Wilson loops in the Large-N limit of QCD

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Contents

Smeared Wilson loops in Euclidean matter-free SU(N) 4D gauge theory

- Wilson loops are gauge invariant, geometrical and form a complete set of observables.
- Classically, an untraced Wilson loop is an $\mathrm{SU}(N)$ matrix.
- In the Path Integral these unitary matrices become random compact variables.
- New UV singularities undermine this naive view.
- Smearing restores it at the cost of introducing a resolution length scale, $\sqrt{s}$.

$$F^f_{\mu,s} = D^{\text{adjoint}}_\nu F^f_{\mu,\nu} \quad \text{with} \quad A^f_\mu(x, s=0) = B^f_\mu(x).$$

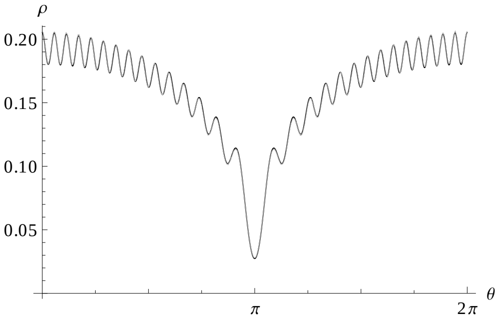
- Smearing becomes diffusion in loop space:

$$\partial_s \mathrm{Tr}\left\langle W_f(\mathscr{C}, s)\right\rangle = \oint_\sigma \frac{\delta^2 \, \mathrm{Tr}\left\langle W_f(\mathscr{C}, s)\right\rangle}{\delta x^2_\mu(\sigma)} \equiv \hat{L}\, \mathrm{Tr}\left\langle W_f(\mathscr{C}, s)\right\rangle .$$

- At $N = \infty$ the MM equations fix the asymptotic behavior as $s \to 0$.

The infinite N transition point

- The single eigenvalue distribution of smeared Wilson loops undergoes a “compactification” transition on the unit circle at $N = \infty$.
- Below is an example at $N = 29$ of a 6×6 smeared Wilson loop of size 0.6 Fermi



QCD string

- Nobody knows what the QCD string is, but the MM equations motivate the dream.
- At $N = \infty$ the QCD string should become free.
- The QCD free string is described by a 2D FT on a disk.
- This description should provide a well converging semi-classical expansion for large loops.
- For large loops one may employ effective string theory instead.
- The effective string theory is known, at least for smooth loops.
- It reproduces terms in an asymptotic expansion in inverse loop size.
- As any effective theory, it does this order by order, admitting the maximal number of free parameters that is allowed by symmetries.
- Effective string theory is not very specific: purportedly it applies to a larger class of theories, beyond 4D nonabelian gauge theories.

Long term Project Objective

Accepting the facts I outlined, calculate approximately the string tension σ at $N = \infty$ in terms of the perturbative scale Λ_{QCD} by matching EST and PT for some simple Wilson loop in 4D.

Introduction

- We study Wilson loop operators $W(\mathscr{C})$ in 4D Euclidean SU(N) pure gauge theory. $\mathscr{C}$ is a closed, non-selfintersecting, continuous curve in $\mathbb{R}^4$, with a finite number of kinks $\longrightarrow$ rectangles on the lattice.
- Perimeter and corner divergences of W require smearing.
- Extra (continuous) smearing parameter s represents effective thickness of $\mathscr{C}$; $\sqrt{s}$ is the observer's resolution.
- At large-N a transition separates a weakly coupled short distance regime from a strongly coupled long distance regime.

- Small loops are insensitive to the compact nature of SU(N), but for large loops where confinement holds, the full group is explored.
- At the transition point the spectral gap of the Wilson loop closes. It is natural to match PT and a long distance description at the transition.
- Traces of Wilson loops are smooth through the transition even at $N = \infty$.

- We first wish learn how to connect the two regimes using numerics.
- This work used the lattice to learn how well an effective string description works close to the transition.

