Singularities, Matter & Gauge backgrounds

Matter in F-theory:

I) Codimension-two structure of ell. fibrations
II) Coupling to gauge backgrounds & counting zero-modes

This talk:

ad I) Complications due to "unresolvable singularities"
Focus on CY_3 [Grassi, TW to appear]
[Arras, Grassi, TW '16]

ad II) Matter multiplicities on smooth CY_4 (& CY_5)
[Mayrhofer, Bies, TW '17]
[+ Pehle '14]

I) F-theory on ell. CY_3 w/ singularities

I.1) $\mathbb{Q}$-factorial terminal singularities on 3-folds ($\mathbb{Q}$-ftS)

Consider X complex algebraic 3-fold,
singular, $\underbrace{\text{Gorenstein}}$
$= K_X$ is Cartier divisor

- Resolution : $f : \tilde{X} \longrightarrow X$ bimeromorphic (birational)

$\tilde{X}$ smooth
$\tilde{X}$ & $f^{-1}(X)$ differ along exceptional locus

f is called small resolution if it is isomorphism in codim 1

big — " ——— not "

- $K_{\tilde{X}} = f^* K_X + \sum_i a_i E_i$ E_i : exceptional divisors

$\uparrow$
discrepancies

If for a (= any) big resolution

$\forall i$: $a_i > 0$ X has at worst terminal singularities

$a_i \geq 0$ canonical

$a_i > -1$ klt

$a_i \geq -1$ log terminal

- If $K_{\tilde{X}} = f^* K_X$, the resolution is crepant.

Note: • 3-fold terminal sing. are isolated

• Gorenstein log terminal sing. are analytic hypersurface singularities

• **Def:** X is $\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier (i.e. $m\, D_{\text{Weil}} = D_{\text{Cartier}}$, $m \in \mathbb{Z}$)

Gorenstein terminal $\longrightarrow$ $\mathbb{Q}$-factorial $\Leftrightarrow$ factorial ($m = 1$)

(crepant)

$\Uparrow$

• **Fact:** A small resolution of klt singularities exists iff X is not $\mathbb{Q}$-factorial.

Example: Conifold singularity of CY_3:

• A big resolution always exists & $a_i > 0 \ \forall i$
$\Rightarrow$ terminal singularity

• If CY_3 not $\mathbb{Q}$-factorial globally,
$\exists$ small resolution, which is crepant

$$CY_3: \quad z_1 z_2 + z_3 z_4 = 0 \qquad \text{globally}$$

• If CY_3 is $\mathbb{Q}$-factorial globally,
$\nexists$ small resolution, and hence no crepant resolution at all

$$CY_3: \quad z_1 z_2 + z_3 z_4 + \underbrace{z_1^3 + z_2^3}_{\text{global obstruction}} = 0$$

$\Longrightarrow$ $\boxed{\text{If } X \text{ has } \mathbb{Q}\text{-fact. terminal singularities, then no crepant resolution exists.}}$

I.2) Relevance for elliptic CY_3 $\left(= \begin{matrix} \text{Gorenstein, } K_X = \mathcal{O}_X, \\ h^{i,0} = 0 \quad i = 1,2 \end{matrix} \right)$

- Weierstrass model $\quad y^2 = x^3 + f x z^4 + g z^6 \quad$ over B_2
- If $\operatorname{ord}(f,g) \not\geq (4,6)$ in codim.-one on B_2, then — possibly after birational trafo of B_2 — $\exists$ flat, crepant, partial resolution up to occurrence of $\mathbb{Q}$-factorial terminal singularities ($\mathbb{Q}$-ftS) (necessarily over codim-two on B_2)

- $\mathbb{Q}$-ftS occur often:

Consider most generic Weierstrass model w/ discriminant

$\Sigma = \Sigma_0 \cup \Sigma_1 \qquad \Sigma_0 : I_1\text{-locus}$

$\qquad\qquad\qquad\qquad \Sigma_1 : (f,g) \not\geq (4,6)$ in codim. 1

Among the 23 different families of such models (Grassi, Morrison 2000 & 2011), 3 have $\mathbb{Q}$-ftS:

i) $\Sigma_1 : I_1 \qquad I_1 \times I_1 \to I_2 \qquad$ conifold, $\mathbb{Q}$-factorial

ii) $\Sigma_1 : II \qquad I_1 \times II \to III$

$\left(z^3 + \sum_{i=1}^{3} x_i^2 = 0 \right)$

iii) $\Sigma_1 : I_{2k+1}^{n.s.} \quad (sp(k))$

$$I_1 \times I_{2k+1}^{\text{n.s.}} \to I_{2k+2}$$ conifold after partial resolution

Case i) governs local behavior of Jacobian of genus-one fibration ([Braun, Morrison 2014])

& studied by [Braun, Collinucci, Valandro 2013]

Many more examples (tuned) [Arras, Grassi, TW 2016]

<u>I.3) Mathematical & Physics Properties</u>

- On singular space, ordinary cohomology does in general <u>not</u>

  a) have Poincaré duality $\left(H^i(X,\mathbb{Q}) \sim H^{6-i}(X,\mathbb{Q})\right)$

  $\Rightarrow$ no non-degenerate intersection pairing

  b) support a pure Hodge structure & Hodge duality (relevant for counting of moduli)

<u>ad a)</u> [Grassi, TW to appear]

If X CY_3 w/ $\mathbb{Q}$-fact. terminal sing., the rational Poincaré duality holds (w.r.t. simplicial cohomology)

$$\boxed{H^i(X,\mathbb{Q}) \simeq H^{6-i}(X,\mathbb{Q})}$$

(In fact: holds for X projective, $\mathbb{Q}$-factorial 3-fold w/ klt isolated hypersurface sing. & $h^2(X,\mathcal{O}_X)=0$)

Consequence: Perform partial crepant resolution so X has only $\mathbb{Q}$-fts. Then

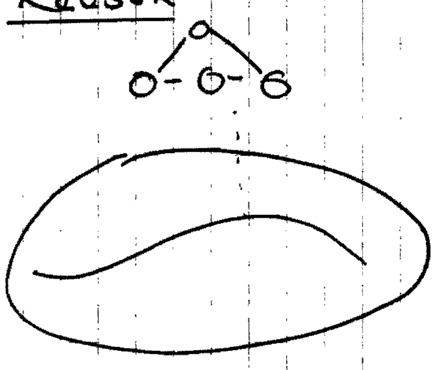
Relation between

codim-one fibres $\longleftrightarrow$ non-abelian Lie algebra

codim-two fibres $\longleftrightarrow$ weight lattice / representations R

is not affected by $\mathbb{Q}$-fts.

Reason:



E_i resol divisors

$\mathbb{P}^1_j$ fibral curves

$E_i \cdot \mathbb{P}^1_j = -C_{ij}$ due to existence of non-deg. intersection pairing $H_2(X)$ & $H_4(X)$

& similarly for codim-two

ad b) • Hodge structure + Hodge duality exists only if X is locally analytically $\mathbb{Q}$-fact., not only globally (RHM)

• Can still define complex structure moduli:

$\exists$ smoothing X_t of X w/

$$CxDef(X) = CxDef(X_t) \overset{\text{Hodge on } X_t}{=} \frac{1}{2} b_3(X_t) - 1$$

$$= \frac{1}{2}\left(b_3(X) + \sum_p m(p)\right) - 1$$

where p : $\mathbb{Q}$-ft singular point

$m(p)$: Milnor number of p

$\left(f(x_i)=0 \Rightarrow m(p) = \dim \mathbb{C}[x_i] \big/ \left\langle \frac{\partial f}{\partial x_i} = 0 \right\rangle \right)$

In general only $b_3(X) + \sum_p m(p) \in 2\mathbb{Z}$ due to lack of Hodge str. & Hodge duality

Interpret: $$\mathrm{CxDef}(X) = -1 + \underbrace{\tfrac{1}{2}\Big(b_3(X) - \sum_p m(p) \Big)}_{\text{unlocalised}} + \underbrace{\sum_p m(p)}_{\text{localised}}$$

