Lattice calculation of isospin breaking corrections to hadronic observables

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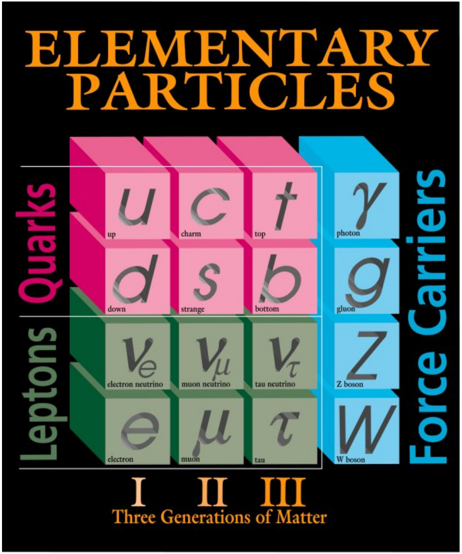
isospin symmetry

- among the questions left open by the standard model there is the origin of flavour
- the two lightest quarks, the up and the down, have different masses and different electric charges
- nevertheless

$$\frac{m_d - m_u}{\Lambda_{QCD}} \ll 1$$

$$(e_u - e_d)\alpha_{em} \ll 1$$

- for these reasons the group of rotations in this bidimensional (complex) "flavour" space is a *good* and *very useful* approximate symmetry of the real world



Fermilab 95-759

isospin symmetry

- rotations in the bidimensional flavour space

$$\begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix} \mathbf{e}^{-\mathbf{i}\alpha^{\mathbf{I}} \frac{\sigma^{\mathbf{I}}}{\mathbf{2}}} \begin{pmatrix} D[U] + m_{ud} & 0 \\ 0 & D[U] + m_{ud} \end{pmatrix} \mathbf{e}^{\mathbf{i}\alpha^{\mathbf{I}} \frac{\sigma^{\mathbf{I}}}{\mathbf{2}}} \begin{pmatrix} u \\ d \end{pmatrix}$$

- the two light quarks are into an $SU(2)$ doublet and hadrons can be classified according to the representations of the "angular momentum" algebra

- from isospin symmetry combined with parity we know, for example, that an even number of pseudoscalar mesons cannot scatter (trough QCD) into an odd number of pseudoscalar mesons,

$$K^0 \longrightarrow \pi\pi \underbrace{\longrightarrow \pi\pi\pi}_{\text{forbidden}} \qquad \langle \pi\pi | H_W^{\Delta S=1} | K^0 \rangle = \begin{cases} A_0 \; e^{i\delta_0} \\ A_2 \; e^{i\delta_2} \end{cases}$$

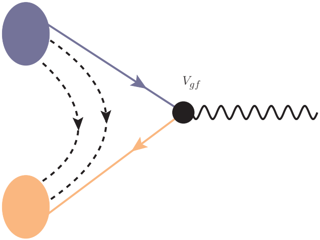
- where the strong phases δ_0 and δ_2 coincide with the scattering phases

- unexplained experimental evidence $A_0 \gg A_2$, the so called $\Delta I = 1/2$ rule

- ...

why isospin breaking?

$$\hat{V}^{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$



except for the ones in the third row, CKM matrix elements can be extracted by (semi)leptonic decay rates, according to

$$V_{gf} = \frac{\text{experiment}}{\text{theory}}$$

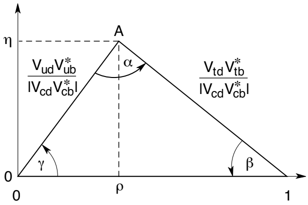
why isospin breaking?

Unitarity of the CKM matrix implies several relations among the different couplings, three of these are the so-called unitarity triangles:

$$V_{ud}V_{us}^{\star} + V_{cd}V_{cs}^{\star} + V_{td}V_{ts}^{\star} = 0$$

$$V_{us}V_{ub}^{\star} + V_{cs}V_{cb}^{\star} + V_{ts}V_{tb}^{\star} = 0$$

$$\mathbf{V_{ud}V_{ub}^{\star} + V_{cd}V_{cb}^{\star} + V_{td}V_{tb}^{\star} = 0}$$



the unitarity triangle is the scalar product of the d-column times the b-column of the CKM matrix we do have a lot of precise experimental measurements in the quark flavour sector of the standard model that, combined with CKM unitarity (first row), allow us to **measure** hadronic matrix elements

M.Antonelli et al. Eur.Phys.J.C69 (2010)
G.Colangelo talk at Lattice2012

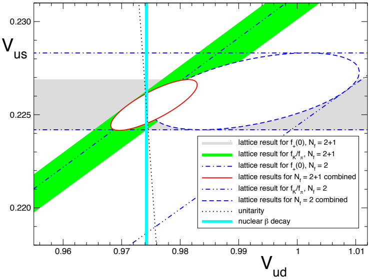
$$\begin{cases} \left|\frac{V_{us}F_K}{V_{ud}F_\pi}\right| = 0.2758(5) \\ \\ \left|V_{us}F_+^{K\pi}(0)\right| = 0.2163(5) \end{cases} \qquad \begin{cases} |V_{ud}|^2 + |V_{us}|^2 = 1 \\ \\ |V_{ud}| = 0.97425(22) \end{cases}$$

where $|V_{ud}|$ comes by combining 20 super-allowed nuclear β-decays and $|V_{ub}|$ has been neglected because smaller than the uncertainty on the other terms, combine to give

$$|V_{us}| = 0.22544(95)$$

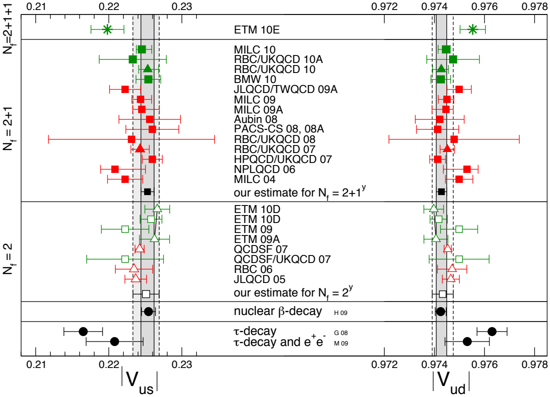
$$F_+^{K\pi}(0) = 0.9595(46)$$

$$\frac{F_K}{F_\pi} = 1.1919(57)$$



lattice QCD is **still** needed to **postdict** these quantities and, in case, to falsify the standard model concerning theoretical predictions, and lattice QCD in particular, these matrix elements are among the well known quantities

FALG Eur.Phys.J. C71 (2011)
G.Colangelo talk at Lattice2012



$$F_+^{K\pi}(0) = 0.956(8) \qquad \sim 0.8\% \qquad\qquad \frac{F_K}{F_\pi} = 1.193(5) \qquad \sim 0.5\%$$

to do better we should include effects that we have been neglecting up to now. . .