Effective String Theory

EST gives an asymptotic expansion for a large loop, starting from the minimal area configuration employing Nambu-Goto + boundary terms. It can be phrased for a dilated loop ($\mathscr{C} \to \rho\,\mathscr{C}$), asymptotically as $\rho \to \infty$. The string tension $\sigma > 0$ is used to set the scale of the EST; it is s-independent. We assume that we stop at an order in ρ^{-1} for which all terms that would vanish when $s \to 0$ can be dropped and set $\sigma = 1$.

$$\log(W(\rho\,\mathscr{C})) \sim -\rho^2 \mathrm{Area}_{\min}(\mathscr{C}) + \Gamma_P \rho\, \mathrm{Length}(\mathscr{C}) + \Gamma_1(\mathscr{C})\log(\rho) + \Gamma_2(\mathscr{C}) + \Gamma_K(\mathscr{C}) + \Gamma_3(\mathscr{C})/\rho^2 + \Gamma_4(\mathscr{C})/\rho^3 + \Gamma_5(\mathscr{C})/\rho^4 + \mathscr{O}(1/\rho^5).$$

- Γ_P is a non-universal number independent of $\mathscr{C}$; it is s-dependent and diverges as $s^{-1/2}$ when $s \to 0$. $\Gamma_{1,2,3,5}$ are universal scale invariant functions of ρ.
- $\Gamma_4(\mathscr{C})$ is scale-invariant and universal up to a non-universal coefficient.
- $\Gamma_K(\mathscr{C}) = \sum_{\mathrm{kinks}} F(\dot{x}_+ \cdot \dot{x}_-|_{\mathrm{kink}})$ with the parametrization choice $\dot{x}^2 = 1$; $F()$ is s-dependent and diverges (as $\log s$ at leading order in PT) when $s \to 0$.
- An assumption implicitly made in numerical tests of EST is that $F(\gamma)$ depends on $\mathscr{C}$ *only* through γ.

Simulation parameters

- Averages of smeared rectangular ($L \times L$, $L \times (L+1)$, $L \times 2L$) Wilson loops on a symmetric hypercubic lattice were obtained from 160 uncorrelated gauge fields.
- The gauge action was of the single-plaquette Wilson type.
- We mainly used couplings $0.359 \leq b = \frac{\beta}{2N^2} \leq 0.369$, spaced by $\Delta b = 0.001$; gauge fields at neighboring b's were separated by 500 complete $\mathrm{SU}(2)$ updates and 500 complete over-relaxation passes.
- Set of N-values used was $7, 11, 13, 19, 29$.
- Measurements were done at different smearing levels, mostly in the range $0.2 \leq S \leq 0.4$. (lattice smearing: $\partial_S U_\mu = -\frac{1}{2}\left(V_\mu - V_\mu^\dagger - \frac{1}{N}\,\mathrm{Tr}\left(V_\mu - V_\mu^\dagger\right)\right)U_\mu$, with ordered 1×1 loops V_μ.)
- Our Wilson loops on the lattice are products over links l round rectangles of sides $L_{1,2}$.

$$W_N(L_1, L_2, b, S, V) = \frac{1}{N}\langle \mathrm{Tr} \prod_{l \in \mathscr{C}} U_l \rangle\,.$$

- Because of exponentiation, the fits were applied to

$$w_N(L_1, L_2, b, S, V) = -\log W_N(L_1, L_2, b, S, V)\,.$$

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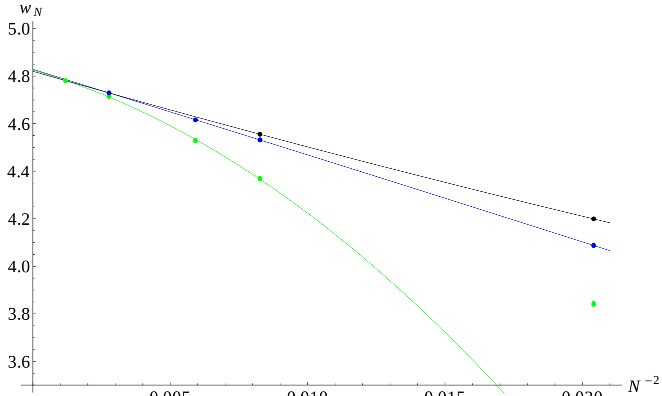
Square loops and their infinite-N,V limits

