu\alpha} R^\alpha_{\nu} + \frac{1}{18} RR_{\mu\nu} + \frac{1}{30} \Box R_{\mu\nu} + \frac{1}{10} \nabla_\mu \nabla_\nu R \left. \right\} - \frac{1}{12} g_{\mu\nu} \hat{R}_{\alpha\beta} \hat{R}^{\alpha\beta} + \frac{1}{6} R \hat{\nabla}_\mu + \frac{1}{6} \hat{R}_{\mu\alpha} \hat{R}^\alpha_{\nu} + \frac{1}{6} \hat{R}_{\nu\alpha} \hat{R}^\alpha_{\mu} - \frac{1}{6} \nabla_\mu \nabla^\alpha \hat{R}_{\alpha\nu} - \frac{1}{6} \nabla_\nu \nabla^\alpha \hat{R}_{\alpha\mu} \right\} \]
\[ p = 2n - 1. \]

\[
\nabla_\mu \frac{\mathbb{1}}{\Box} = \frac{\sqrt{g}}{8(n-4)\pi^2} \left( \frac{1}{12} \nabla_\mu R \mathbb{1} - \frac{1}{6} \nabla^\nu \hat{R}_{\nu\mu} \right)
\]
\[
p = 2n - 2
\]

\[
\nabla_\mu \nabla_\nu \frac{I}{g} = -\frac{\sqrt{g}}{8(n-4)\pi^2} \left[ \frac{1}{6} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{2} \bar{R}_{\mu\nu} \right]
\]

\[
\nabla_\alpha \nabla_\beta \nabla_\mu \nabla_\nu \frac{I}{g} = -\frac{\sqrt{g}}{8(n-4)\pi^2} \left( \frac{1}{6} \frac{1}{4} \left\{ \frac{2}{6} R_{\mu\beta\nu\alpha} - \frac{2}{6} R_{\nu\beta\mu\alpha} + g_{\mu\nu} \left( \frac{1}{6} R_{\alpha\beta} I - \frac{1}{2} \bar{R}_{\alpha\beta} \right) + g_{\alpha\nu} \left( \frac{1}{6} R_{\mu\alpha\beta} I + \frac{1}{2} \bar{R}_{\mu\alpha\beta} \right) \right\} \right)
\]

\[
\nabla_\alpha \nabla_\beta \nabla_\mu \nabla_\nu \nabla_\lambda \frac{I}{g} = -\frac{\sqrt{g}}{8(n-4)\pi^2} \left( \frac{1}{6} \left\{ g_{\mu\nu\alpha\beta} \bar{R}_{\alpha\lambda} + g_{\mu\nu\alpha\sigma} \bar{R}_{\beta\lambda} + g_{\mu\alpha\beta\sigma} \bar{R}_{\nu\lambda} \right\} \right)
\]
Symmetrized outer products of metric tensors

\[ g^{(0)} = 1 \]
\[ g^{(1)}_{\mu\nu} = g_{\mu\nu} \]
\[ g^{(2)}_{\mu\nu\alpha\beta} = g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} + g_{\mu\nu}g_{\alpha\beta} \]
\[ g^{(3)}_{\mu\nu\alpha\beta\sigma\lambda} = g_{\mu\nu}g^{(2)}_{\alpha\beta\sigma\lambda} + g_{\mu\alpha}g^{(2)}_{\nu\beta\sigma\lambda} + g_{\mu\beta}g^{(2)}_{\nu\alpha\sigma\lambda} + g_{\mu\alpha}g^{(2)}_{\nu\beta\lambda} + g_{\mu\beta}g^{(2)}_{\nu\alpha\lambda} + g_{\mu\nu}g^{(2)}_{\alpha\beta\lambda} \]
\[ g^{(n+1)}_{\mu_1...\mu_{2n+2}} = \sum_{i=2}^{2n+2} g_{\mu_1\mu_i}g^{(n)}_{\mu_2...\mu_{i-1}\mu_{i+1}\mu_{2n+2}} \]
\[ \hat{B}_{\alpha\beta} = \frac{1}{24} R_{\alpha\beta} + \frac{1}{8} \hat{R}_{\alpha\beta} \]
\[ p = 2n - 4 \]

\[ \nabla_{\mu_1} \cdots \nabla_{\mu_{2n-4}} \mathbb{I}^n = -\frac{\sqrt{g}}{8(n - 4)\pi^2} \frac{g^{(r - 2)}_{\mu_1 \cdots \mu_{2n - 4}}}{2^{n - 2}(n - 1)!} \]