F_K/F_π & $F_+^{K\pi}(q^2)$ beyond the isospin limit

- it is useful to *divide* the isospin breaking effects into strong and electromagnetic ones,

$$\underbrace{m_u \neq m_d}_{\textbf{QCD}} \qquad\qquad \underbrace{e_u \neq e_d}_{\textbf{QED}}$$

- in the particular and (lucky) case of these observables, the correction to the isospin symmetric limit due to the difference of the up and down quark masses (**QCD**) can be estimated in *chiral perturbation theory*,

$$\begin{cases} F_+^{K\pi}(0) = 0.956(8) \qquad \sim 0.8\% \\ \\ \left(\dfrac{F_+^{K^+\pi^0}(q^2)}{F_+^{K^0\pi^-}(q^2)} - 1 \right)_{QCD} = 0.029(4) \end{cases}$$

A. Kastner, H. Neufeld Eur.Phys.J.C57 (2008)

$$\begin{cases} \dfrac{F_K}{F_\pi} = 1.193(5) \qquad \sim 0.5\% \\ \\ \left(\dfrac{F_{K^+}/F_{\pi^+}}{F_K/F_\pi} - 1 \right)_{QCD} = -0.0022(6) \end{cases}$$

V. Cirigliano, H. Neufeld Phys.Lett. B700 (2011)

- we need first principle lattice QCD calculations to avoid uncertainties coming from the effective theory
- but the home message is: reducing the error on these quantities without taking into account isospin breaking is useless. . .

RM123

JHEP 1204 (2012)

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the gauge configurations

β	am^L_{ud}	am^L_s	L/a	N_{conf}	a (fm)	$Z_P(\overline{MS}, 2GeV)$
3.80	0.0080	0.0194	24	150	0.0977(31)	0.411(12)
	0.0110		24	150		
3.90	0.0030	0.0177	32	150	0.0847(23)	0.437(07)
	0.0040		32	150		
	0.0040		24	150		
	0.0064		24	150		
	0.0085		24	150		
	0.0100		24	150		
4.05	0.0030	0.0154	32	150	0.0671(16)	0.477(06)
	0.0060		32	150		
	0.0080		32	150		
4.20	0.0020	0.0129	48	100	0.0536(12)	0.501(20)
	0.0065		32	150		

- gauge configurations for this study have been taken from the gauge ensembles made publicly available by the ETMC collaboration
- **caveat:** the Twisted Mass discretization breaks isospin at finite lattice spacing
- we have been working in a mixed-action setup by introducing $O(a^2)$ errors coming from violations of unitarity
- in what follows I shall illustrate our method without discussing these technical details by thinking to a isospin-symmetric lattice regularization

isospin breaking on the lattice

- the calculation of QED isospin breaking effects on the lattice it has been done for the first time in
  Duncan, Eichten, Thacker, Phys. Rev. Lett. 76 (1996)
- QED is treated in the quenched approximation in its "non-compact" formulation
- because the photons are massless and unconfined this approach may introduce large finite volume effects...
- we shall come back on QED effects later in this talk

- the calculation of QCD isospin breaking effects on the lattice poses a theoretical problem

$$
\begin{aligned}
Z &= \int DU D\psi \; e^{-S_g[U]+S_f[U;m_u,m_d]} \\
&= \int DU \; e^{-S_g[U]} \; \underbrace{\det(D[U]+m_u)\;\det(D[U]+m_d)}_{\text{must be real and } >0}
\end{aligned}
$$

- if $m_u \neq m_d$ but very light, this can be only achieved by recurring to non (ultra) local and, consequently, very expensive fermion formulations
- furthermore the effect is very small and it can be extremely difficult to see it with limited statistical accuracy

our QCD isospin breaking on the lattice

- our idea is to calculate QCD isospin corrections **at first order** in $\Delta m_{ud} = (m_d - m_u)/2$:

$$
\begin{aligned}
S &= \bar{u}\,(D[U] + m_u)\,u + \bar{d}\,(D[U] + m_d)\,d \\
&= \underbrace{\bar{u}\,(D[U] + m_{ud})\,u + \bar{d}\,(D[U] + m_{ud})\,d}_{S_0} - \overbrace{\frac{m_d - m_u}{2}(\bar{u}u - \bar{d}d)}^{\boldsymbol{\Delta m_{ud}\hat{S}}}
\end{aligned}
$$

- the calculation of an observable proceeds as follows

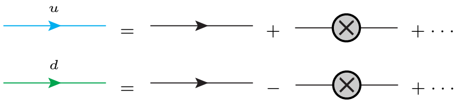
$$
\begin{aligned}
\langle\mathcal{O}\rangle - \boldsymbol{\Delta}\langle\mathcal{O}\rangle &= \frac{\int DU\ e^{-S_g[U]-S_0[U]+\Delta m_{ud}\hat{S}}\ \mathcal{O}}{\int DU\ e^{-S_g[U]-S_0[U]+\Delta m_{ud}\hat{S}}} = \frac{\int DU\ e^{-S_g[U]-S_f^0[U]}\ (1+\Delta m_{ud}\hat{S})\ \mathcal{O}}{\int DU\ e^{-S_g[U]-S_f^0[U]}\ (1+\Delta m_{ud}\hat{S})} \\
&= \langle\mathcal{O}\rangle + \boldsymbol{\Delta m_{ud}}\langle\hat{\mathbf{S}}\ \mathcal{O}\rangle - \underbrace{\Delta m_{ud}\langle\hat{S}\rangle}_{=0}
\end{aligned}
$$

our QCD isospin breaking on the lattice

- to insert $\bar{u}u - \bar{d}d$ within a correlation function amounts (after fermionic Wick contractions) to calculate **the same observables** but with **light propagators squared**

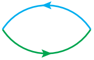
$$
\begin{aligned}
\mathcal{S}_u &= \frac{1}{D[U]+m_{ud}-\Delta m_{ud}} &= \frac{1}{D[U]+m_{ud}} + \frac{\Delta m_{ud}}{(D[U]+m_{ud})^2} \\
\mathcal{S}_d &= \frac{1}{D[U]+m_{ud}+\Delta m_{ud}} &= \frac{1}{D[U]+m_{ud}} - \frac{\Delta m_{ud}}{(D[U]+m_{ud})^2}
\end{aligned}
$$

- relations that can be represented diagrammatically as

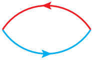


our QCD isospin breaking on the lattice: notation

in the following, two-point functions of pseudoscalar mesons will be represented graphically as

$$C_{\pi^+\pi^-}(t,\vec{p}) \;=\; -$$


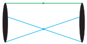
$$= \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle\, \bar{u}\gamma_5 d(x)\;\bar{d}\gamma_5 u(0)\,\rangle$$

$$C_{K^+K^-}(t,\vec{p}) \;=\; -$$


$$= \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle\, \bar{u}\gamma_5 s(x)\;\bar{s}\gamma_5 u(0)\,\rangle$$

