Entanglement, Holography and Geometry

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Entanglement is what makes quantum mechanics different from classical mechanics.

not entangled $|\uparrow\uparrow\rangle$

entangled
(EPR pair) $\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$

Entanglement: correlations between outcomes of simultaneous, acausal measurements.

To quantify the amount of entanglement between two subsystems A and B entanglement entropy is a useful notion.

~number of EPR pairs between A and B

Definition:

Given a state ρ on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ the entanglement entropy of A wrt B is

$$S(A) = -\mathrm{Tr}(\rho_A \log \rho_A) \qquad \rho_A = \mathrm{Tr}_{\mathcal{H}_B} \rho$$

ρ_A is the reduced density matrix

$$\rho = \rho_{ab}^{a'b'} |a\rangle|b\rangle\langle b'|\langle a'| \Longleftarrow \rho_A = \rho_{ab}^{a'b}|a\rangle\langle a'|$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$\Downarrow$$

$$\rho_A = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow|$$

$$\Downarrow$$

$$S(A) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = \log 2$$

Quantum information theory is (among others) about the interpretation and uses of this notion of entropy.

Eigenvalues of ρ_A are p_i :

$$S_A = -\sum_i p_i \log p_i$$

Classical interpretation (Shannon):

$$S_A = \lim_{N\to\infty} \frac{1}{N} \log \begin{pmatrix} & N & \\ p_1 N \, p_2 N & \ldots & p_k N \end{pmatrix}$$

This is the amount of information per bit in a string where the probability that i appears is p_i .

Entanglement entropy inequalities:

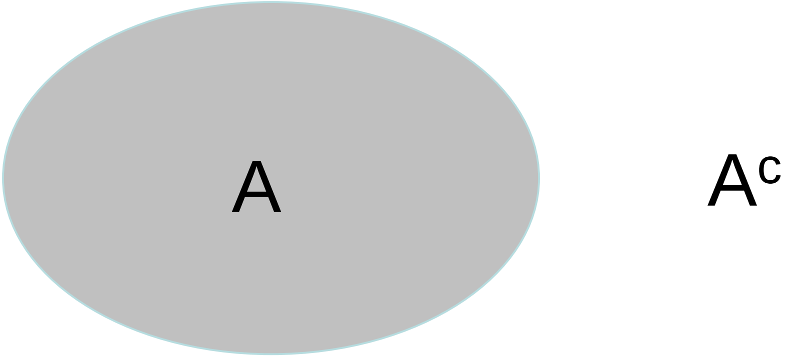
$$|S(A) - S(B)| \leq S(A \cup B) \leq S(A) + S(B)$$

$$S(A \cup B) + S(A \cap B) \leq S(A) + S(B)$$

strong subbadditivity

Entanglement entropy in quantum field theory

Typical situation: consider degrees of freedom associated to a spatial domain and its complement



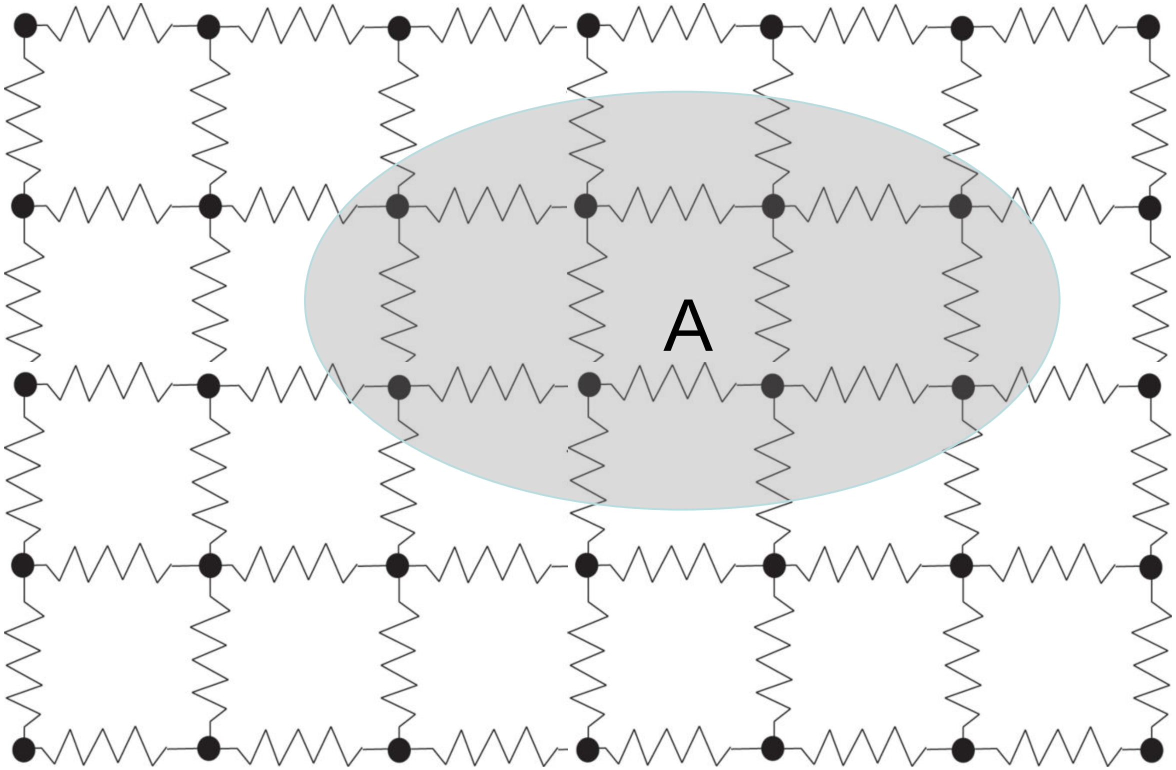
$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$$

Bombelli, Koul, Lee, Sorkin `86
Srednicki `93

(various caveats)

Entanglement entropy = infinite in continuum field theory.

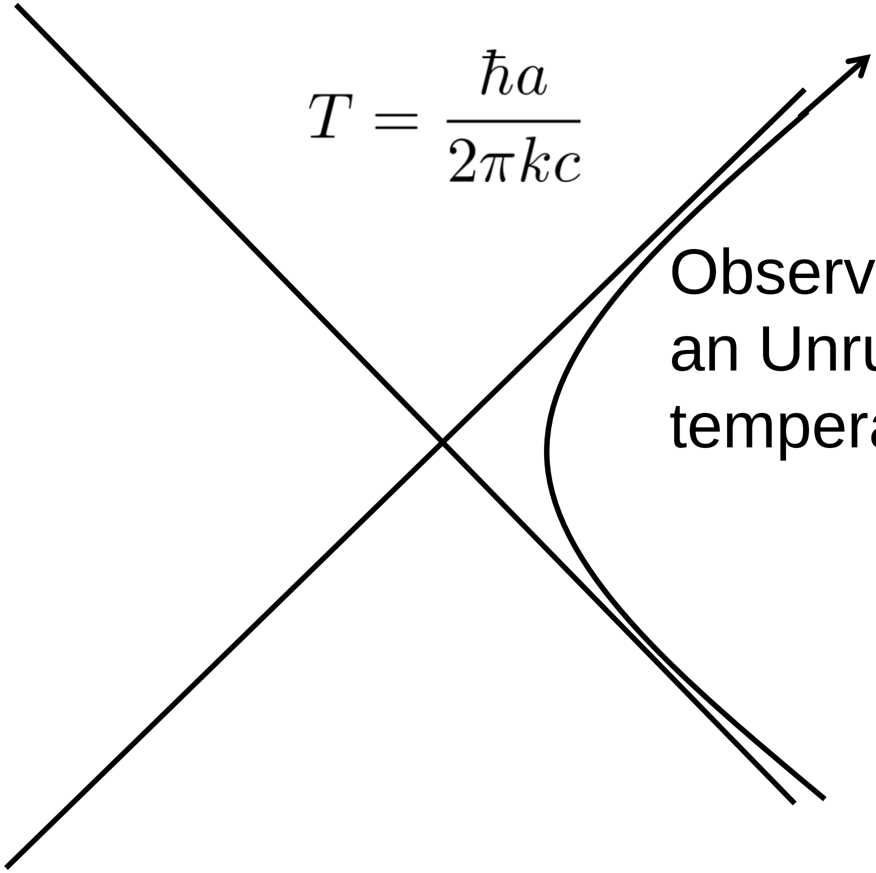
Needs to be regulated: short distance regulator a.



$$S(A) = \frac{\mathrm{area}(A)}{a^{D-2}} + \dots$$

Another example of entanglement: accelerated observers

Accelerated observers do not see left wedge. Only have access to reduced density matrix.



