Global anomalies in 8d and universality of F-theory

Iñaki García-Etxebarria

MAX-PLANCK-GESELLSCHAFT

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

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String theory vs. Quantum Field Theory

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This is too wide a question at this point, so I will ask instead whether all **supersymmetric** QFTs admit an embedding in String Theory.

Known results for supersymmetric universality

Once we assume supersymmetry we can make progress:

- In more than 11 dimensions every susy Lagrangian theory includes excitations of helicity larger than two (except, possibly, for F-theory viewed as a theory of signature $(10, 2)$ [Vafa '96]).

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- In 10d we have IIA and IIB sugra in the $\mathcal{N} = 2$ sector. Pure $\mathcal{N} = 1$ gauge theory is anomalous. If we couple it to supergravity we can cancel the anomalies using the Green-Schwarz mechanism [Green, Schwarz '84]. This is possible for the gauge algebras

$$\mathfrak{g} \in \{\mathfrak{e}_8 \oplus \mathfrak{e}_8, \mathfrak{so}(32), \mathfrak{e}_8 \oplus \mathfrak{u}(1)^{248}, \mathfrak{u}(1)^{496}\} . \quad (1)$$

  The first two possibilities are realized by string theory, while the second two are more subtly inconsistent [Adams, DeWolfe, Taylor '10]. So string theory is also universal in 10d.

String universality below 10d

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In this talk I will describe some progress in narrowing this gap in the $d = 8$ case (and thus also $d = 9$).

(In 6d local anomalies are powerful again. Here we have many choices, both in the field theory and the string theory sides. The status of 6d universality is not clear yet, but important progress is being made regularly. [Kumar, Taylor '09], [. . .])

String compactifications down to 9d

There are four known components of the $\mathcal{N} = 1$ moduli space one can construct this way (for a detailed analysis see [Aharony, Komargodski, Patir '07])

- **Rank 2 (a):**
  - M-theory on the Klein bottle.
- **Rank 2 (b):**
  - IIA with $O8^+$ and $O8^-$.
- **Rank 10:**
  - M-theory on Möbius band.
  - CHL string. [Chaudhury, Hockney, Lykken '95]
- **Rank 18:**
  - M-theory on the cylinder.
  - Heterotic on S^1.
  - IIA with two $O8^-$ planes and 16 D8s.

String compactifications down to 8d

We obtain three possible $\mathcal{N}=1$ 8d theories by putting the previous $\mathcal{N}=1$ theories on an S^1. The resulting theories are neatly described in IIB language (on $T^2/(\mathcal{I}\Omega(-1)^{F_L})$):

- **Rank 4:** IIB with two $O7^-$ and two $O7^+$.
- **Rank 12:** IIB with three $O7^-$, one $O7^+$ and 8 D7s.
- **Rank 20:** IIB with four $O7^-$ and 16 D7s.

All these cases can also be described in F-theory, possibly with frozen singularities. (For a detailed discussion of the moduli spaces and dual pictures, see [de Boer, Dijkgraaf, Hori, Keurentjes, Morgan, Morrison, Sethi '01] and [Taylor '11].)

So we are really asking whether F-theory is universal in 8d.

Non-abelian enhancements

$\mathcal{N}=1$ theories in 8d have a complex scalar in the vector multiplet. Giving a generic vev to these scalars costs no energy, and breaks the gauge algebra to $\mathfrak{u}(1)^{\mathrm{rk}}$. The set of all vacua accessed in this way is the *Coulomb branch*.

At certain points in the Coulomb branch there can be non-abelian enhancements. The enhancements in the known backgrounds are to $\mathfrak{su}(N)$, $\mathfrak{so}(2N)$, $\mathfrak{sp}(N)$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$.

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We will aim to explain why the other algebras

$$\mathfrak{so}(2N+1) \qquad ; \qquad \mathfrak{f}_4 \qquad \text{and} \qquad \mathfrak{g}_2$$

do not appear.

Summary of results

We find that 8d $\mathcal{N}=1$ theories with algebra $\mathfrak{f}_4$ and $\mathfrak{so}(2N+1)$ for $N \geq 3$ do not exist quantum mechanically, due to an anomaly.

We find no anomaly for $\mathfrak{su}(N)$, $\mathfrak{so}(2N)$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ and $\mathfrak{g}_2$.

We find no ordinary global anomaly for $\mathfrak{sp}(N)$ (associated to π_8), but there is an anomaly of a more subtle kind. We conjecture that it is cancelled by coupling to an appropriate TQFT (the *topological Green-Schwarz mechanism*), but we have not been able to write the TQFT down, so perhaps these theories are ultimately inconsistent.