We need $\lim_{N\to\infty}\left(\lim_{V\to\infty} w_N(L,b,S,V)\right)$ for square $L\times L$ loops.
Large-N reduction provides a shortcut but to apply it requires tests of finite-volume effects. In general, we have:

$$w_N(V) = w_\infty(V) + \frac{a_1(V)}{N^2} + \frac{a_2(V)}{N^4} + \dots$$

- Method 1)
  At fixed N, use volumes sufficiently large for finite-volume effects to be negligible at that N ($V = 24^4$, 18^4, 14^4, 12^4 for $N = 7$, 11, 19, 29, resp.) and set $V = \infty$.
- Method 2)
  First take $N \to \infty$ at fixed V by fitting the data including $1/N^2$ and $1/N^4$ corrections.
  Large-N reduction says that in the confining regime $w_\infty(V) = w_\infty(V = \infty)$.
  - 2a) $w_\infty(V = 12^4)$ from $N = 11$, 13, 19, 29
  - 2b) $w_\infty(V = 14^4)$ from $N = 7$, 11, 13, 19

Square loops: infinite-N,V limits



Plots of $w_N(L=9, b=0.368, S=0.4, V)$ as a function of $1/N^2$:

$V=12^4$, $V=14^4$, $V=24^4$ (at $N=7$), $V=18^4$ (at $N=11$).

Square loops and their infinite-N,V limits

- We obtained reasonable values of χ^2/N_{dof} for the fits, except for the $V = 12^4$ case at $b \leq 0.361$. Perhaps this reflects the $N = \infty$ bulk transition.
- We found agreement between the three results within our statistical errors of about 0.1%.
- Truncating the expansion at order $1/N^2$ results in very large χ^2/N_{dof}. Hence, we cannot set $a_2/N^4 = 0$.
- The $N = 29$ result is crucial for the $V = 12^4$ case; however, using $N = 7$ too would require an $1/N^6$ correction.
  As expected, when V gets closer to the critical size at which a Polyakov-Z_N breaks, higher N's are needed in order to extract $\lim_{N,V\to\infty} w_N(V)$.
- The computation time goes as $\sim N^3 V$.
  method 2a) [$V = 12^4$] is 1.75 times more expensive than method 2b) [$V = 14^4$]; method 1) [large V's] is 2.5 times more expensive than method 2b).

  Could save computer time, but to be confident about $\lim_{N,V\to\infty} w_N(L, b, S, V)$ we needed agreeing results from 1), 2a) and 2b).

Infinite-N string tension from square loops.

- Consider $L \times L$ loops at fixed b, S. $w_\infty(L) \equiv \lim_{N,V\to\infty} w_N(L,b,S,V)$.

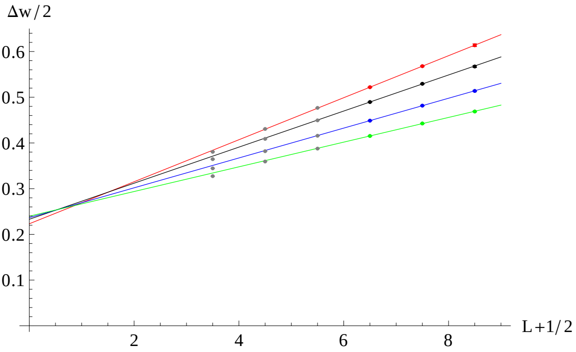
$$w_\infty(L) + \frac{1}{4}\log L^2 = c_1 + c_2 L + \sigma L^2 + \mathcal{O}\left(\frac{1}{\sigma L^2}\right).$$

- The log term comes from the effective string description; including it gives good fits while excluding it gives bad fits.
  We have also did a separate global fit to $L \times L$, $L \times L+1$, and $L \times 2L$ loops and obtained agreement with the EST value $1/4$.
- Neglecting corrections of order $\frac{1}{\sigma L^3}$, we fit

$$\frac{1}{2}\left(w_\infty(L+1) - w_\infty(L) + \frac{1}{2}\log\left(1+\frac{1}{L}\right)\right) = \sigma\left(L+\frac{1}{2}\right) + \frac{c_2}{2} + \mathcal{O}\left(\frac{1}{\sigma L^3}\right)$$

  to a straight line as a function of $L+\frac{1}{2}$ to determine σ and the perimeter coefficient c_2.
- Next, we fit $w_\infty(L) + \frac{1}{4}\log L^2 - \sigma L^2 - c_2 L$ to a constant, c_1.
- For the b and S values we use, the 5×5 loops fall in the vicinity of the large-N transition; smaller loops have a gap in their eigenvalue distribution. In our fits we use square loops with $6 \le L \le 9$.