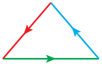
nucleon two-point functions as,

$$C^{\pm}_{pp}(t,\vec{p}) \;=\; -$$


$$+$$


$$= \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle \left[\epsilon_{abc}(\bar{u}_a C\gamma_5 \bar{d}_b^T)\bar{u}_c \frac{1\pm\gamma^0}{2}\right](x)\;\left[\epsilon_{def}\frac{1\pm\gamma^0}{2} u_d(u_e^T C\gamma_5 d_f)\right](0)\,\rangle$$

three point functions as

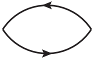
$$C^{\mu}_{K^0\pi^-}(t;\vec{p}_K,\vec{p}_\pi) = -$$


$$= \sum_{\vec{x},\vec{y}} e^{-i\vec{p}_\pi\cdot\vec{x}} e^{-i\vec{p}_K\cdot(\vec{x}-\vec{y})} \langle\, \bar{d}\gamma^5 s(\vec{y},T/2)\;\bar{s}\gamma^\mu u(\vec{x},t)\;\bar{u}\gamma^5 d(0)\,\rangle$$

our QCD isospin breaking on the lattice: notation

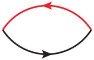
in the following, two-point functions of pseudoscalar mesons will be represented graphically as

$$C_{\pi\pi}(t,\vec{p}) \quad = \quad -$$



$$= \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle\, \bar{u}\gamma_5 d(x)\ \bar{d}\gamma_5 u(0)\,\rangle$$

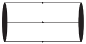
$$C_{KK}(t,\vec{p}) \quad = \quad -$$



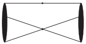
$$= \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle\, \bar{u}\gamma_5 s(x)\ \bar{s}\gamma_5 u(0)\,\rangle$$

nucleon two-point functions as,

$$C^{\pm}_{NN}(t,\vec{p}) \quad = \quad -$$



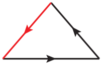
$$+$$



$$= \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle \left[\epsilon_{abc}(\bar{u}_a C\gamma_5 \bar{d}_b^T)\bar{u}_c \frac{1\pm\gamma^0}{2}\right](x) \left[\epsilon_{def}\frac{1\pm\gamma^0}{2} u_d (u_e^T C\gamma_5 d_f)\right](0)\,\rangle$$

three point functions as

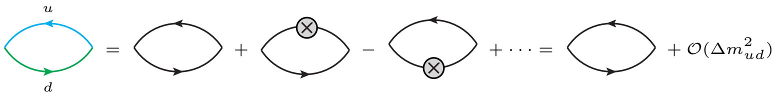
$$C^{\mu}_{K\pi}(t;\vec{p}_K,\vec{p}_\pi) = -$$



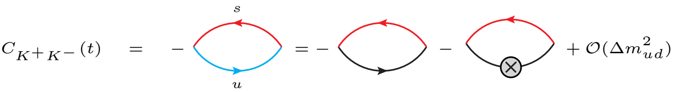
$$= \sum_{\vec{x},\vec{y}} e^{-i\vec{p}_\pi\cdot\vec{x}} e^{-i\vec{p}_K\cdot(\vec{x}-\vec{y})} \langle\, \bar{d}\gamma^5 s(\vec{y},T/2)\ \bar{s}\gamma^\mu u(\vec{x},t)\ \bar{u}\gamma^5 d(0)\,\rangle$$

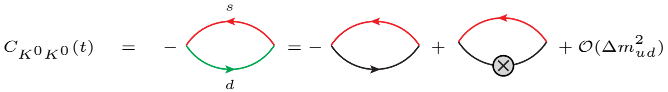
our QCD isospin breaking on the lattice: two point functions

- at first order in Δm_{ud} pion mass and decay constants don't get a correction (here $\pi^{\pm}$ but it works also for π^0 because $\langle \pi \| \hat{S} \| \pi \rangle = \langle 1, I_3 \| 1, 0 \| 1, I_3 \rangle = 0$)



- the kaons do get a correction





- this means that at first order ($\delta_{.}$ stays for relative variation while $\Delta_{.}$ for absolute variation),

$$\delta_u \left(\frac{F_K}{F_\pi} \right) = \frac{\Delta_u F_K}{F_K} - \frac{\Delta_u F_\pi}{F_\pi} = \frac{F_K - F_{K^+}}{F_K}$$

what do we expect from "corrected" correlation functions?

let's consider the euclidean correlation function in the **full perturbed** theory, $C_{K^0 K^0}(t)$, and in the **symmetric unperturbed** theory, $C_{KK}(t)$:

$$
\begin{aligned}
C_{K^0 K^0}(t) &= \sum_{\vec{x}} \langle \bar{d}\gamma_5 s(\vec{x},t)\, \bar{s}\gamma_5 d(0)\rangle = \sum_n \langle 0|\bar{d}\gamma_5 s(0)|n^\Delta\rangle \langle n^\Delta|\bar{s}\gamma_5 d(0)|0\rangle\; e^{-E_n^\Delta t} \\
&= \frac{G_{K^0}^2}{2E_{K^0}}\; e^{-E_{K^0} t} + \cdots
\end{aligned}
$$

$$
C_{KK}(t) = \frac{G_K^2}{2E_K}\; e^{-E_K t} + \cdots
$$

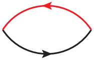
where the fact that the leading exponential is the same **is not obvious** and follows from the fact that our perturbation $\hat{S}$ is **flavour diagonal** (e.g. does not happen for insertions of the weak hamiltonian)

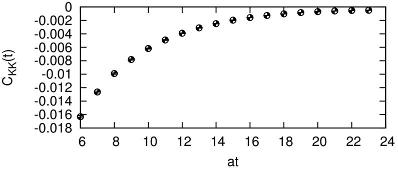
by using non degenerate perturbation theory (I_3 is conserved), we have

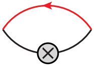
$$
E_{K^0} = E_K + \Delta E_K = E_K + \Delta m_{ud}\langle K|\hat{S}|K\rangle
$$

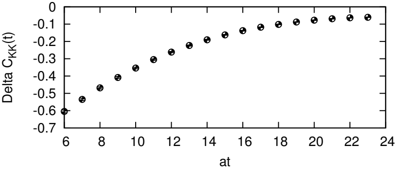
$$
|K^0\rangle = |K\rangle + |\Delta K\rangle = |K\rangle + \Delta m_{ud} \sum_{n\neq K} |n\rangle \frac{\langle n|\hat{S}|K\rangle}{E_K - E_n}
$$

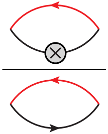
what do we expect from “corrected” correlation functions?