A motivating puzzle: $\mathfrak{so}(2N+1)$ in 8d?

In the IIB picture, in perturbation theory, we also have the possibility of putting "half" a D7 on top of the $O7^-$ plane. This would lead to a $\mathfrak{so}(2N+1)$ gauge algebra in 8d.

This seems problematic non-perturbatively:

- There is no monodromy associated to $\mathfrak{so}(2N+1)$ in the Kodaira classification.
- There is no natural "frozen" flux in the F-theory realization that could lead to this. [Hyakutake, Imamura, Sugimoto '00] [de Boer, Dijkgraaf, Hori, Keurentjes, Morgan, Morrison, Sethi '01] [Bergman, Gimon, Sugimoto '01] [Tachikawa '15]
- A D3 probe has a global $SU(2)$ anomaly. [Hyakutake, Imamura, Sugimoto '00] [Witten '82].

The first two arguments are potentially a limitation of model building tools. The last one is more serious, in that it can signal an inconsistency.

Review of anomalies

Consider a (Lagrangian) theory $\mathcal{T}$ with some global symmetry G. We can introduce a background connection A_G for G, and compute the path integral

$$Z(A_G) = \int [D\psi] e^{-S(A_G,\psi)} \tag{2}$$

where ψ are some fundamental fields. (Only the fermionic fields, and the connection they couple to, matter for my discussion.)

Denote by $\mathcal{M}$ the space of all physically inequivalent A_G. We have an anomaly whenever $Z(A_G)$ is not well defined as a function on the manifold $\mathcal{M}$:

- Non-invariance under small loops (curvature) in $\mathcal{M}$: *local anomaly*.
- Non-invariance under parallel transport for non-trivial loops in $\mathcal{M}$: *global anomalies*.

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This means that for Lagrangian theories anomalies are at most phases: for any field ψ in a representation R, we can include an extra field $\tilde{\psi}$ in a rep $\overline{R}$ (and with an action which is the conjugate of that for ψ), and then the full matter content can be made massive. So

$$Z(A_G) = Z_\psi(A_G)Z_{\tilde{\psi}}(A_G) = Z_\psi(A_G)\overline{Z_\psi(A_G)} = \left|Z_\psi(A_G)\right|^2 . \quad (3)$$

Since the $\psi + \tilde{\psi}$ theory is gappable, we have that $\left|Z_\psi(A_G)\right|$ is a well defined function on $\mathcal{M}$.

Review of anomalies

In general, $Z(A_G)$ is a section of some bundle over $\mathcal{M}$. If the bundle is non-trivial the theory is still consistent; we say that we have a 't Hooft anomaly. For example, the $SU(4)_R$ symmetry of $\mathcal{N} = 4$ $SU(N)$ SYM has such an anomaly in 4d ($\mathrm{Tr}(F_R^3) \neq 0$), but the theory is fine, and the symmetry is unbroken.

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We will consider the case in which there are no local anomalies. How do we detect a possible global anomaly?

The "traditional" global anomaly

Consider a symmetry transformation $g\colon \mathbb{R}^d \to G$. We impose that $g \to 1$ at infinity, so it leads to a proper gauge symmetry. The resulting set of transformations are topologically classified by maps $S^d \to G$ up to continuous deformations, i.e. by $\pi_d(G)$.

Now, for any choice of $[g] \in \pi_d(G)$, pick a representative g and consider the family of (not pure gauge) connections

$$A_G(g;t) = f(t)g^{-1}dg \tag{4}$$

for some smooth $f(t)$ such that $f(-\infty) = 0$ and $f(+\infty) = 1$. This defines a loop in the space of connections (modulo gauge transformations). So there is a global anomaly if

$$\frac{Z(A_G(g;+\infty))}{Z(A_G^g(g;-\infty))} = \frac{Z(0^g)}{Z(0)} = e^{i\mathcal{A}} \neq 1\,. \tag{5}$$

The "traditional" global anomaly

We will be interested in the case in which the fermions are *real*. This means that the mass coupling

$$m\psi\psi = 0 \tag{6}$$

does not break G, but it identically vanishes. But we can add an extra copy of the fermions, and introduce a mass coupling

$$m\psi_1\psi_2 \neq 0 \tag{7}$$

This implies that $Z(A_G)^2$ is well defined, so the anomaly is $\mathbb{Z}_2$-valued (i.e. $e^{i\mathcal{A}} = \pm 1$ at most).