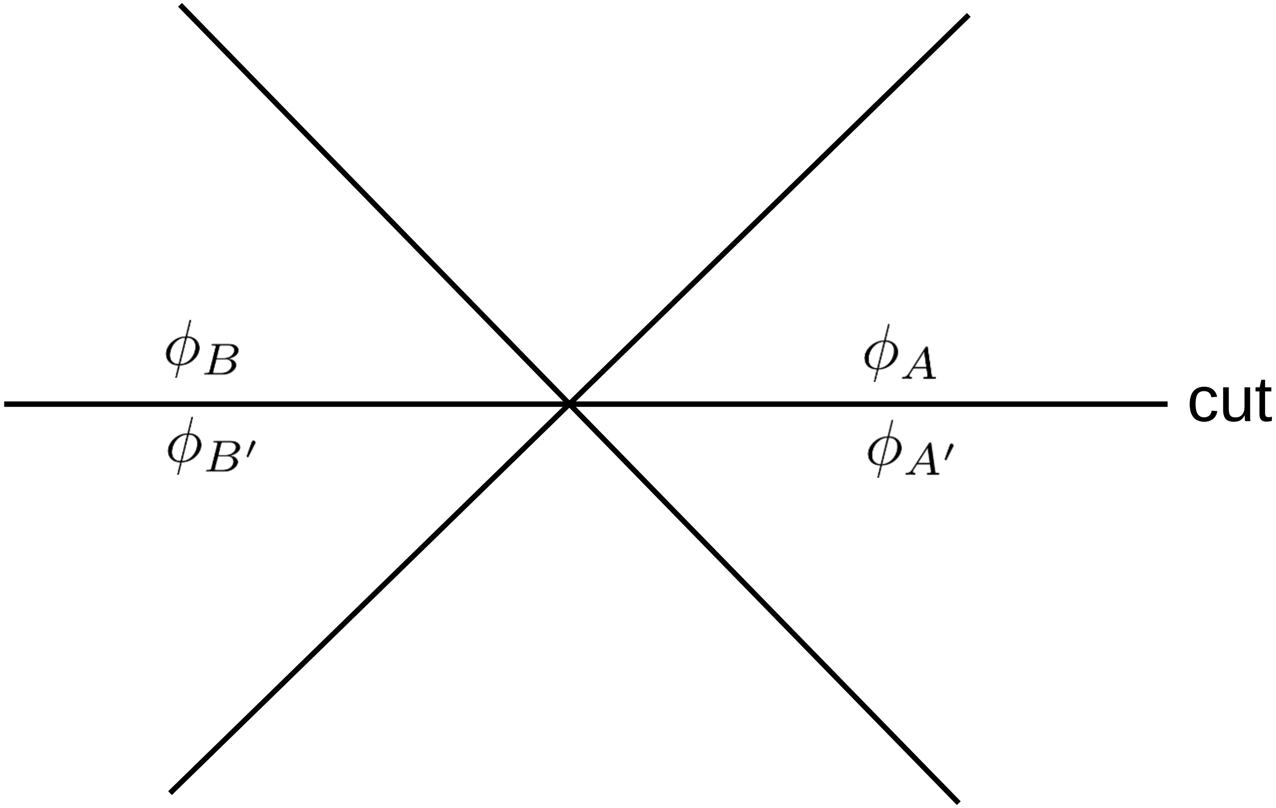
$$T = \frac{\hbar a}{2\pi k c}$$

Observer measures an Unruh temperature

$$|0\rangle_M \;= \sum_E e^{-\frac{E}{2T}} |E\rangle_L |E\rangle_R$$

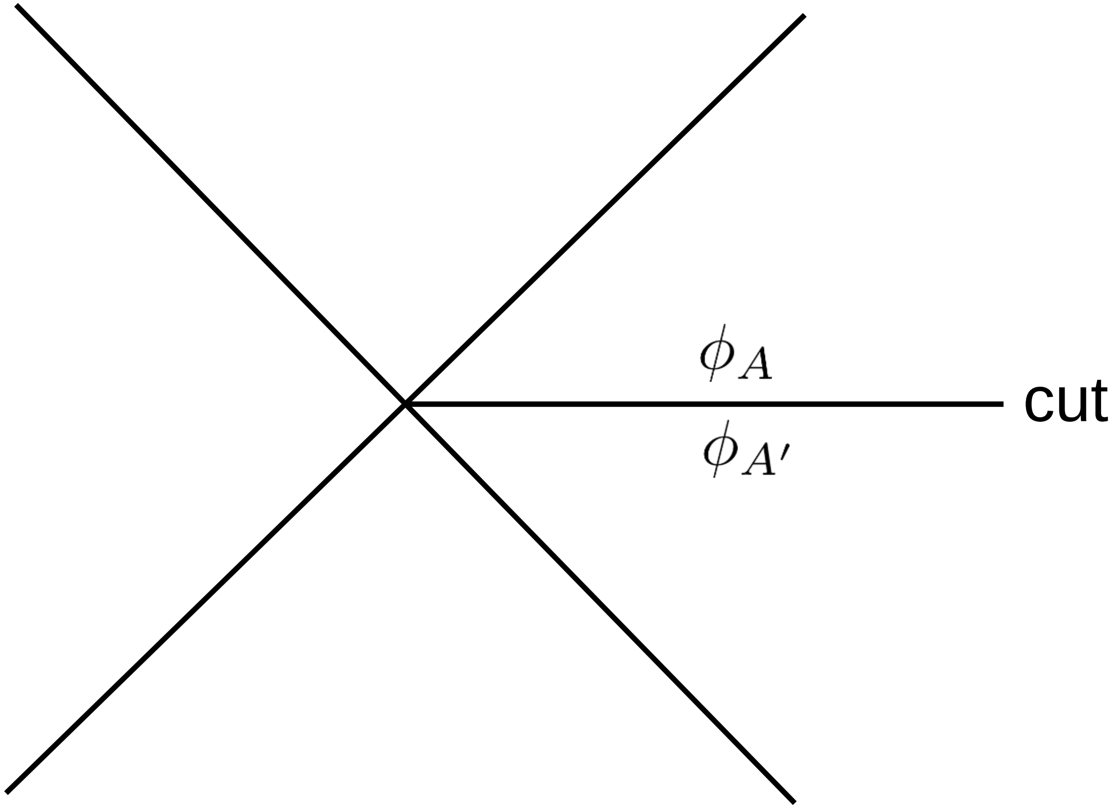
Path integral with boundary conditions computes transition elements.

$$|0\rangle\langle 0| = \sum_{\phi_A,\phi_{A'},\phi_B,\phi_{B'}} Z_{ABA'B'} |\phi_{B'}\rangle|\phi_{A'}\rangle\langle\phi_A|\langle\phi_B|$$

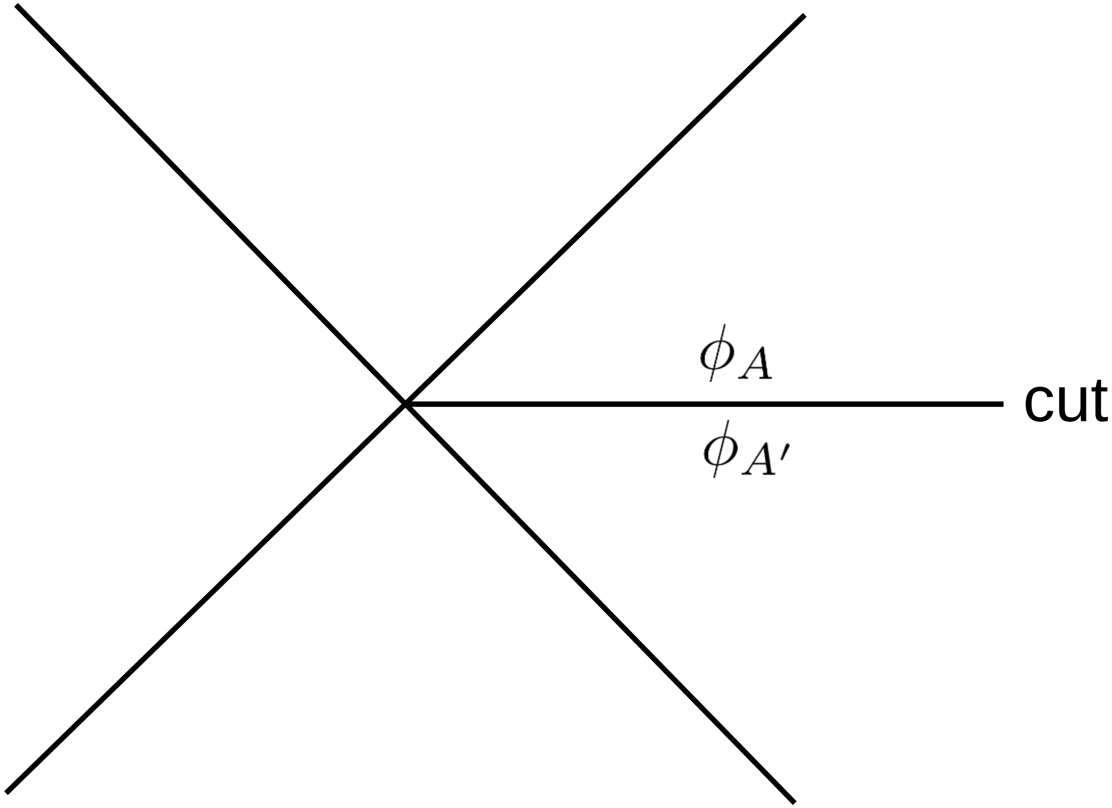


Path integral with boundary conditions computes transition elements.