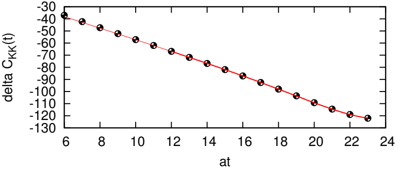
$-$  $= \frac{G_K^2}{2E_K}\, e^{-E_K t}$



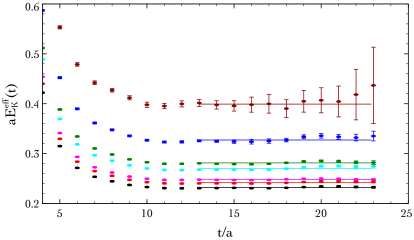
 $= \frac{G_K^2}{2E_K}\, e^{-E_K t} \left[\frac{\Delta(G_K^2/2E_K)}{G_K^2/2E_k} - t\Delta E_K \right]$



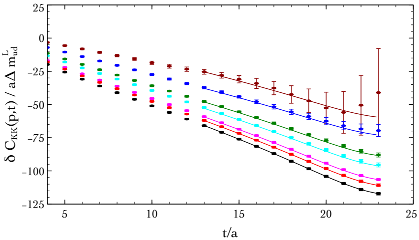
$-$  $= \delta \left(\mathbf{\frac{G_K^2}{2E_K}} \right) - \mathbf{\Delta E_K t}$



our QCD isospin breaking on the lattice: kaons two point functions



$$E^2_K(p) = M^2_K + p^2$$



$$\Delta E_K(p) = \frac{M_K \Delta M_K}{\sqrt{M^2_K + p^2}}$$

- by considering pseudoscalar-pseudoscalar correlators and by taking into account the finite time extent of the lattice, we fit correlations at different $\vec{p}$ according to,

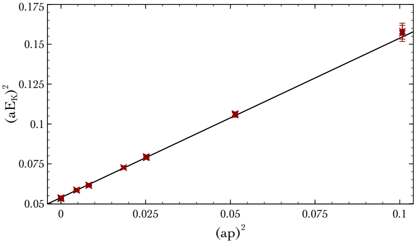
$$\delta C_{KK}(\vec{p}, t) = \delta \left(\frac{G^2_K e^{-E_K T/2}}{2E_K} \right) + \Delta E_K (t - T/2) \tanh \left[E_K (t - T/2) \right] + \ldots$$

- and extract F_K and δF_K according to

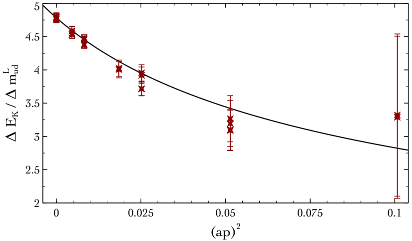
$$F_K = (m_s + m_{ud}) \frac{G_K}{M^2_K} \qquad \delta F_K = \frac{\Delta m_{ud}}{m_s + m_{ud}} + \delta G_K - 2\delta M_K$$

our QCD isospin breaking on the lattice: kaons two point functions

are we sure that the slopes correspond to ΔE_K?



$$E_K^2(p) = M_K^2 + p^2$$



$$\Delta E_K(p) = \frac{M_K \Delta M_K}{\sqrt{M_K^2 + p^2}}$$

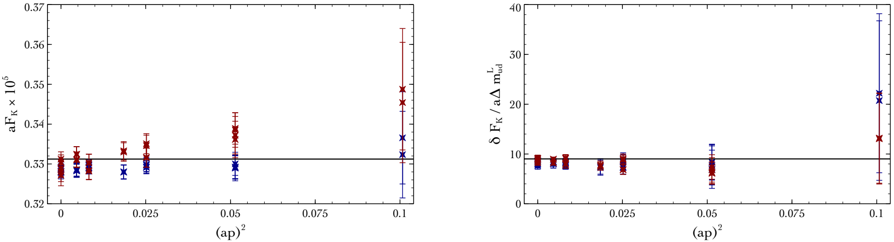
- the solid lines are not fitted, but theoretically predicted by using calculated M and ΔM
- this kind of accuracy on kinematics at $p \neq 0$ is possible thanks to the use of twisted boundary conditions

G.M. de Divitiis, R. Petronzio, N.T. Phys.Lett. B595 (2004)

$$\psi(x+L) = e^{i\theta}\psi(x) \qquad \longrightarrow \qquad p = \frac{\theta}{L} + \frac{2\pi n}{L}$$

our QCD isospin breaking on the lattice: kaons two point functions

are we sure that the intercepts correspond to δF_K?



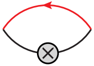
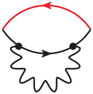
$F_K(p) = F_K(M_K^2)$ $\qquad$ $\delta F_K(p) = \delta F_K(M_K^2)$

- the solid lines are not fitted, but drawn by using $F_K(p=0)$ and $\delta F_K(p=0)$
- this kind of accuracy on kinematics at $p \neq 0$ is possible thanks to the use of twisted boundary conditions
  G.M. de Divitiis, R. Petronzio, N.T. Phys.Lett. B595 (2004)

$$\psi(x+L) = e^{i\theta}\psi(x) \qquad \longrightarrow \qquad p = \frac{\theta}{L} + \frac{2\pi n}{L}$$

extracting $[m_d - m_u]^{QCD}$: QED corrections

- in order to extract $2\Delta m_{ud}^{QCD} = [m_d - m_u]^{QCD}$ we need experimental inputs and we cannot neglect QED corrections
- If we work at first order in the QED coupling constant and Δm_{ud} and neglect terms of $\mathcal{O}(\alpha_{em}\Delta m_{ud})$, *some* of the relevant Feynman diagrams entering kaons two point functions are

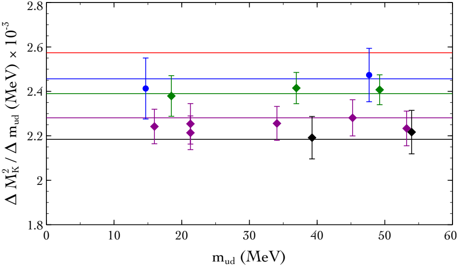
$\Delta C_{KK}(t) =$  $- \dfrac{e_d^2 - e_u^2}{2}$  $- e_s \dfrac{e_d - e_u}{2}$  $+ \cdots$

- the electromagnetic corrections to $C_{KK}(t)$ are logarithmically divergent, corresponding to the renormalization of the quark masses, and the separation of QED and QCD effects is ambiguous (**prescription dependent**)
- **in the chiral limit** QED corrections to $M^2_{K^0} - M^2_{K^+}$ and $M^2_{\pi^0} - M^2_{\pi^+}$ are the same (Dashen's theorem)
- **beyond the chiral limit** violations to Dashen's theorem are parametrized in term of small parameters