Lattice string tension



Plots of $\frac{\Delta w}{2} = \frac{1}{2}\left(w_\infty(L+1) - w_\infty(L) + \frac{1}{2}\log\left(1+\frac{1}{L}\right)\right)$ [obtained with method 1)] at $S = 0.4$ and $b = 0.36$, 0.362, 0.365, 0.368

Error bars are not visible in the plot. Straight lines show linear fits (using $6 < L + \frac{1}{2} < 9$).

Continuum limit

- σ does not depend on smearing parameter S within statistical errors which decrease with increasing S.
- Extrapolations to continuum was carried out with scale $\xi_c(b)$ ($\approx L_c$ where center symmetry breaks)

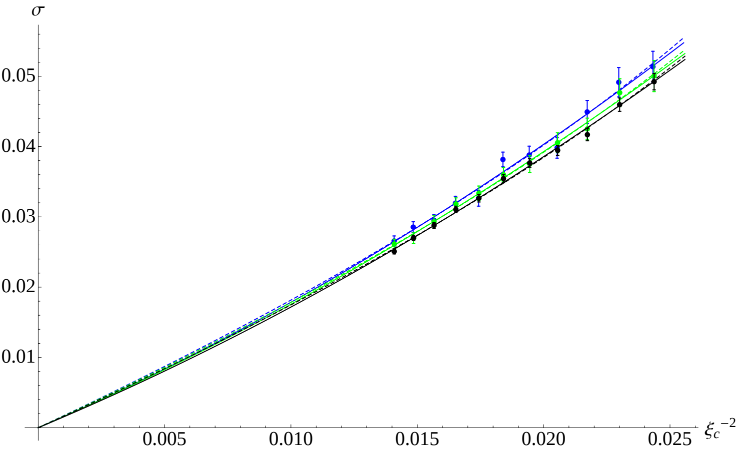
$$\xi_c(b) = 0.26 \left(\frac{\bar\beta_1}{\bar\beta_0^2} + \frac{b_I(b)}{\bar\beta_0} \right)^{-\frac{\bar\beta_1}{2\bar\beta_0^2}} \exp\left[\frac{b_I(b)}{2\bar\beta_0} \right] \exp\left[\frac{\bar\beta_2}{2\bar\beta_0^2 b_I(b)} \right]$$

$b_I(b) = \lim_{N,V\to\infty} b\, W_N(L=1, b, S=0, V)$ is the tadpole improved coupling and the coefficients $\bar\beta_i = \beta_i / N^{i+1}$ for large N.
- We employed two different fits for the relation between $\sigma(b)$ and $\xi_c(b)$

$$\sigma(b) = \frac{d_0}{\xi_c(b)^2} + \frac{d_1}{\xi_c(b)^4}, \qquad \frac{1}{\xi_c(b)^2} = f_0^{-1}\sigma(b) + f_1 \sigma(b)^2$$

- We got for the infinite-N continuum string tension:
$\lim_{b\to\infty} \sigma(b)\xi_c^2(b) = 1.6(1)(3)$
Sources of systematic error: two fits d_0, f_0; ranges $0.359 \le b \le 0.369$ and $0.362 \le b \le 0.367$; different methods for $\lim_{N,V\to\infty}$.

Continuum limit



String tension from $\lim_{N,V\to\infty} w_N(L,b,S,V)$ obtained with:

method 1) [large V's], method 2a) [$V = 12^4$] method 2b) [$V = 14^4$].

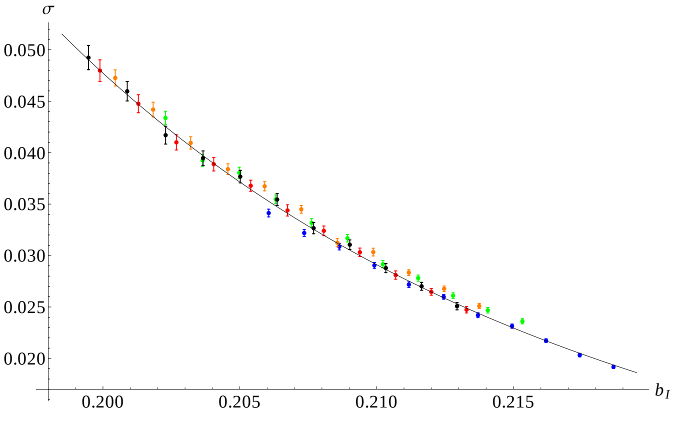
Solid and dashed lines: different fit functions ($0.359 \le b \le 0.369$)