$$\rho_A = \sum_{\phi_A, \phi_{A'}} Z_{AA'} |\phi_{A'}\rangle\langle\phi_A|$$



$$\rho_A = \sum_{\phi_A, \phi_{A'}} Z_{AA'} |\phi_{A'}\rangle\langle\phi_A| = e^{2\pi \frac{\partial}{\partial \varphi_E}} = e^{-2\pi H_{\text{obs}}}$$



Unfortunately, it is in general very hard to compute entanglement entropy in a qft (even in a free qft).

Entanglement entropy is also not an observable.

So why care?

- Probe of ground states (e.g. diagnostic of Fermi surfaces).
- Probe of topological phases with no local order parameter.
- Helps determine properties of critical points for e.g. spin chains.
- Helps construct variational ansatze for ground states.
- Useful notion in quantum information theory.
- Sheds light on nature of thermalization/entropy production.
- Can be computed for strongly coupled field theories with a holographic dual.
- Seems to play a fundamental role in quantum gravity and holography.

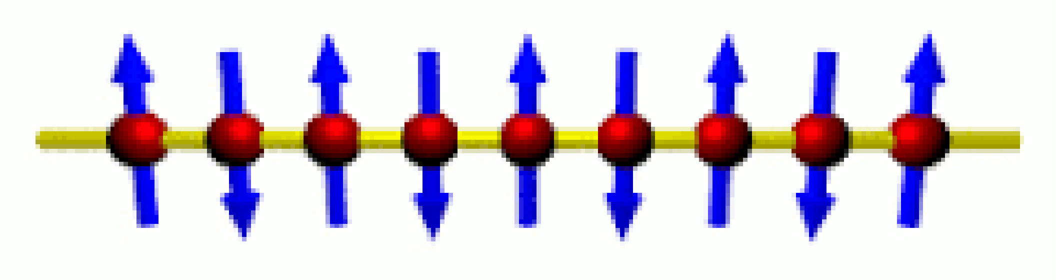
Exception: 1+1 dimensional conformal field theory.

$$S = \frac{c}{3} \log\left(\frac{\ell}{a_{UV}}\right)$$

Holzhey, Larsen, Wilczek `94
Cardy, Calabrese `07

ℓ: length of interval

c: central charge



Compute entanglement entropy as function of ℓ and extract c

Topological entanglement entropy In 2+1 dimensions.