CxDefs

[More precisely: $$\mathrm{CxDef} = -\tfrac{1}{2} + \tfrac{1}{2}\Big(b_3(X) + \sum_p m(p) - 2\tau(p) \Big) + \sum_p \underbrace{\tau(p)}_{\text{Tyurina-number}}$$

For all cases known to us $m(p) = \tau(p)$]

- Kähler moduli
  - $H^2(X) = H^{1,1}(X)$ (Srinivas) X CY_3 w/ $\mathbb{Q}$-ftS

$$\mathrm{KaDef}(X) = b_2(X) = h^{1,1}(X) \quad \checkmark$$

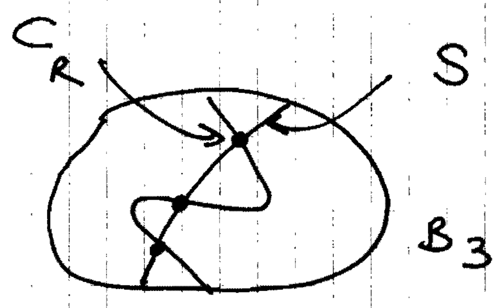
$\Rightarrow H - V + 29T = 273 \Rightarrow 30 K_B^2 + \frac{1}{2}\Big(\chi_{top}(X) + \sum_p m(p)\Big) - \#_{ch} - \dim(\mathrm{adj})_{G,\,charged} + \sum_p m(p) \; (\tau(p))$

Next step:
- Generalisation to CY_4?
- In particular: Counting of "localsed CxDef" on CY_4

II) Counting charged massless matter on ell. CY_4, smooth

II.1) Local picture

Donagi, Wijnholt
Beasley, Heckman, Vafa 2008
Watari et al.
...



surface S : gauge algebra
curve C_R : localised matter repr. R

Focus on localised matter (bulk matter similar):

4d $\mathcal{N}=1$ chiral multiplet in repr. R $\psi^i \in H^0(C_R, \sqrt{K_{C_R}} \otimes L_R)$

" anti-chiral " " " " $\tilde{\psi}^i \in H^1(C_R, \sqrt{K_{C_R}} \otimes L_R)$

where:

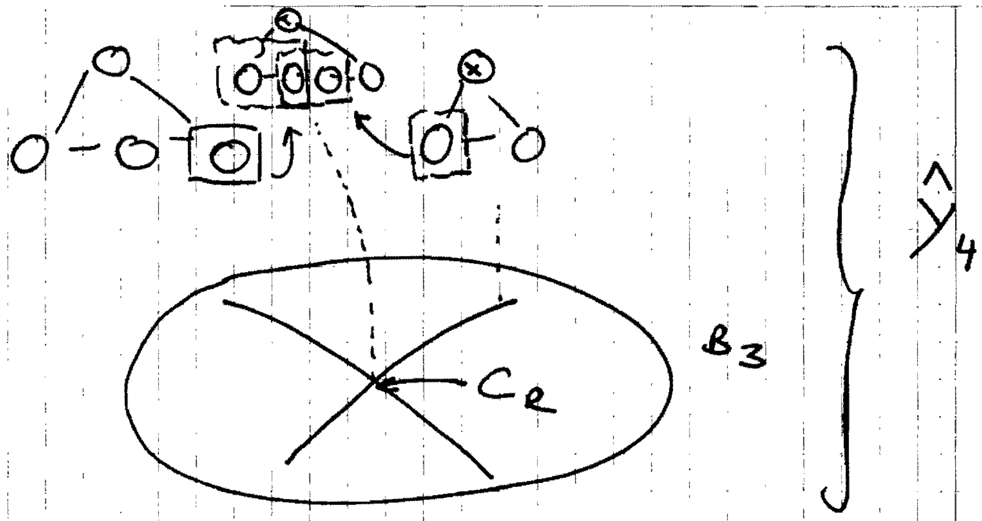
* $\sqrt{K_{C_R}}$ is the spin bundle of C_R compatible with holomorphic embedding of C_R into B_3
* L_R is a gauge bundle / sheaf depending on gauge background to which $\overset{(\sim)}{\psi}{}^i$ couple

<u>Qu:</u> How describe gauge background globally on elliptic fibration Y_4 & compute induced charged matter?