$\varepsilon_\gamma = 0.7(5)$ **from FLAG: Eur.Phys.J. C71 (2011) our prescription, for the time being**

$$\left[M^2_{K^0} - M^2_{K^+}\right]^{QCD} = \left[M^2_{K^0} - M^2_{K^+}\right]^{exp} - (1+\varepsilon_\gamma)\left[M^2_{\pi^0} - M^2_{\pi^+}\right]^{exp} \quad = \quad 6.05(63) \times 10^3 \text{ MeV}^2$$

$$\varepsilon_\gamma = 0 \quad \longrightarrow \quad 5.16 \times 10^3 \text{ MeV}^2$$



$$[m_d - m_u]^{QCD} \, (\overline{MS}, 2GeV) = 2\Delta m_{ud}^{QCD}$$

$$= 2.35(8)(24) \text{ MeV}$$

chiral perturbation theory formulae can be derived from known results

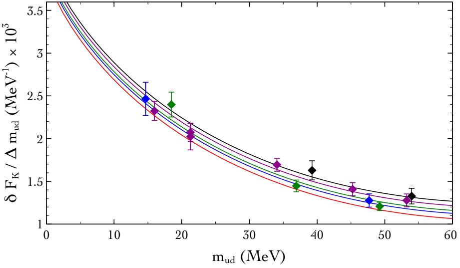
$n_f = 2+1$: Gasser and Leutwyler Nucl. Phys. B250(1985)
non unitary $n_f = 2$: S.Sharpe Phys. Rev. D56(1997)

$$\frac{\Delta M_K^2}{\Delta m_{ud}} = B_0 \Big\{ 1 + 2(m_{ud} + m_s)\hat{B}_0(2\alpha_8 - \alpha_5) + 4m_{ud}\hat{B}_0(2\alpha_6 - \alpha_4)$$

$$+ \hat{B}_0 m_s \log(2\hat{B}_0 m_s) + \hat{B}_0 \frac{m_s + m_{ud}}{m_s - m_{ud}} \left[m_s \log(2\hat{B}_0 m_s) - m_{ud} \log(2\hat{B}_0 m_{ud}) \right] \Big\}$$

where α_i are low energy constants and $\hat{B}_0 = 2B_0/(4\pi F_0^2)$

calculating δF_K^{QCD}: chiral-continuum extrapolations



$$\left[\frac{F_{K^+}/F_{\pi^+}}{F_K/F_\pi} - 1\right]^{QCD} = -0.0039(3)(2)$$

$$\epsilon_\gamma = 0 \longrightarrow -0.0032(3)$$

to be compared with

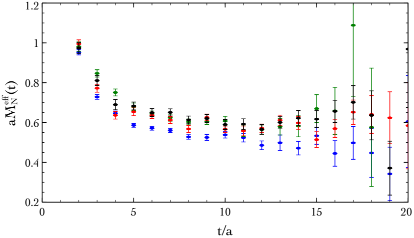
$$\left[\frac{F_{K^+}/F_{\pi^+}}{F_K/F_\pi} - 1\right]^{\chi pt} = -0.0022(6)$$

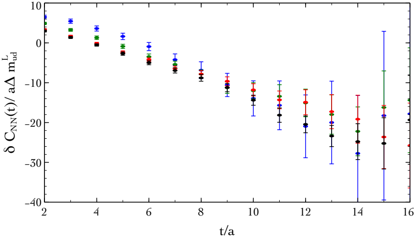
chiral perturbation theory formulae can be derived from known results

$n_f = 2+1$: **Gasser and Leutwyler Nucl. Phys. B250(1985)**
non unitary $n_f = 2$: **S.Sharpe Phys. Rev. D56(1997)**

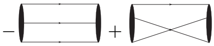
$$\frac{\delta F_K}{\Delta m_{ud}} = \frac{B_0}{2}\left\{\alpha_5 - \hat{B}_0 \frac{m_s + m_{ud}}{m_s - m_{ud}}\left[m_s \log(2\hat{B}_0 m_s) - m_{ud}\log(2\hat{B}_0 m_{ud})\right]\right\}$$

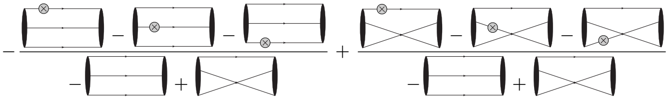
where α_i are low energy constants and $\hat{B}_0 = 2B_0/(4\pi F_0^2)$





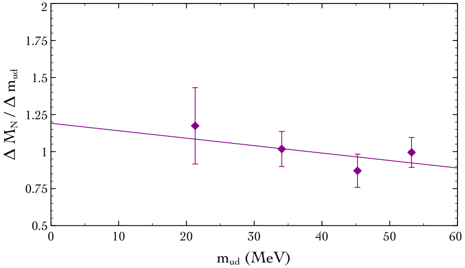
the calculation of the neutron-proton mass difference proceeds along the same lines as in the K^0-K^+ case

$C_{NN}(t) \quad =$  $= W_N e^{-M_N t} + \cdots$

$\delta C_{NN}(t) \quad =$ 

$= \quad \delta W_N - t\Delta M_N + \cdots$

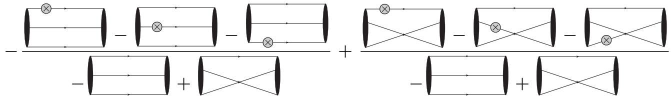
calculating $M_n - M_p$



$$[M_n - M_p]^{QCD} = 2\Delta m_{ud}^{QCD} \left[\frac{\Delta M_N}{\Delta m_{ud}}\right]^{QCD}$$

$$= 2.8(6)(3)\ \text{MeV}$$

$$\delta C_{NN}(t) =$$



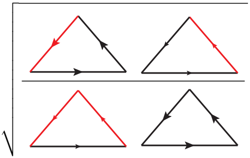
$$= \delta W_N - t\Delta M_N + \cdots$$

- here the results are at fixed lattice spacing $a = 0.085$ fm.
- correlators have been computed by "Gaussian smearing" sink operators

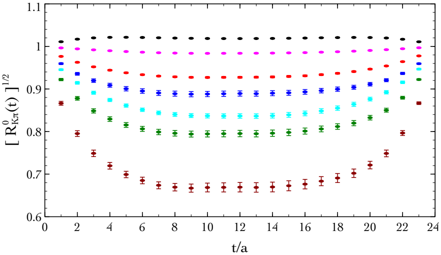
calculating $\delta f_+^{K\pi}(q^2)$

form factors parametrizing semileptonic decays can be calculated with good precision by considering **double ratios** of three point correlation functions