Levin, Wen `05
Kitaev, Preskill ‘05

Entanglement entropy for a disc with radius ℓ

$$S = \alpha \frac{\ell}{a_{UV}} - \gamma + \dots$$

area law

universal coefficient

General structure for spherical domain

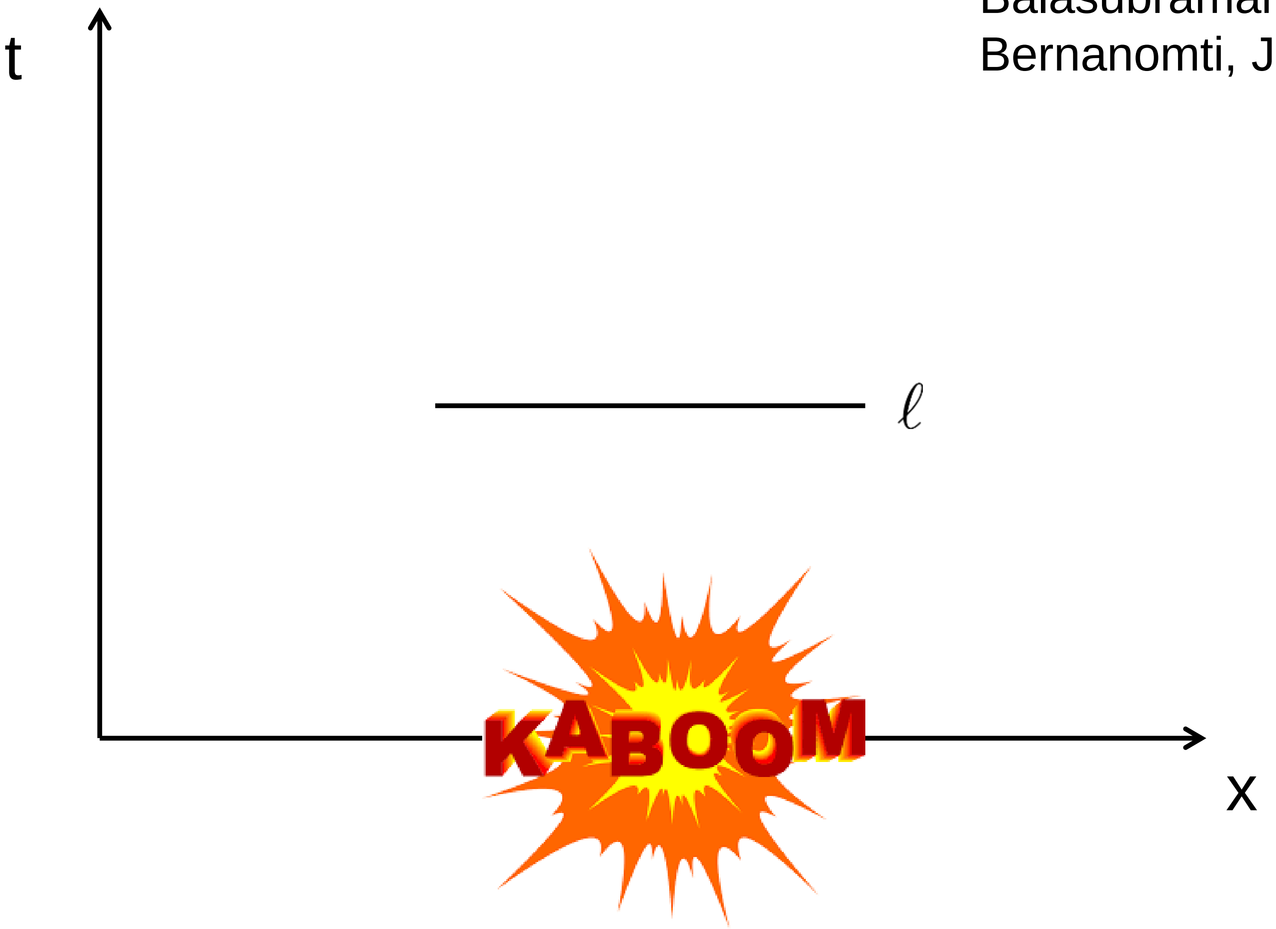
d=even
c~ conformal anomaly
c-theorem

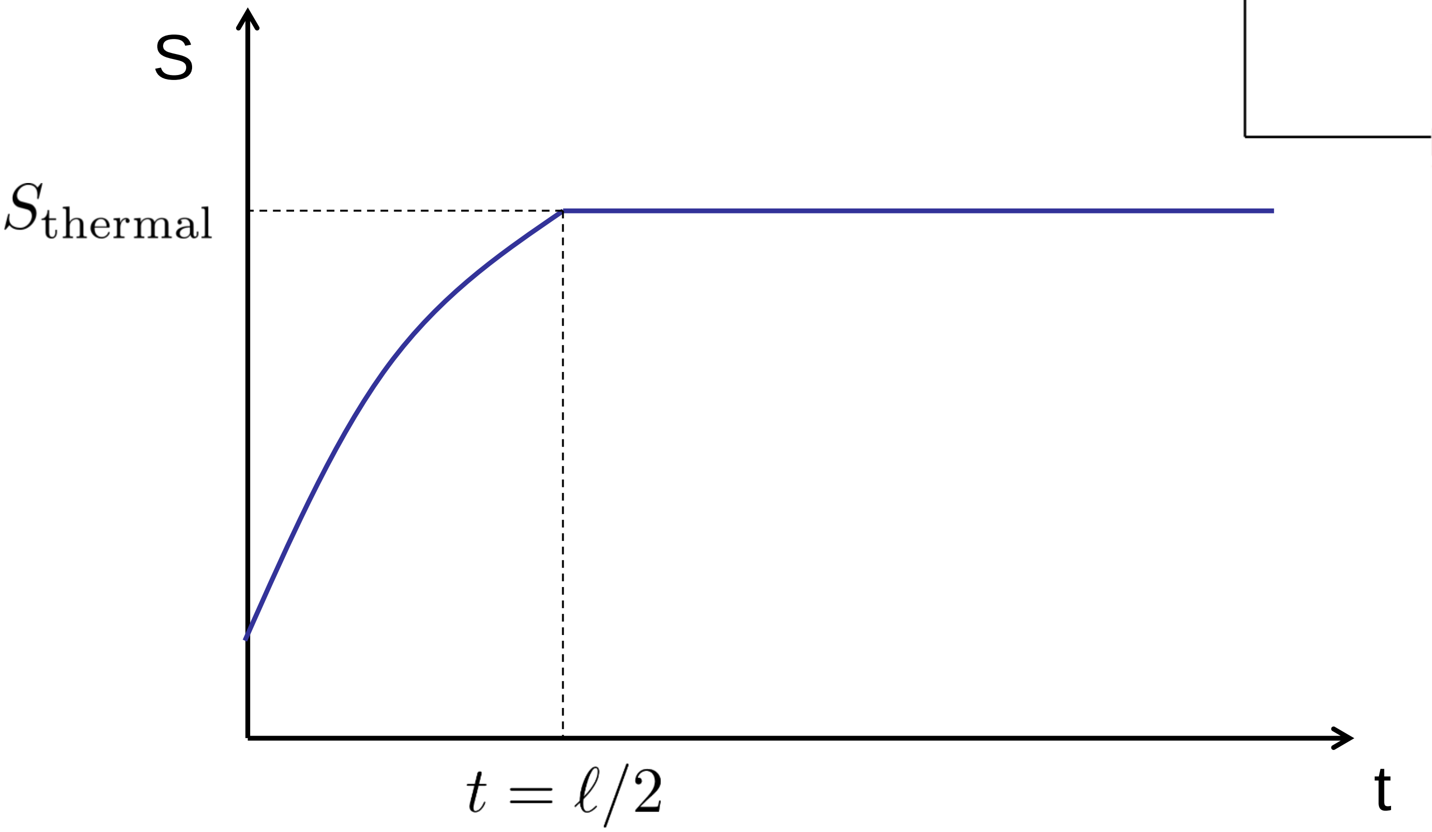
$$S = a_{d-2}\left(\frac{\ell}{a_{UV}}\right)^{d-2} + a_{d-4}\left(\frac{\ell}{a_{UV}}\right)^{d-4} + \ldots + \left\{ \begin{array}{c} c\log\frac{\ell}{a_{UV}} \\ c \end{array} \right. + \ldots$$

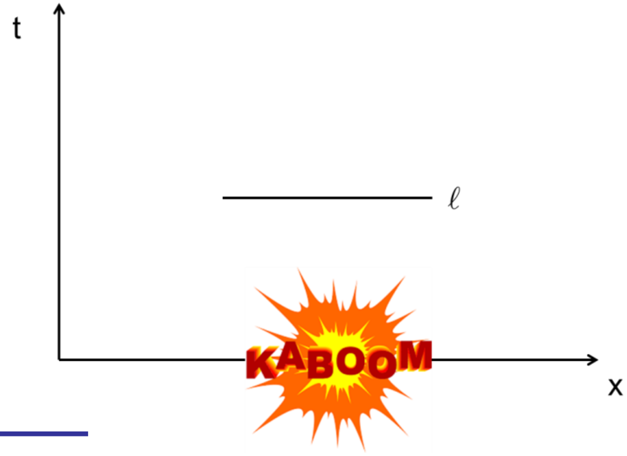
d=odd
c~ F
F-theorem

Thermalization after a quench

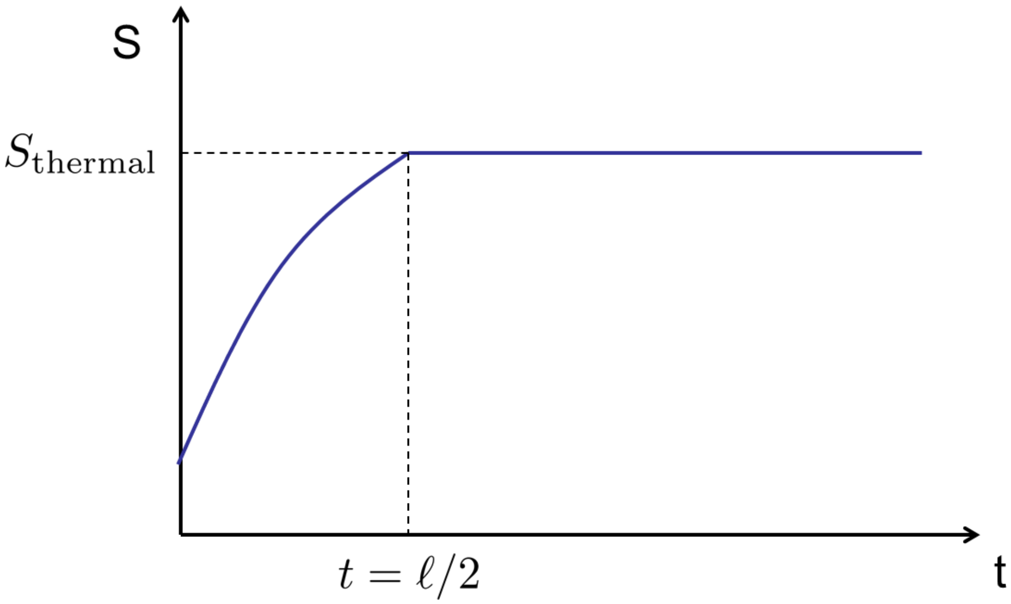
Cardy, Calabrese `07
Abajo-Arrastio, Aparicio, Lopez ‘10
Balasubramanian, Bernanomti, JdB et al `11