Continuum string tension: Wilson and Polyakov loops

- In terms of $\Lambda_{\overline{MS}}$, our result is $\sigma/\Lambda^2_{\overline{MS}} = 3.4(2)(6)$.
- From Polyakov loop correlators, the $N = \infty$ continuum result is [Allton, Teper, Trivini (2008)]: $\sigma/\Lambda^2_{\overline{MS}} = 3.95(3)(64)$.
- Independent study [Gonzalez-Arroyo, Okawa (2012) → arXiv:1206.0049] of rectangular Wilson loops: $\sigma/\Lambda^2_{\overline{MS}} = 3.63(3)$ (with the same cont-extr. method).
- The systematic errors are too large to claim evidence for a difference from Polaykov case, but at the statistical level there is a discrepancy.

String tension at finite N

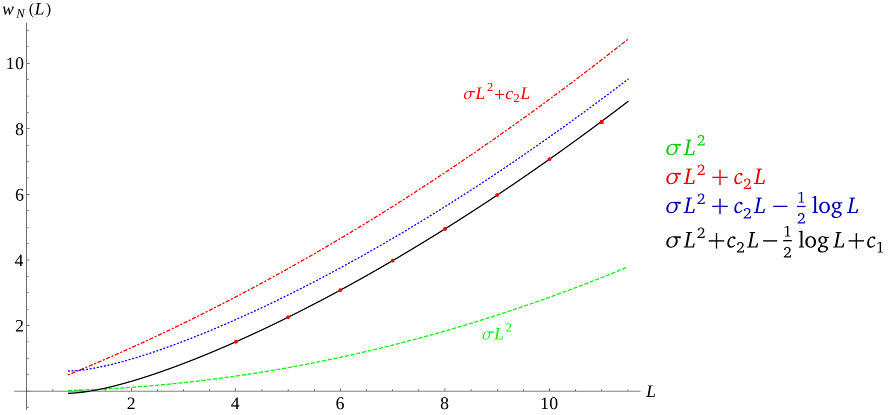
- In order to get a feel for the commutativity of $N \to \infty$ and $b \to \infty$ we determine the string tension σ_N at $N = 7, 11, 19, 29$.
- At fixed b: $\sigma_N(b) = \sigma_\infty(b) + \frac{h(b)}{N^2}$ but $h \approx 10\sigma_\infty$.
- However, finite-N corrections get largely absorbed by the improved coupling $b_I(b, N)$ leaving a weak N-dependence in the continuum limits



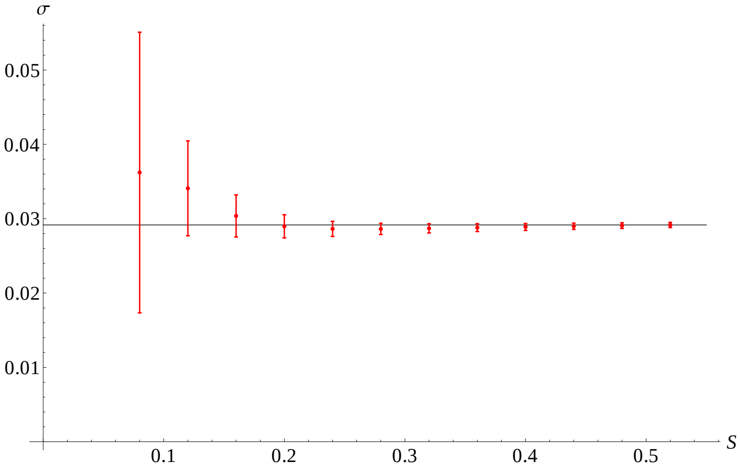
$N = 7$
$N = 11$
$N = 19$
$N = 29$
$N = \infty$
(from w_∞)

Various contributions to the Exponent

Plot of $w_N(L) = -\log W_N(L)$ for $N = 11$ on $V = 18^4$ at $b = 0.365$ and $S = 0.28$. Obtaining a plot like this was the main objective in our recent work. The fit parameters used to plot the analytic functions were obtained from the data at $6 \le L \le 9$. They are $\sigma = 0.02863$, $c_2 = 0.6041$, $c_1 = -0.6788$. The large-N transition is at $w_\infty(L) = 2$.