The “traditional” global anomaly

Consider an example: 4d Weyl fermion ψ_1 in the fundamental of $SU(2)$. This is a real fermion (the mass term is allowed, but it identically vanishes).

Famously [Witten ’82], this system has a global anomaly:

$$Z(0) = -Z(0^g) \tag{8}$$

for $[g]$ the non-trivial generator of $\pi_4(SU(2)) = \mathbb{Z}_2$.

The Elitzur and Nair approach

In his original argument, Witten relates the anomaly to the mod 2 index in the five dimensional mapping torus for (S^4, g), by viewing $A_G(t)$ as a connection on the mapping torus. This is hard to generalize. An easier to generalize argument is due to Elitzur and Nair. [Witten '83] [Elitzur, Nair '84]

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The basic trick is embedding $SU(2)$ into $SU(3)$, and deform (using the fact that $\pi_4(SU(3)) = 0$) $A_G(t)$ into an interpolating pure gauge connection in $SU(3)$

$$B_G(t) = 0^{f_t} = f_t^{-1} df_t \tag{9}$$

for f_t some homotopy in $SU(3)$ between 1 and $g \in SU(2)$.

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To go to $SU(3)$ we need to add a $SU(2)$ singlet, since under $SU(3) \supset SU(2)$ we have $\mathbf{3} \to \mathbf{2} \oplus \mathbf{1}$. Note that

$$Z_{\mathbf{3}} = Z(A_G) Z_{\mathbf{1}} \tag{10}$$

has the same $SU(2)$ anomaly as our original partition function.

The Elitzur and Nair approach

We now view the anomaly

$$Z_{\mathbf{3}}(0^g) = e^{i\mathcal{A}} Z_{\mathbf{3}}(0) \tag{11}$$

as coming from the local anomaly we obtain by a series of $SU(3)$ transformations. By the usual descent arguments ($\delta\omega = d\mathcal{A}$, with $d\omega = I_{d+2}$), we have

$$\mathcal{A} = \pi \int_{B_{d+1}} \omega(0^g) - \omega(0) = \pi \int_{B_{d+1}} \omega(0^g) \tag{12}$$

with B_{d+1} the $d+1$ ball with boundary S^d (for us, $d+4$), and $\omega(A)$ is the Chern-Simons density for a connection A.

The Elitzur and Nair approach

For groups without local anomalies (such as $SU(2)$) the Chern-Simons functional ω is invariant, so we can contract the boundary to a point in order to compute the anomaly

$$\mathcal{A} = \int_{B_{d+1}} \omega(0^g) \to \mathcal{A} = \int_{S^{d+1}} \omega(0^g)\,. \tag{13}$$

So we can view the above anomaly as a homomorphism from appropriate homotopy classes to $\mathbb{R}$:

$$\mathcal{A}\colon \pi_{d+1}(SU(3)/SU(2)) \to \mathbb{R}\,. \tag{14}$$

The Elitzur and Nair approach

The issue is finding the normalization for the homomorphism. Consider the short exact sequence

$$0 \to SU(2) \to SU(3) \to SU(3)/SU(2) \to 0 \tag{15}$$

which induces (since $\pi_4(SU(3)) = 0$)

$$\ldots \to \underbrace{\pi_5(SU(3))}_{\mathbb{Z}} \xrightarrow{\alpha} \underbrace{\pi_5\left(\frac{SU(3)}{SU(2)}\right)}_{\mathbb{Z}} \xrightarrow{\beta} \underbrace{\pi_4(SU(2))}_{\mathbb{Z}_2} \to 0\,. \tag{16}$$

By exactness, α is multiplication by 2, and β is reduction modulo 2.

The homomorphism in the $SU(3)$ case is well understood. We have that $\int_{B^{d+1}} \omega(0^g) = \int_{B^{d+1}} \mathrm{tr}\big((g^{-1}dg)^{d+1}\big)$. So $\mathcal{A}(f) = 2\pi$ with f the generator of $\pi_5(SU(3))$.

Which implies $\mathcal{A}(g) = \pi$, so there is an anomaly in this case.

The Elitzur and Nair approach

All this was a fairly roundabout argument, but it generalizes easily when we know enough about homotopy groups of Lie groups.

We have $\pi_8(G) \neq 0$ for

$$G \in \{SU(2), SU(3), SU(4), SO(7) \ldots SO(10), SO(N), G_2, F_4\}\,.$$

So only these groups can have global anomalies of the kind we are computing.