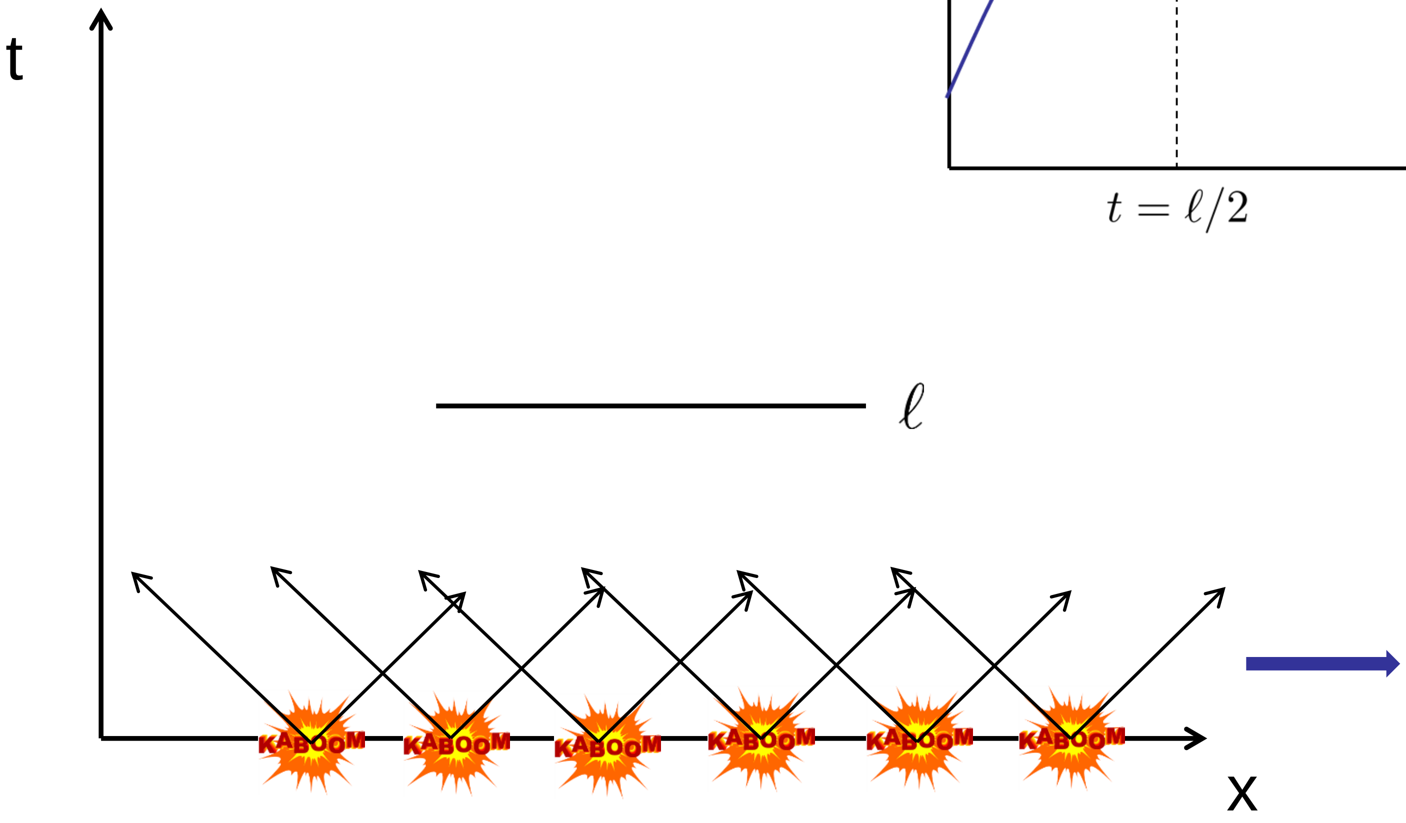



S
S_{thermal}
$t = \ell/2$
t


t
ℓ
KABOOM
x

Explanation





Entangled pairs of quasi-particles

A similar picture exists in higher dimensions for those field theories with a holographic dual.

If these capture the qualitative dynamics of thermalization after heavy ion collisions get an interesting picture:

Thermalization proceeds from the UV to the IR and not the other way around.

But many issues and subtleties…

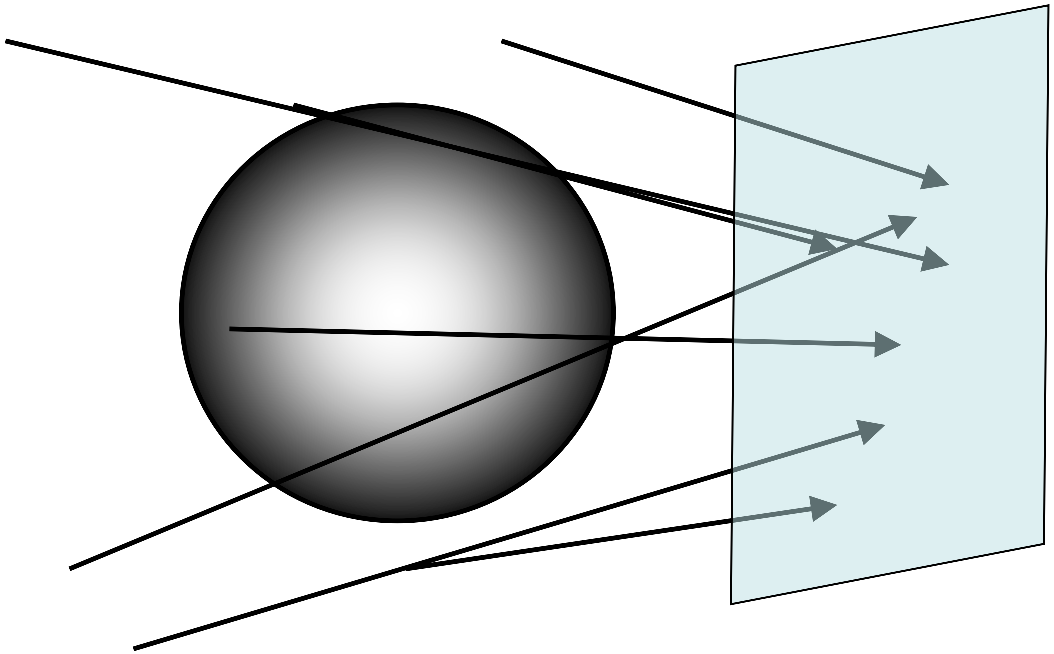
HOLOGRAPHY

Quantum gravitational degrees of freedom behave like ordinary degrees of freedom in one dimension less.

‘t Hooft, Susskind

Inspired by Bekenstein-Hawking entropy of a black hole

$$S = \frac{Ac^3}{4G\hbar}$$



Holographic screen

There is something very strange about the degrees of freedom in quantum gravity:

Normal local degrees of freedom: S ~ volume

Quantum gravity degrees of freedom: S ~ area!!

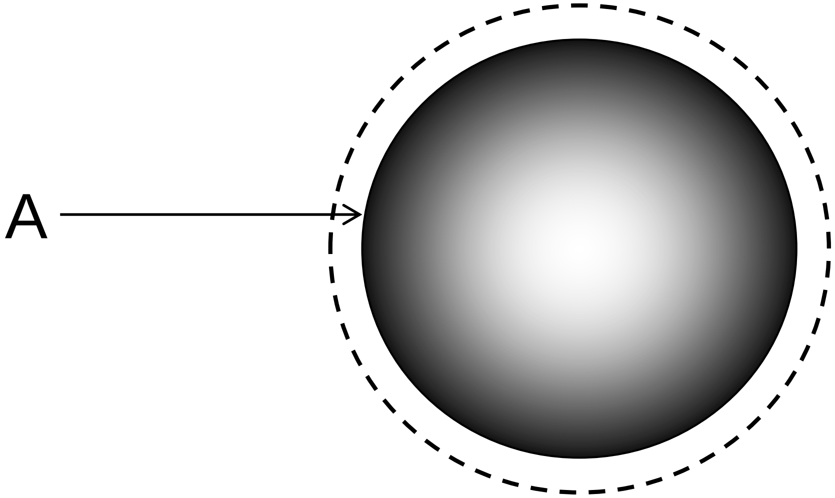
The fundamental degrees of freedom of quantum gravity must be highly non-local and represent some sort of quantum geometries.

The standard local picture of gravity emerges from these non-local degrees of freedom only after we do a suitable averaging over them (similar to what one does in thermodynamics).

In this sense, spacetime and gravity are emergent phenomena.

Holography and quantum gravity.

Black holes: entropy = entanglement entropy?



$$S_{\mathrm{BH}} = \frac{A}{4G}$$

$$S_{\mathrm{EE}} \sim \frac{A}{4G} \quad (a \sim \ell_P)$$

Cooperman, Luty `03

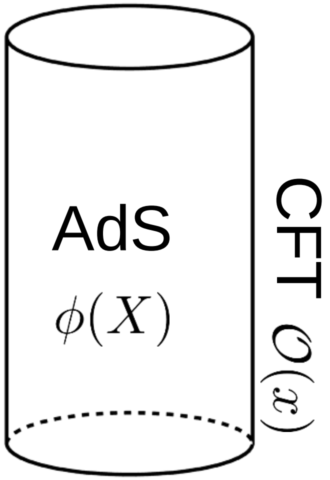
Quantum gravity and holography made precise: AdS/CFT.

Precise equivalence between a conformal field theory in d dimensions and quantum gravity in d+1 dimensions.

Matches nicely with entropy of black holes.

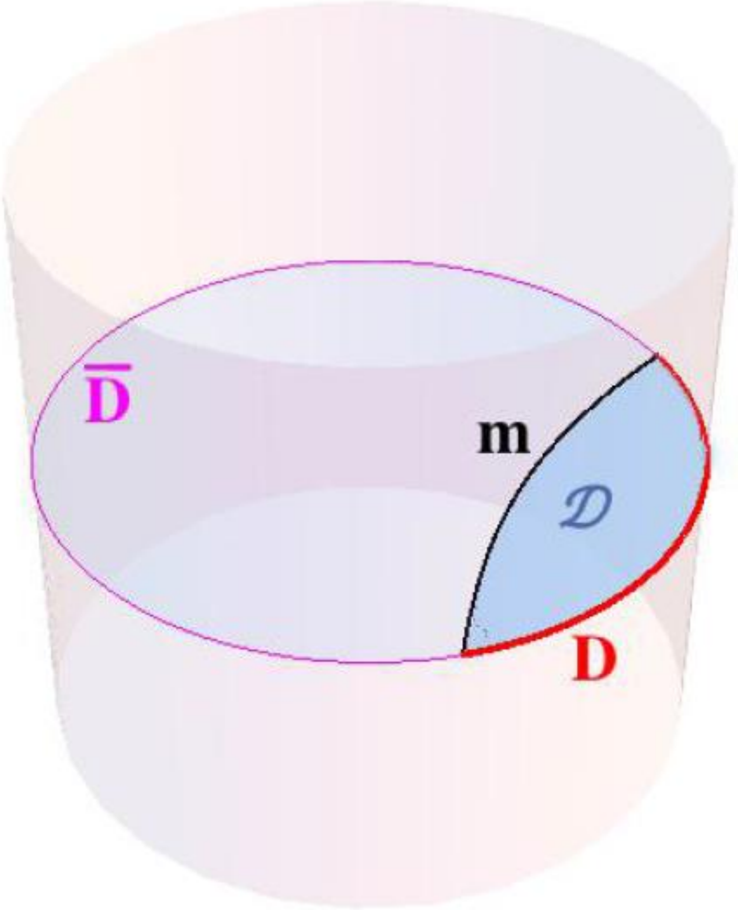
Bulk fields vs boundary operators

$$\phi(X)_{\mathrm{bulk}} \simeq \int d^{\mathrm{d}}x K(X,x)\mathcal{O}(x)_{\mathrm{boundary}}$$



Important question: How does locality come about?