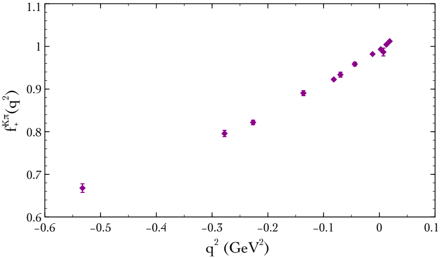
$$\frac{\langle \pi | V^\mu_{su} | K \rangle}{2\sqrt{E_\pi E_K}} =$$



$$=$$



and



$$\langle \pi | V^0_{su} | K \rangle = (E_K + E_\pi) f_+^{K\pi} + (E_K - E_\pi) f_-^{K\pi}$$

$$\langle \pi | \vec{V}_{su} | K \rangle = (\vec{p}_i + \vec{p}_f) f_+^{K\pi} + (\vec{p}_i - \vec{p}_f) f_-^{K\pi}$$

calculating $\delta f_+^{K\pi}(q^2)$

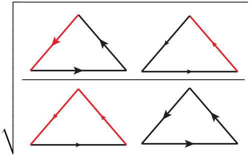
in order to calculate QCD isospin breaking corrections to $K \longrightarrow \pi\ell\nu$ form factors one needs to calculate,

$$\langle\pi|\mathsf{T}\left\{\int d^4x\; S^3(x;\mu)\; V^{\mu}_{su}\right\}|K\rangle \qquad \longrightarrow \qquad \left\{\begin{array}{l} \langle\bar{K}|\mathsf{T}\left\{\int d^4x\; H_W^{\Delta S=1}(x;\mu)\; H_W^{\Delta S=1}(0;\mu)\right\}|K\rangle \\ \\ \langle\pi|\mathsf{T}\left\{\int d^4x\; H_W^{\Delta S=1}(x;\mu)\; V^{\mu}_{em}\right\}|K\rangle \end{array}\right.$$

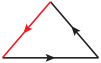
a key difference with respect to the calculation of long distance effects for $K \longrightarrow \pi\nu\nu$ and K-$\bar{K}$ mixing is that the isospin breaking correction does not induce the decay of the kaon...

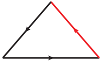
by using perturbation theory it can be shown that the isospin breaking corrections to the matrix elements is given by (all t-**dependent** and wave function contributions **cancel**)

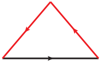
$$\delta\left\{\frac{\langle\pi|V^{\mu}_{su}|K\rangle}{2\sqrt{E_\pi E_K}}\right\} \quad = \quad \delta\left\{\sqrt{}\right\}$$

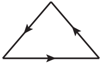


$$= \quad \frac{1}{2}\left\{\delta\;\cdot\; + \;\delta\;\cdot\; - \;\delta\;\cdot\; - \;\delta\;\underbrace{\cdot}_{=0}\right\}$$



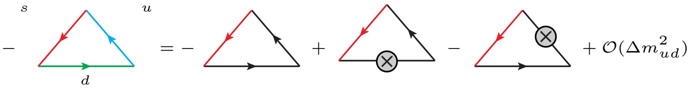




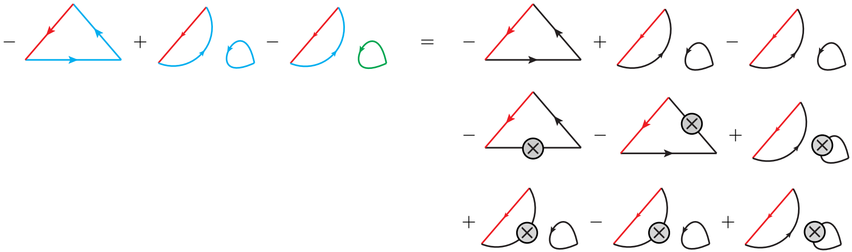


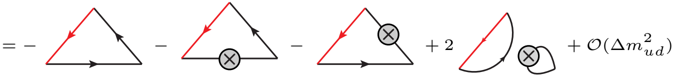
calculating $\delta f_+^{K\pi}(q^2)$

the diagrammatic expansion in the $K^0 \to \pi^- \ell\nu$ is

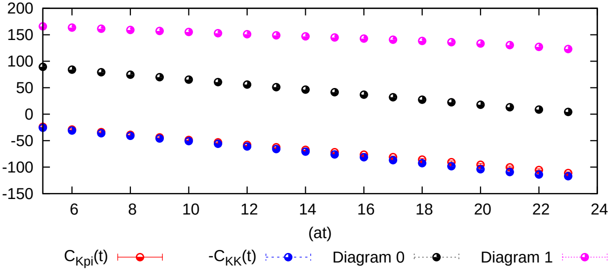


and is different, because of the **disconnected diagrams**, from the $K^+ \to \pi^0 \ell\nu$ case





calculating $\delta f_+^{K\pi}(q^2)$

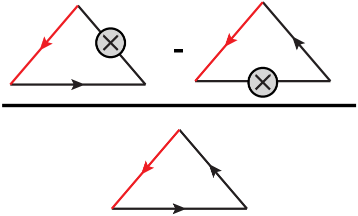


what do we expect from corrected three point correlation functions?

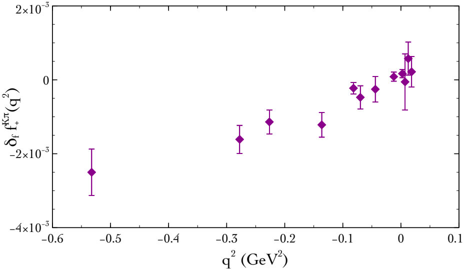
$$C^{\mu}_{K\pi}(t) = Z^{\mu}_{K\pi}\; e^{-E_K t}\; e^{-E_\pi(T-t)}$$

$$\Delta C^{\mu}_{K\pi}(t) = \left(\Delta Z^{\mu}_{K\pi} - Z^{\mu}_{K\pi}\Delta E_K t\right)\; e^{-E_K t}\; e^{-E_\pi(T-t)}$$

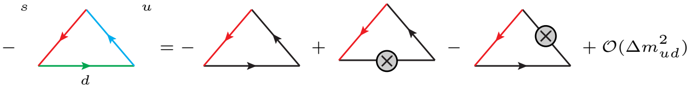
$$\delta C^{\mu}_{K\pi}(t) = \left(\delta \mathbf{Z}^{\mu}_{\mathbf{K}\pi} - \mathbf{\Delta E_K t}\right)$$



calculating $\delta f_+^{K\pi}(q^2)$



$$\left[\frac{f_+^{K^0\pi^-}(0) - f_+^{K\pi}(0)}{f_+^{K\pi}(0)}\right]^{QCD} = 0.85(18)(1) \times 10^{-4}$$