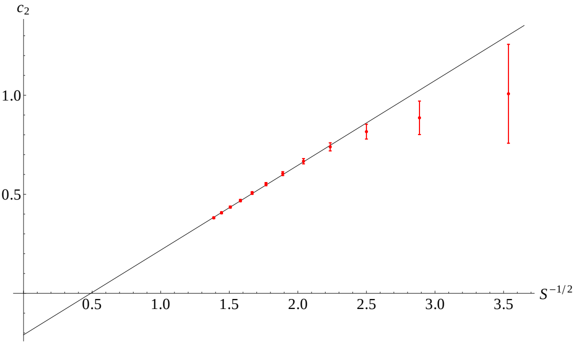


Smearing dependence of string tension



Example for $N = 11$, $b = 0.365$, $V = 18^4$.
σ determined using square loops with $6 \leq L \leq 9$.

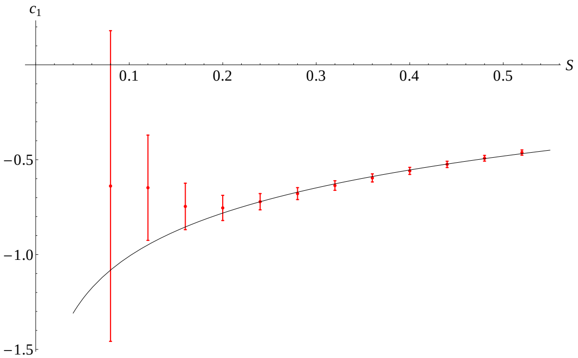
Smearing dependence of perimeter term



Perimeter coeff. c_2 (for $N = 11$, $b = 0.365$). Fit: $c_2 = -0.2097 + 0.4279/\sqrt{S}$.

- There is no divergence as $S \to 0$ on the lattice obviously. We see a window where the behavior that would cause a divergence in the continuum is evident.
- Tree-level PT perimeter term is $\frac{g^2 C_F}{2} \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{2l}{\sqrt{s}}$. In the above example: $\to$ $\frac{g^2 N}{4\pi} \approx 1.08$.

Smearing dependence of the rho-independent term



- S-dependence of the L-independent term c_1 is consistent with a $\log(S)$, $S \to 0$ divergence. For $N = 11$, $b = 0.365$: $c_1 = -0.2538 + 0.3278 \log S$ in the continuum window.
- Corner div. at tree-level: $\frac{g^2 C_F}{2} \frac{1}{\pi^2} \log s$ In the above case, $\frac{g^2 N}{4\pi} \approx 1.03$.

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Shape-dependence: non-square loops

- Study the shape dependence of the size-independent term (c_1) in $w_N = -\log W_N$.
- Scaling-invariant shape parameter for rectangular $L_1 \times L_2$ loops:

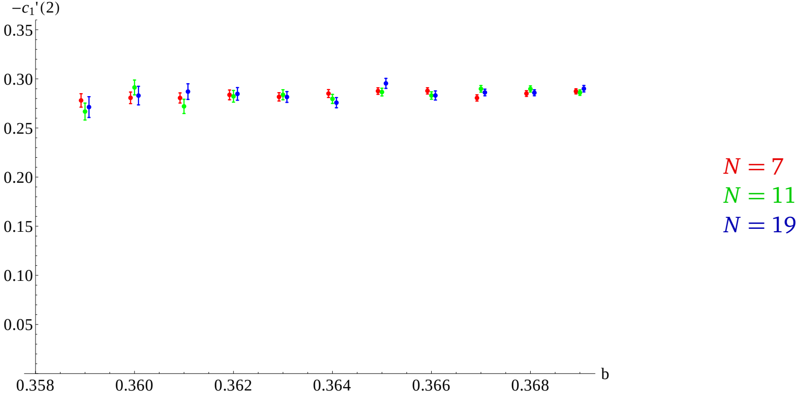
$$\zeta = \frac{L_1}{L_2} + \frac{L_2}{L_1} .$$

- At fixed b, S, V, and fixed finite N, we expect

$$w_N(L_1, L_2) + \frac{1}{4}\log L_1 L_2 = c_{1,N}(\zeta) + c_{2,N}\frac{L_1 + L_2}{2} + \sigma_N L_1 L_2 + \mathcal{O}\left(\frac{1}{\sigma_N L_1 L_2}\right) .$$