Entanglement entropy?



Ryu-Takayanagi proposed that

$$S(D) = \frac{\text{Area}(m)}{4G}$$

m: extremal surface

A proof of this statement has been given (Lewkowycz, Maldacena `13)

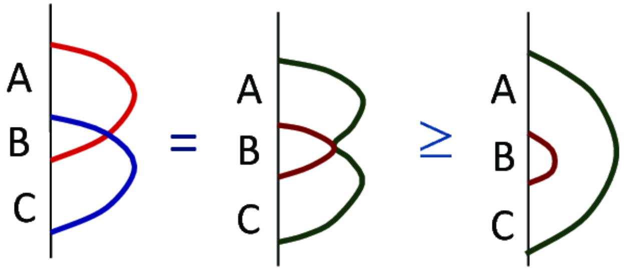
Therefore entanglement entropy is a direct probe of space-time geometry. Can perhaps probe the (non)locality of quantum gravity.

Entanglement entropy is very easy to compute, purely geometric computation!

Entanglement inequalities become geometric inequalities.

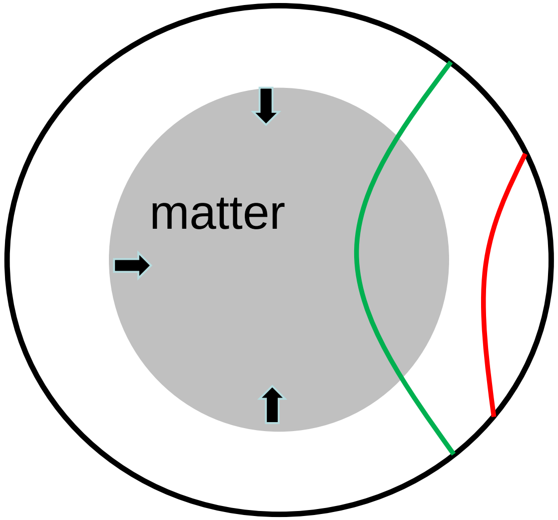
UV cutoff = radial cutoff.

Universal result for 2d CFT’s is reproduced.

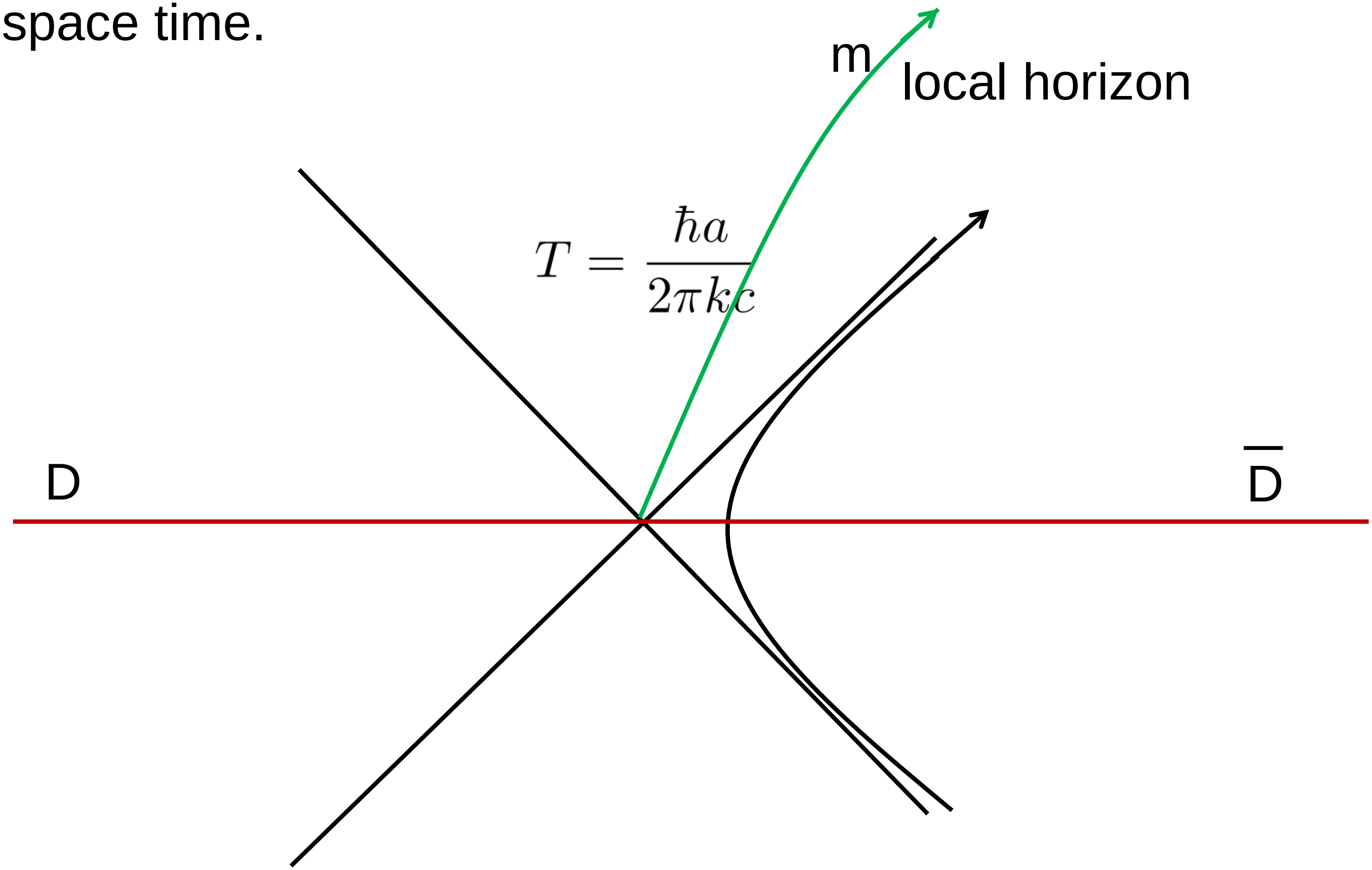


$$\Rightarrow \quad S_{A\cup B}+S_{B\cup C}\geq S_{A\cup B\cup C}+S_B$$

Thermalization picture is confirmed.



Important message: entanglement is crucial to get a smooth space time.



$$T = \frac{\hbar a}{2\pi k c}$$

$$|0\rangle_M = \sum_E e^{-\frac{E}{2T}} |E\rangle_L |E\rangle_R$$

Van Raamsdonk `09

Changing the nature of the entanglement easily leads to a divergent expectation value of the energy momentum tensor on the “horizons”.

In some sense, this is a manifestation of the “firewall”.

Almheiri, Marold, Polchinski, Sully

Postulating that entanglement entropy is computed by minimal area surfaces implies the linearized Einstein equations.

Faulkner, Guica, Hartman, Myers, van Raamsdonk `13

Important ingredient: first law of entanglement entropy

$$\delta S = -\delta \mathrm{Tr}(\rho \log \rho) = -\mathrm{Tr}(\delta\rho \log \rho) = -\delta \langle \log \rho \rangle$$

For conformal field theories:

$$\delta S(B) = 2\pi \int_B d^{d-1}x' \;\frac{R^2 - |\vec{x} - \vec{x}'|^2}{2R}\, \langle T_{tt}(\vec{x}') \rangle \; .$$

Is quantum gravity a local theory?