Example: $\mathcal{N}=1$ $\mathfrak{f}_4$ in 8d

For instance, in 8d for F_4 [Conlon '66]

$$0 \to \underbrace{\pi_9(E_6)}_{\mathbb{Z}} \xrightarrow{\alpha} \underbrace{\pi_9\left(\frac{E_6}{F_4}\right)}_{\mathbb{Z}} \xrightarrow{\beta} \underbrace{\pi_8(F_4)}_{\mathbb{Z}_2} \to 0\,. \tag{17}$$

Since $\mathbf{27} \to \mathbf{26} \oplus \mathbf{1}$ for $E_6 \supset F_4$, we find that the fundamental $(\mathbf{26})$ of F_4 has a $\mathbb{Z}_2$ discrete anomaly, by the same arguments as before.

The adjoint $\mathbf{78}$ of E_6 is free of local anomalies, and decomposes as $\mathbf{26} \oplus \mathbf{52}$, so the adjoint of F_4 also has a $\mathbb{Z}_2$ discrete anomaly!

Results

The other cases are also tractable. For the cases with $\pi_8(G) \neq 0$ we find:

$$G \in \{SU(2), SU(3), SU(4), SO(2N+1), SO(2N+2), G_2, F_4\}$$

with $N \geq 3$. Red means that we have proven the theory inconsistent, blue that we found no inconsistency from this particular check.

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An important loophole: we could potentially couple to a TQFT that forbids the problematic gauge bundles, as in [Seiberg '10].

For example, we could have a theory with gauge group G/C, with an anomaly coming from bundles with non-trivial Stiefel-Whitney class. We can "fix" this by coupling to a TQFT that effectively removes the problematic bundles [Kapustin, Seiberg '14].

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- This mechanism cannot cancel the traditional anomaly.
- We give an explicit 3d example of such "fixing" by coupling to a TQFT in the paper ($SU(N)/\mathbb{Z}_N$ with an adjoint).
- We have not been able to construct the right TQFT in the problematic 8d cases. Maybe our own mathematical limitations, or maybe it does not exist.

Anomalies on $S^4 \times \mathbb{R}^{d-4}$ with an instanton

We consider a spacetime of the form $S^4 \times \mathbb{R}^{d-4}$. Choose a decomposition of $G \supset (SU(2) \times H)/\mathcal{C}$. Now put a single instanton of the $SU(2)$ factor on the S^4.

Assume that the original theory had a fermion in a representation

$$R_G \to \bigoplus_{n \geq 1} (\mathbf{n}_{SU(2)} \otimes R_H^n) \,. \tag{18}$$

A fermion in the representation $\mathbf{n}_{SU(2)}$ gives rise to $N_n = \frac{1}{6}(n^3 - n)$ zero modes, so there is an effective theory in $\mathbb{R}^{d-4}$, with gauge group H, and fermions in the representation

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The original theory is anomalous if this effective theory in $\mathbb{R}^{d-4}$ is.

Results of the analysis on $\mathbb{R}^4 \times S^4$

The analysis is straightforward, and for the most part reproduces the results of the analysis in $\mathbb{R}^8$.

There is one exception. Consider the decomposition $USp(2N) \supset USp(2) \times USp(2N-2)$. The adjoint decomposes as

$$\mathsf{Adj} \rightarrow (\mathbf{2} \otimes (\mathbf{2N-2})) \oplus (\mathsf{Adj} \otimes \mathbf{1}) \oplus (\mathbf{1} \oplus \mathsf{Adj}) \,.$$

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So there is a Witten anomaly!

Conclusions

We reduced (a little bit) the gap between what you seemed to be able to do in $d = 8$ $\mathcal{N} = 1$ QFT (everything!) to what you seem to be able to do in string theory (very little!).

Local anomalies are absent, but global anomalies are powerful enough to say some things.

We find that:

- $\mathfrak{so}(2N+1)$ and $\mathfrak{f}_4$ are anomalous, as you might have suspected.
- $\mathcal{N} = 1$ $\mathfrak{g}_2$ seems to be omalous in 8d, to the extent that we checked. But no known construction!
- An unexpected anomaly for $\mathfrak{sp}(2N)$, i.e. the O7$^+$! Potentially fixable, but we don't quite know how.

Future directions

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- The very specific choices of rank: $\mathrm{rk}(G) \in \{4, 12, 20\}$.
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- In this context sometimes one can really prove absence of global anomalies. We should do so.
- The 9d analysis is important, and remains to be done.

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And this was all about anomalies, what about the swampland?
[Vafa '05]