Here: $\hat{Y}_4$ smooth $\rightarrow$ only "abelian" background

II.2) Gauge background – global

Charged matter $\longleftrightarrow$ M2 branes along fibral curves in dual M-theory on smooth ell. fibration $\hat{Y}_4$



- repr. R $\longleftrightarrow$ weight vector $\beta^a(R)$ $\qquad a = 1, \dots, \dim(R)$

  $\longleftrightarrow$ M2 brane on fibral curve $\mathcal{C}^a$

  $\longleftrightarrow$ matter surface S^a_R

$$\pi_a : \mathcal{C}^a \hookrightarrow S^a_R \to C_R$$

- M2 couples to gauge potential C_3 : $S_{M2} \supset 2\pi \int_{M2} i^* C_3$

  $\Rightarrow$ Gauge background induced by C_3 background

- Aim: Parametrise i) field strength / flux background / curvature " $G_4 = dC_3$ "

  ii) flat backgrounds / "Wilson lines" " $\oint_{3\text{-cycle}} C_3$ "

Both information is encoded in the Deligne cohomology group $H^4_D(\hat{Y}_4, \mathbb{Z}(2))$:

$$0 \longrightarrow J^2(\hat{Y}_4) \longrightarrow H^4_D(\hat{Y}_4, \mathbb{Z}(2)) \xrightarrow{c_2} H^{2,2}_{\mathbb{Z}}(\hat{Y}_4) \longrightarrow 0$$

$$\underset{\mathcal{A}_G}{\Psi}$$

[Donagi '98]

i) $c_2(\mathcal{A}_G) = G_4 \in H^{2,2}_{\mathbb{Z}}(\hat{Y}_4)$ "curvature" of gauge background

ii) c_2 is <u>surjective</u>, but not injective with kernel:

• $J^2(\hat{Y}_4) := H^3(\hat{Y}_4, \mathbb{C}) \Big/ \underset{\overset{\|}{0}}{H^{3,0}(\hat{Y}_4,\mathbb{C})} \oplus H^{2,1}(\hat{Y}_4,\mathbb{C}) \oplus H^3(\hat{Y}_4,\mathbb{Z})$

Jacobian ⟷ flat gauge backgrounds

Note: $J^2(\hat{Y}_4)$ studied recently by • Intriligator, Jockers, Morrison, Mayr, Plesser

• Greiner / Grimm

• <u>Practical parametrisation:</u> via Chow group $CH^2(\hat{Y}_4)$

$CH^p(\hat{Y}_4)$ = group of alg. codim-p cycles modulo rational equivalence

rational eqn – generalisation of linear equiv. to higher codim. cycles:

$$\Gamma^{(k)} \sim 0 \text{ if } \Gamma^{(k)} = \sum_i \Gamma^{(k)}_i$$

k: C-dimension

$\underset{\text{div}(r_i)}{\overset{\|}{\Gamma^{(k)}_i}}$ = pole/zero of invertible rational function r_i on (k+1)-dim. subvariety W_i of total space

Crucial point: $\exists$ refined cycle map $\hat{\gamma}_2$

$$\begin{tikzcd} A_G \in CH^2(\hat{Y}_4) \arrow[d, "\hat{\gamma}_2"'] \arrow[dr, dashed, "\gamma_2"] & \\ H^4_D(\hat{Y}_4, \mathbb{Z}(2)) \arrow[r, "c_2"'] & H^{2,2}_{\mathbb{Z}}(\hat{Y}_4) \\ \mathcal{A}_G \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & \end{tikzcd}$$

w/ properties:

1) well-defined: (if $\Gamma_1 \sim \Gamma_2$, $\hat{\gamma}_2(\Gamma_1) \sim \hat{\gamma}_2(\Gamma_2)$)