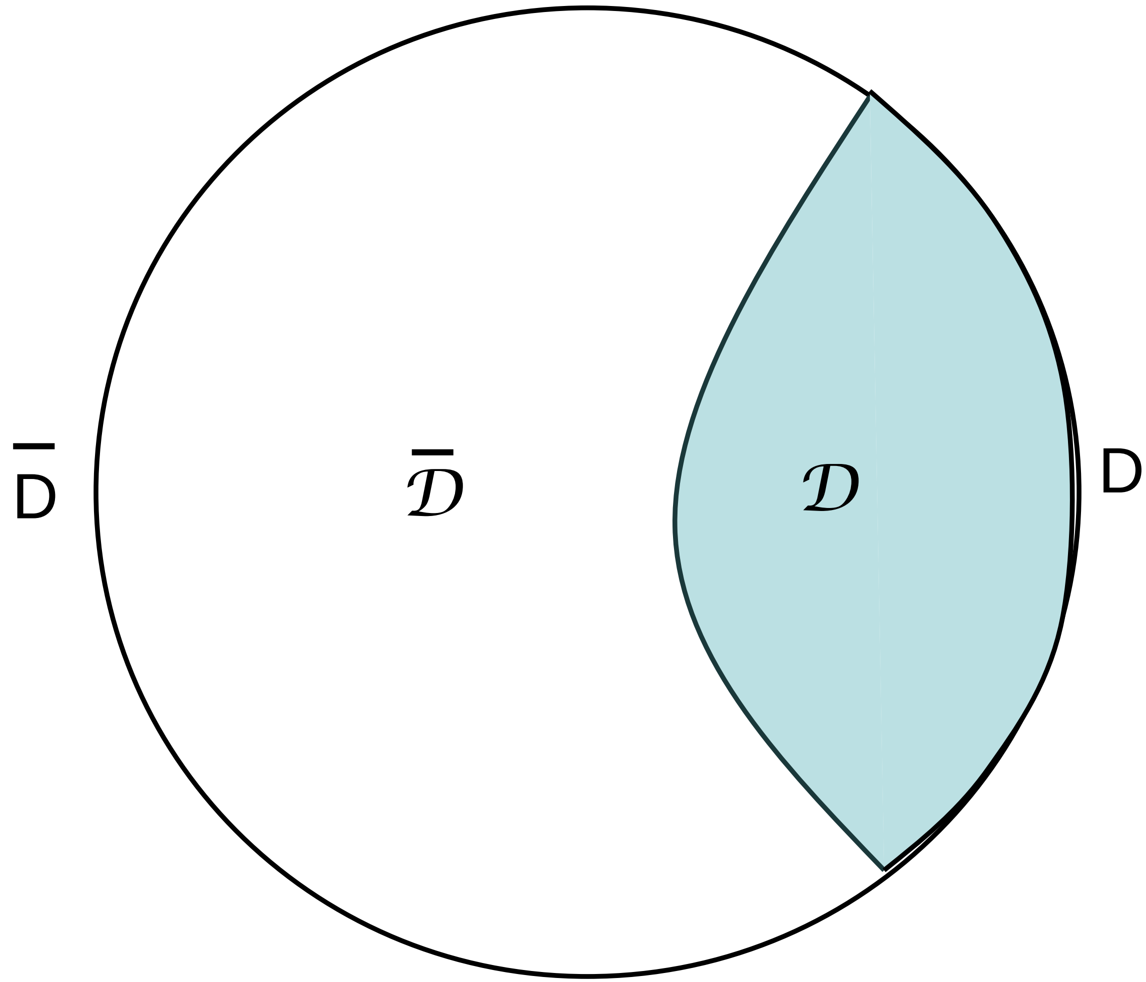
If the degrees of freedom of quantum gravity were approximately local, one should be able to compute their entanglement between some spatial domain and its complement.

This requires a factorization

$$\mathcal{H} = \mathcal{H}_{\mathrm{outside}} \otimes \mathcal{H}_{\mathrm{inside}}$$

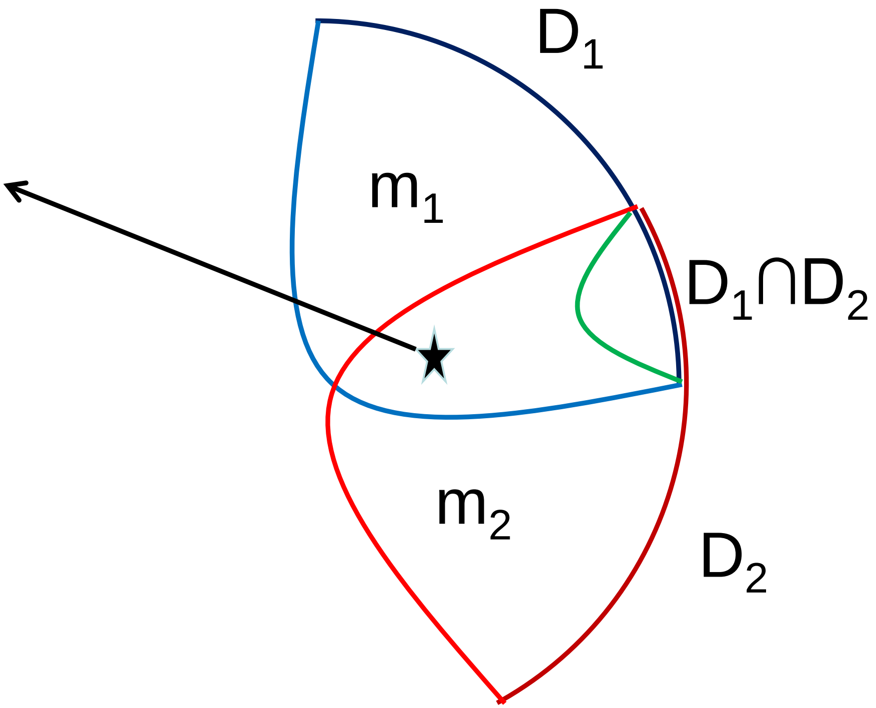
Such a factorization is often used when computing Hawking radiation, when discussing the information loss paradox, and in many arguments pertaining to the (non)existence of firewalls.

$$\mathcal{H} = \mathcal{H}_D \otimes \mathcal{H}_{\bar{D}} \simeq \mathcal{H}_{\mathcal{D}} \otimes \mathcal{H}_{\bar{\mathcal{D}}}$$



A local operator here would act entirely in D_1 but also entirely in D_2; but it does not act in $D_1 \cap D_2$. This is a contradiction. Local operators do not exist?

Connection to quantum error correction?



Local operators only act properly on a subset of the degrees of freedom: the so-called “code subspace”.

Almheiri, Dong, Harlow `14
Mintun, Polchinski, Rosenhaus `15

Black hole information paradox I

Low energy effective field theory applied to black hole creation and evaporation predicts a loss of unitarity.

Breakdown of effective field theory either (i) through the creation of a firewall or (ii) loss of locality.

How can a pure state harbor enough entanglement to be able to reconstruct smooth spacetime across a horizon? Only needs to be smooth for low-energy observers? Description of local physics is state-dependent? (Papadodimas-Raju)