- Using σ_N and $c_{2,N}$ obtained from square loops, we determine $c_{1,N}\left(\zeta = \frac{5}{2}\right)$ from $L \times 2L$ loops and compare it with $c_{1,N}(2)$
- From square and almost square $L \times L \pm 1$ loops, we obtain $c'_{1,N}(2)$.
- Allowing the coefficient of the $\log L_1 L_2$ term to become a fit parameter and expanding $c_1(\zeta)$ around $\zeta = 2$, we simultaneously fit $L \times L$, $L \times L + 1$, and $L \times 2L$ loops which confirms the expected value of $1/4$ and previous results for ζ-dependence of c_1.

Example for $-c'_{1,N}(2)$ as a function of b at $S = 0.4$:



- No significant dependence on b, N, S
- The effective string prediction is $c'_1(2) \approx -0.162276$.
- Similar deviation from effective string theory were observed by Gonzalez-Arroyo and Okawa [$\rightarrow$ arXiv:1206.0049]

More about shape dependence

For a rectangular loop in tree-level continuum perturbation theory:

$$w_N^{\mathrm{PT}}(l_1,l_2,s)=\frac{g^2C_2}{2}\left[\frac{1}{(2\pi)^{\frac{3}{2}}}\left(\frac{l_1+l_2}{\sqrt{s}}\right)+\frac{1}{\pi^2}\log\left(\frac{s}{l_1l_2}\right)+h_0\left(\frac{l_2}{l_1}\right)+\mathcal{O}\left(\frac{s}{l_i^2}\right)\right]$$

(h_0 has an integral representation in terms of error functions).

- Terms divergent as $s\to 0$ (outside the reach of eff. string theory) enter additively in $w_N=-\log W_N$.
- The difference from the effective-string prediction for the shape-dependent term might be explained if we assume that the measured w_N is given by a sum of separate effective-string and tree-level PT contributions. This would require $\frac{g^2N}{4\pi}\approx 0.49$, in disagreement with previous coupling estimates.

Higher orders in EST might conspire to produce constancy of deviation. Using results from work on Z_2 strings by the Turin group we estimated various higher order contributions in EST. The outcome is that the corrections are quite sizable and therefore the discrepancy might be immaterial. In that case, the constancy of the difference between data and the string answer as the physical size of the loop is varied must be viewed as an accident.

Contents

Does EST work in the presence of kinks at zero smearing ?

- In any D≥ 3 EST the logarithmic term in ρ comes from the corners.
- In 4D PT the corners make log-log contributions to w_N after RG improvement.
- Leading Log perturbation theory indicates that the ρ dependence of kinks of angle $\gamma = \dot{x}_+ \cdot \dot{x}_-$ is determined by the anomalous dimension $\Gamma(\gamma)$. This is a consequence of UV-IR FT factorization.
- Once the term of order $\log\log\rho$ enters, the entire EST framework is put under question.
- At the minimum, corners would require a multiplicative factor in EST which might be represented as an off-shell vertex operator with some free parameters.
- For my project kinks should be avoided if possible.
- On the other hand, corners provide an opportunity to make progress on the QCD string problem.

Contents

Does EST work in the presence of kinks at nonzero smearing ?

- Smearing eliminates the UV divergences associated with kinks.
- UV-IR Factorization may be lost for $s > 0$.
- In principle, smearing dependence may violate the universality of EST predictions.
- Perhaps, diffusion in loops space could be incorporated into EST. This might be a difficult problem.
- Another open problem is to work out the consequences of smearing in $\mathcal{N} = 4$ YM and see if there is a dual and what the latter has to do with diffusion in loops space.

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Conclusions

- Within everybody's estimates of the systematic errors, the results for the large-N string tension from Wilson loops agree with those obtained from Polyakov loop correlators.
- The leading order stringy parametrization for Wilson loops holds relatively well all the way down to the large-N transition point.
- The applicability of full EST to smeared $\mathrm{SU}(N)$ Wilson loops with kinks is less clear.
- The shape dependence of planar Wilson loops presents an interesting case for testing the limitations of the effective string approach. For this, further numerical checks are required (loops with different combinations of corners, with self-intersections,...).
- It would be useful to find a large-N phase transition for Polyakov loop correlators in 4D $\mathrm{SU}(N)$ YM because these are accessible to LGT and have no kinks.