2) surjective if Hodge conj. holds

3) not necessarily injective

Then

$$c_2 \circ \underbrace{\hat{\gamma}_2(A_G)}_{\mathcal{A}_G} = G_4 = \underset{\gamma_2(A_G)}{\underset{\parallel}{[A_G]}}$$

is 4-form flux

<u>Message:</u> Understanding / constructing 4-cycles up to rational equivalence $\longrightarrow$ Parametrising gauge backgrounds, in principle even beyond G_4 (but possibly redundant)

<u>Note:</u> A_G subject to constraints in agreement w/ known constraints on G_4 to parametrise / lift to 7-brane gauge background in F-theory $\rightarrow$ see later

II.3) From global back to local gauge backgrounds

Intuitively: L_R from integrating C_3 over fibre $\mathcal{e}^a$ of S^a_R

cf. "cylinder map" for spectral covers

Donagi, Wijnholt 2008

Watari et al.

Schäfer-Nameki et al.

Realised by considering intersection of A_G w/ S^a_R & projection onto base:

Schematically: $$A_G \circ_{i_{R,a}} S^a_R \in CH_0(S^a_R)$$

intersection product in Chow ring!

"Integration over fibre": $$D_R := \pi_{a*}(A_G \circ_{i_{R_a}} S^a_R) \in CH_0(C_R) \cong CH^1(C_R)$$

$\Rightarrow$ Chow class of points on C_R defines a sheaf on C_R

$$L_R := \mathcal{O}_{C_R}(D_R) \quad \text{is correct gauge background}$$

& $H^i(C_R, L_R \otimes \sqrt{K_{C_R}})$ counts massless 4d $\mathcal{N}=1$ chirals

Note: $c_1(L_R) = [D_R] \in H^{1,1}(C_R)$

Chiral index
$$\chi = h^0(C_R, L_R \otimes \sqrt{K_{C_R}}) - h^1(C_R, L_R \otimes \sqrt{K_{C_R}})$$
$$= \int_{C_R} c_1(L_R)$$
$$= [A_G]\cdot[S^a_R] = \int_{\tilde{Y}_4} \gamma(A_G)\cdot\gamma(S^a_R) = \int_{S^a_R} G_4$$

$\Rightarrow$ reproduces well-known expression for chirality based on flux / curvature G_4, but goes beyond this.

Chirality: cf. [Donagi, Wijnholt] '08
[Braun, Collinucci, Valandro],
[Schafer-Nameki, Marsano]
[Krause, Mayrhofer, TW] '11, ...
[Grimm, Hayashi]

<u>Note</u>: Similar expressions derivable for bulk matter with

$$\pi_{i*}(E_i \cdot A_G) \in CH_1(S)$$

E_i: resolution divisors

In general, L_R is <u>not</u> the pullback of a line bundle on B_3 or S $\Rightarrow$ $i_* \mathcal{O}_{C_R}(D_R)$ is a sheaf on B_3

II.4) Systematics of construction

2-cycle class $A_G \in CH^2(\hat{Y}_4)$ to be chosen in agreement w/ conditions on G_4 to define consistent 7-brane gauge background in F-theory:

1) Transversality: i) $G_4 \cdot \pi^*[D_i] \cdot \pi^*[D_j] = 0 \qquad [D_{i,j}] \in H^2(B_3)$

ii) $G_4 \cdot [S_0] \cdot \pi^*[D_i] = 0$

$\uparrow$ class of zero section

2) Quantisation: $G_4 + \frac{1}{2} c_2(\hat{Y}_4) \in H^4(\hat{X}_4, \mathbb{Z}) \cap H^{2,2}(\hat{Y}_4, \mathbb{R})$