Black hole information paradox II

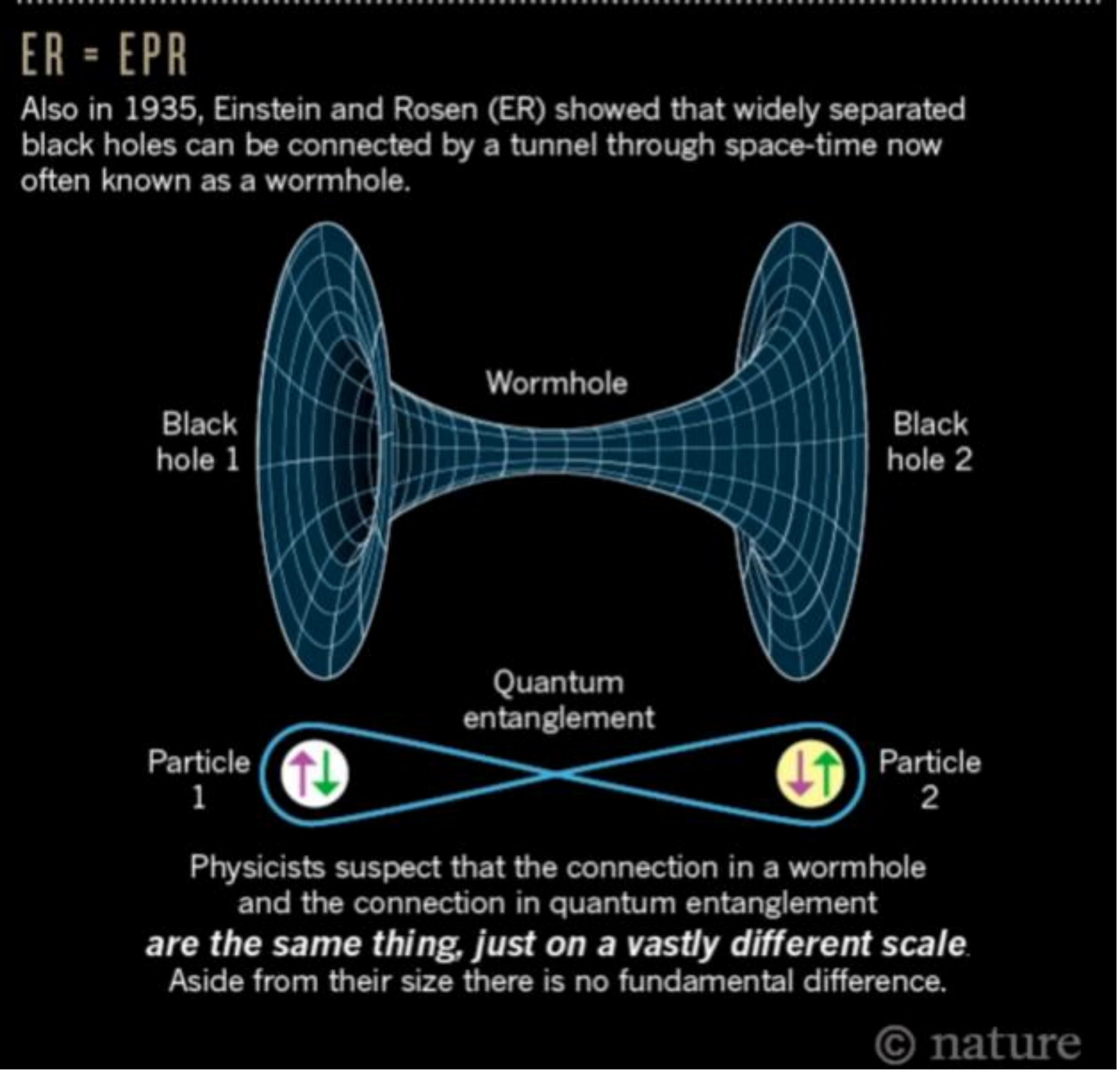
There have been many recent papers discussing how to get the “page curve” for the entropy of radiation emitted by an evaporating black hole.

Non-perturbative gravitational saddles play an important role in these discussions.

In one class of examples, the discussion involves an averaging over theories. (Penington, Stanford, Shenker, Yang)

In another class of examples, this averaging does not seem to be important. (Almheiri, Mahajan, Maldacena, Zhao, Hartman, Shaghoulian, Tajdini)

This has led to a debate: is quantum gravity dual to an ensemble of theories or to a single theory?



ER = EPR

Also in 1935, Einstein and Rosen (ER) showed that widely separated black holes can be connected by a tunnel through space-time now often known as a wormhole.

Physicists suspect that the connection in a wormhole and the connection in quantum entanglement ***are the same thing, just on a vastly different scale***. Aside from their size there is no fundamental difference.

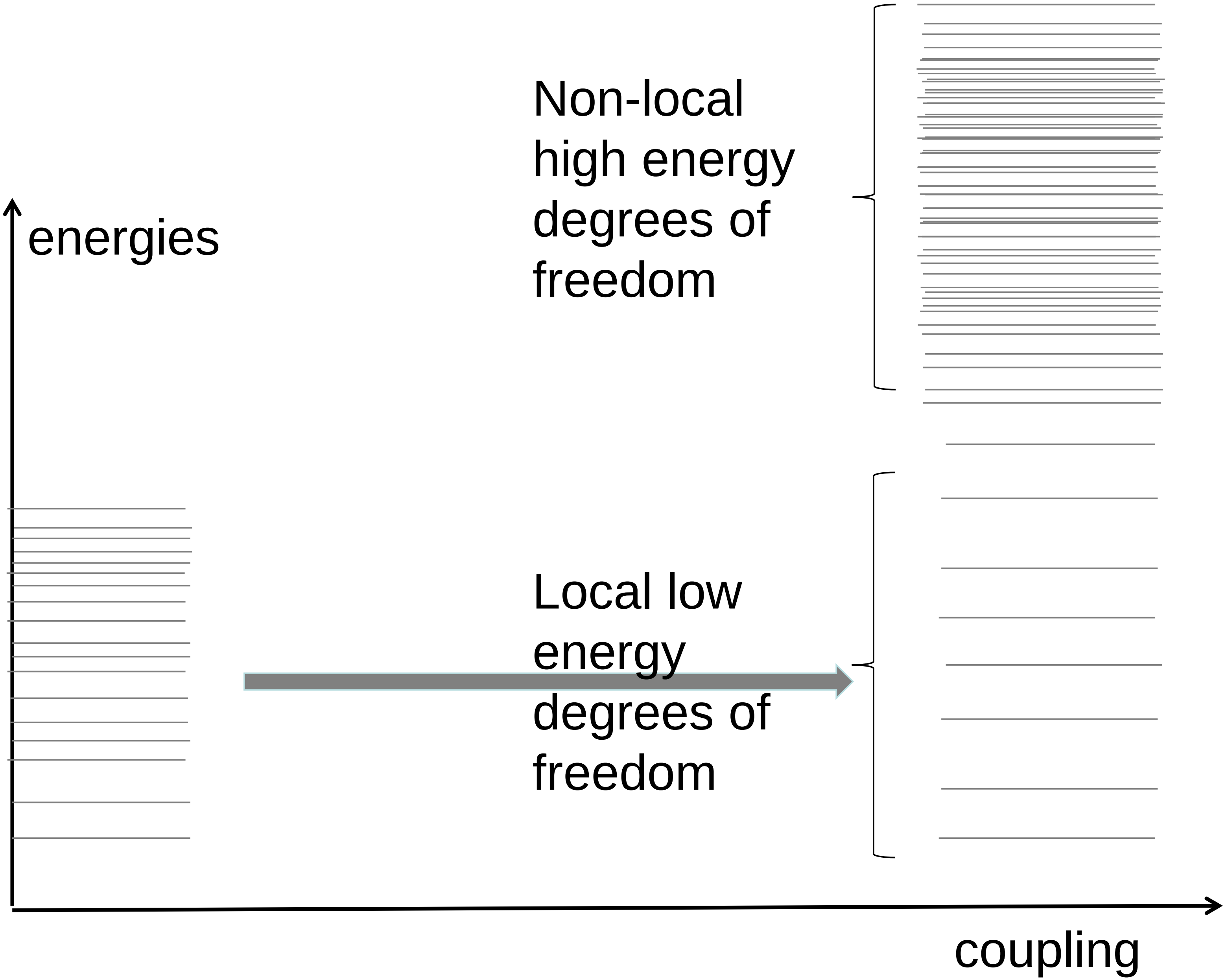
© nature

Gao, Jafferis, Wall ‘16
Maldacena, Stanford, Yang ‘17
JdB, van Breukelen, Lokhande, Papadodimas, Verlinde ‘19

Can use entanglement to engineer wormholes

Upshot

- Entanglement entropy is a natural “observable” for quantum gravity.
- Most fundamental degrees of freedom are non-local, only a small subset (“ripples on the sea”) are local and manifest themselves as local degrees of freedom (code subspace).
- Smooth spacetime requires that the microscopic degrees of freedom be very entangled.
- Standard results in quantum information theory imply e.g. the linearized Einstein equations and many other properties of general relativity.
- Several interesting puzzles remain.



Non-local high energy degrees of freedom

Local low energy degrees of freedom

Outlook:

- Did not talk about: ETH, scrambling, SYK, chaos, complexity
- Also not about various generalizations: Rényi entropies, mutual information, relative entropy, algebraic QFT,….
- Generalize entanglement entropy to deal with non-minimal surfaces? For example differential entropy

Balasubramanian, Chowdhury, Czech, JdB, Heller `14

- Develop better techniques to compute entanglement entropy.
- There seems to be a deep relation between quantum information theory and locality, causality, chaos and unitarity, what is the precise relation? (eg Hartman-Kundu-Tandjini ‘16 vs Faulkner-Lee-Parrikar-Wang ’16; JdB Lamprou ‘19))
- Reformulate gravity purely in terms of entanglement entropy? Implications?
- Berry phases?

Czech,JdB,Ge,Lamprou ‘19

Jacobson `95 `15
JdB, Haehl, Heller, Myers ’16
Czech, Lamprou, McCandlish, Mosk, Sully ‘16