3) If G_4 is not to break gauge invariance w.r.t. non-abelian Lie algebra

$$G_4 \cdot [E_i] \cdot \pi^*[D_i] = 0 \qquad E_i \text{ resolution divisors}$$

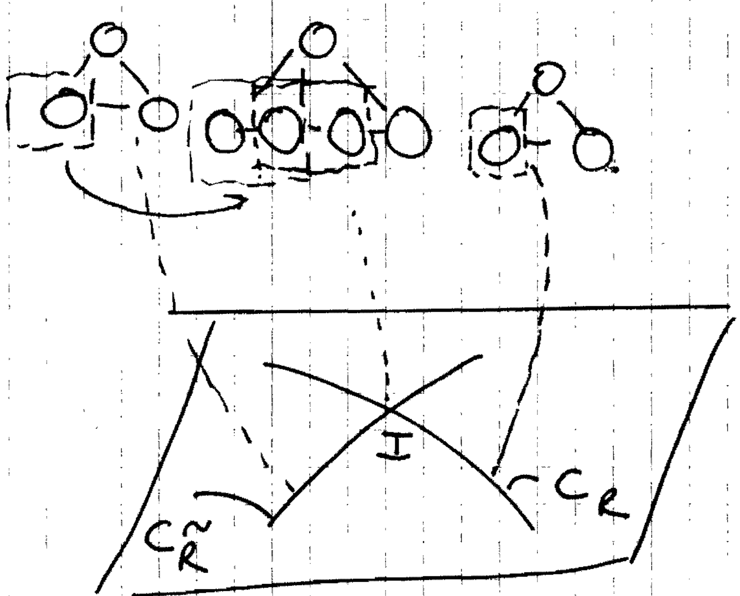
$\Rightarrow$ Analogue of 1) & 3) at level of $CH^2(\hat{Y}_4)$ in [Bies, Mayrhofer, TW '17]

Note: Every matter surface defines itself a gauge background consistent w/ 1) & for 3) need correction term

$$\Gamma_{\tilde{R}} := S^a_{\tilde{R}} + \Delta^a_R \qquad \Delta^a_R = \beta^a_i(\tilde{R})\, C^{-1}_{ij}\, E_j\Big|_{C_{\tilde{R}}}$$

w/ Chow class $A_{\tilde{R}}$ in $CH^2(\hat{Y}_4)$

Note: This is independent of index "a"



Assume transverse intersection of S^a_R & $A_{\tilde{R}}$ over points I

$$S^a_R \cdot_i A_{\tilde{R}} = \sum I \Big(\underbrace{m(I)}_{\text{multiplicity of points}} \cdot \underbrace{m_{\text{fibre}}(I)}_{\text{intersection \# in fibre}} \Big)$$

$$\in CH^1(C_R)$$

$\Rightarrow$ Very explicit construction of sheaves $\mathcal{O}_{C_R}(D_R)$ possible [Bies, Mayrhofer, TW '17]

Explicitly for $SU(5) \times U(1)$ model over $B_3 = \mathbb{P}^3$

$\Rightarrow$ Explicit dependence of cohomology dim. on complex structure moduli determining curves C_R (incl. "jumping lines" along moduli space)

Method in collaboration w/ M. Barakat et al:

D_R on C_R $\longleftrightarrow$ ideal I on $X_\Sigma \supseteq B_3$
toric ambient space

$\mathcal{O}_{C_R}(D_R) = \mathcal{H}om(\mathcal{J}_{X_\Sigma}, \mathcal{O}_{X_\Sigma})$ $\mathcal{J}$: ideal sheaf associated w/ I

Explicitly in language of finitely presented S-modules
- S = coord. ring on X_Σ
- $I \Longleftrightarrow$ relations between generators $\longleftrightarrow$ homom. of S-modules $\equiv$ f.p. graded S-module $\tilde{I}$
- Consider $M = Hom_S(\tilde{I}, S)$ $\equiv$ fp graded S-mod.
- $\tilde{M}$ = sheaf on $\langle x_i = 0 \rangle$ locally given by localisation of M
- $\tilde{M} = \mathcal{O}_{C_R}(D_R)$
- $H^i(\tilde{M}) = Ext^i(N, M)_0$ N: suitably designed fp graded S-module

Practically:
- CAP & GAP give locally free resolution of $\tilde{M}$ w/ input data ideal I (M as locally free resol.)
- Algorithm to determine N w/ above prop.