- in this work we have **not calculated disconnected diagrams**
- we can only show results for the $K^0 \to \pi^- \ell\nu$ case (above)
- this is a quantity that cannot be measured directly and the missing contribution, according to χpt, is expected to be much bigger
- the results given here make us confident on the possibility of completing the calculation by including disconnected diagrams

non-compact QED on the lattice

- in order to perform combined QCD+QED lattice simulations one can use the *non-compact* formulation:

$$
\begin{aligned}
S_{QED} &= \frac{1}{4}\sum_{x;\mu,\nu}\left[\nabla_\mu^+ A_\nu(x) - \nabla_\nu^+ A_\mu(x)\right]^2 \\
&= -\frac{1}{4}\sum_{x;\mu,\nu}\left\{A_\nu(x)\nabla_\mu^-\left[\nabla_\mu^+ A_\nu(x) - \nabla_\nu^+ A_\mu(x)\right] - A_\mu(x)\nabla_\nu^-\left[\nabla_\mu^+ A_\nu(x) - \nabla_\nu^+ A_\mu(x)\right]\right\}
\end{aligned}
$$

- by using a covariant gauge fixing, one gets:

$$
\nabla_\mu^- A_\mu(x) = 0 \qquad \longrightarrow \qquad
\begin{aligned}
S_{QED} &= \frac{1}{2}\sum_x A_\mu(x)\left[-\nabla_\nu^-\nabla_\nu^+\right]A_\mu(x) \\
&= \frac{1}{2}\sum_k A_\mu^\star(k)\left[2\sin(k_\nu/2)\right]^2 A_\mu(k)
\end{aligned}
$$

- note that the zero momentum mode is *not* constrained by any "derivative" gauge fixing, and there is a residual gauge ambiguity to be addressed

$$
\nabla_\mu^-\left[A_\mu(x) + c\right] = \nabla_\mu^- A_\mu(x)
$$

non-compact QED on the lattice: gauge invariance

by assuming that one is able to sample properly the QED gauge potential $A\mu(x)$ (we shall discuss this point in the next few slides), gauge invariance works as follows:

- the QED links are defined by

$$A_\mu(x) \qquad \longrightarrow \qquad E_\mu(x) = e^{-ieA_\mu(x)}$$

- QCD+QED covariant lattice derivatives are defined according to

$$\bar{\psi}(x)\nabla_\mu^+\psi(x) \quad = \quad \bar{\psi}(x)\; E_\mu(x)\; U_\mu(x)\psi(x+\mu) - \bar{\psi}(x)\;\psi(x)$$

- the "exact" gauge invariance is

$$\begin{aligned}
\psi(x) \qquad &\longrightarrow \qquad e^{ie\lambda(x)}\psi(x) \\
\bar{\psi}(x) \qquad &\longrightarrow \qquad \bar{\psi}(x)e^{-ie\lambda(x)} \\
A_\mu(x) \qquad &\longrightarrow \qquad A_\mu(x) + \nabla_\mu^+\lambda(x)
\end{aligned}$$

non-compact QED on the lattice: the american's way

in order to sample the QED gauge potential, the strategy followed by other groups is the following

...
MILC Collaboration, PoS LATTICE2008 (2008) 127
T.Blum et al. Phys. Rev. D82 (2010)
[BMW Collaboration] PoS LATTICE2010 (2010) 121
[T. Ishikawa et al.] Phys. Rev. Lett. 109 (2012)

- choose periodic boundary conditions for the gauge potential,

$$A_\mu(x+L\hat{\nu}) = A_\mu(x) \quad \longrightarrow \quad k_\mu = \frac{2\pi n_\mu}{L} \quad \longrightarrow \quad S_{QED} = \frac{1}{2}\sum_{k\neq 0} A_\mu(k)^\star \left[2\sin(k_\nu/2)\right]^2 A_\mu(k)$$

- the action is quadratic and diagonal in momentum space so, by *excluding the zero momentum mode*, $A_\mu(k)$ can be obtained by an heat-bath algorithm (actually they choose a different gauge, diagonalize the action and perform a gaussian sampling...) and the gauge potential in coordinate space is obtained by (fast) fourier transform

$$A_\mu(x) = \frac{1}{L^4}\sum_{k\neq 0} e^{ikx} A_\mu(k)$$

- it can be shown that the effect of neglecting the zero momentum mode is a *finite volume effect*. classically: add four lagrange multipliers to the action,

$$S_{QED} \quad \longrightarrow \quad \frac{1}{2}\sum_x A_\mu(x)\left[-\nabla_\nu^- \nabla_\nu^+\right] A_\mu(x) + \frac{1}{L^3}\sum_x \xi_\mu A_\mu(x)$$

$$A_\mu(k=0) = \frac{\partial S}{\partial \xi_\mu} = \sum_x A_\mu(x) = 0$$

- at quantum level: this prescription does not affect short distance physics (no new divergences)

non-compact QED on the lattice: our approach

- we want to deal with QED on the lattice at *fixed order* in the expansion with respect to α_{em}
- to this end, we need to expand the lattice action with respect to the electric charge

$$\sum_x \bar{\psi}(x)\left\{D[U,E]-D[U,0]\right\}\psi(x) =$$

$$+\sum_{x,\mu} ieA_\mu(x)\left\{\bar{\psi}(x)U_\mu(x)\frac{W-\gamma^\mu}{2}\psi(x+\mu)-\bar{\psi}(x+\mu)U^\dagger_\mu(x)\frac{W+\gamma^\mu}{2}\psi(x)\right\}$$

$$+\sum_{x,\mu}\frac{e^2}{2}A_\mu(x)A_\mu(x)\left\{\bar{\psi}(x)U_\mu(x)\frac{W-\gamma^\mu}{2}\psi(x+\mu)+\bar{\psi}(x+\mu)U^\dagger_\mu(x)\frac{W+\gamma^\mu}{2}\psi(x)\right\}$$

$$+\dots$$

$$=\sum_{x,\mu}\left\{ieA_\mu(x)V^\mu(x)+\frac{e^2}{2}A_\mu(x)A_\mu(x)T^\mu(x)+\dots\right\}$$

- the "Wilson" contribution is $W=\{1, i\gamma_5\tau^3\}$ in clover and twisted mass QCD respectively
- note: tadpole currents $T^\mu(x)$ are required to have gauge invariance at order e^2
- note: the point split vector current is exactly conserved: $\nabla^-_\mu V^\mu(x)=0$

non-compact QED on the lattice: our approach

let us consider, for example, the following contribution to the mass splittings of the kaons:

$-$  $= \; -\frac{e_s e_u \hat{e}^2}{2} \sum_{x,y} D_{\mu\nu}(x-y)\; T\langle 0|\, \bar{s}(t)\gamma_5 u(t)\; V_s^\mu(x)\; V_u^\nu(y)\; \bar{u}(0)\gamma_5 s(0)\, |0\rangle$

where $D_{\mu\nu}(x-y)$ is the propagator of the gauge potential A_μ: this means that we are also using the QED in its non-compact lattice formulation. now, in order to properly define the lattice propagator of A_μ we must

- fix the QED gauge; we have used

$$\nabla_\mu^- A_\mu(x) = 0 \quad \longrightarrow \quad S_{QED} = \frac{1}{2}\sum_x A_\mu(x)\left[-\nabla_\nu^- \nabla_\nu^+\right] A_\mu(x) = \frac{1}{2}\sum_k A_\mu(k)\,[2\sin(k_\nu/2)]^2\, A_\mu(k)$$

- introduce the infrared regulated photon propagator,

$$\mathtt{P}^\perp \phi(x) = \phi(x) - \frac{1}{V}\sum_y \phi(y)$$

$$D_{\mu\nu}^\perp(x-y) = \left[\mathtt{P}^\perp\, \frac{\delta_{\mu\nu}}{-\nabla_\rho^- \nabla_\rho^+}\, \mathtt{P}^\perp\right](x-y) = \sum_{k\neq 0} \frac{e^{ik(x-y)}}{[2\sin(k_\nu/2)]^2}$$

non-compact QED on the lattice: our approach

we have decided to work directly in coordinate space, thus avoiding fourier transforms, by applying the following stochastic technique

- we extract a set of four independent real fields distributed according to a real Z_2 distribution,

$$\sum_B B_\mu(x) B_\nu(y) = \delta_{\mu\nu}\ \delta(x-y)$$

- for each field we solve numerically the equation

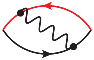
$$[-\nabla_\nu^- \nabla_\nu^+] C_\mu[B;x] = \mathtt{P}^\perp B_\mu(x) \qquad \longrightarrow \qquad \begin{aligned} C_\mu[B;x] &= \left[\frac{1}{-\nabla_\nu^- \nabla_\nu^+} \mathtt{P}^\perp\right] B_\mu(x) \\ &= \left[\mathtt{P}^\perp \frac{1}{-\nabla_\nu^- \nabla_\nu^+} \mathtt{P}^\perp\right] B_\mu(x) \\ &= \sum_z D^\perp(x-z) B_\mu(z) \end{aligned}$$

- by using the properties of the Z_2 noise we thus obtain

$$\sum_B B_\mu(y) C_\nu[B;x] = D^\perp(x-z) \sum_B B_\mu(y) B_\nu(z) = D^\perp_{\mu\nu}(x-y)$$

non-compact QED on the lattice: our approach

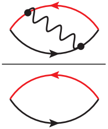
coming back to our example, we get



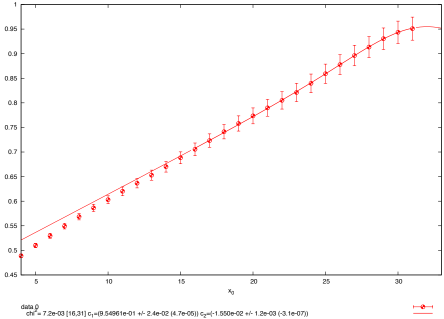
$$- \quad = \quad -\frac{e_s e_u \hat{e}^2}{2} \sum_{x,y} D^{\perp}_{\mu\nu}(x-y)\; T\langle 0|\; \bar{s}(t)\gamma_5 u(t)\; V_s^{\mu}(x)\; V_u^{\nu}(y)\; \bar{u}(0)\gamma_5 s(0)\; |0\rangle$$

$$= \quad -\frac{e_s e_u \hat{e}^2}{2} \sum_{B} \sum_{x,y} B_{\mu}(y) C_{\nu}[B;x]\; T\langle 0|\; \bar{s}(t)\gamma_5 u(t)\; V_s^{\mu}(x)\; V_u^{\nu}(y)\; \bar{u}(0)\gamma_5 s(0)\; |0\rangle$$

does it works?



$$\sim \frac{\partial}{\partial e^2}\left(\frac{G_K^2}{M_K} e^{-M_K}\right) =$$



well, from the numerical point of view it seems to work. ok, what about the physics?

- let's start by considering a two-point correlator in the full theory ($m_u \neq m_d$ and $e_q \neq 0$)

$$C^{full}_{HH}(t) = \langle\, \mathcal{O}_H(t)\ \mathcal{O}^\dagger_H(0)\, \rangle \qquad \longrightarrow \qquad e^{M^{full}_H} = \frac{C^{full}_{HH}(t-1)}{C^{full}_{HH}(t)} + \text{ non leading exps.}$$

where $\mathcal{O}_H$ is an interpolating operator having the quantum numbers of a given hadron H

- if H is a charged particle, the correlator $C_{HH}(t)$ is not QED gauge invariant. for this reason it is not possible, in general, to extract physical informations directly from the residues of the different poles

- on the other hand, the mass of the hadron is gauge invariant and *finite* in the continuum limit, provided that the parameters of the actions have been properly renormalized. it follows that, at any given order in a perturbative expansion with respect to any of the parameters of the action, the ratio $C^{full}_{HH}(t-1)/C^{full}_{HH}(t)$ is both gauge and renormalization group (RGI) invariant

- by expanding the full theory with respect to the isospin symmetric theory ($m_u = m_d$ and $e_q = 0$) and by considering $e_q^2 \sim m_d - m_u = O(\epsilon)$, we shall find expressions of the form

$$C^{full}_{HH}(t) = C_{HH}(t)\left[1 + e^2\,\frac{\partial_{e^2} C_{HH}(t)}{C_{HH}(t)} + \Delta m_{ud}\,\frac{\partial_{\Delta m_{ud}} C_{HH}(t)}{C_{HH}(t)} + \dots\right]$$

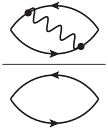
$$M^{full}_H - M_H = -e^2\partial_t\,\frac{\partial_{e^2} C_{HH}(t)}{C_{HH}(t)} - \Delta m_{ud}\ \partial_t\,\frac{\partial_{\Delta m_{ud}} C_{HH}(t)}{C_{HH}(t)} + \dots\ ,$$

where we have defined $\partial_t f(t) = f(t) - f(t-1)$

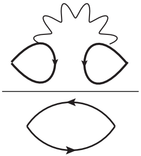
the physics: a first look at the pions

by expanding the two-point function of an interpolating operator having the quantum numbers of the neutral pion, we get

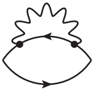
$$M_{\pi^0} - M_{\pi} = -\frac{e_u^2 + e_d^2}{2}\,\hat{e}^2 \partial_t$$



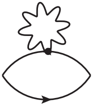
$$+\ \frac{(e_u - e_d)^2}{2}\,\hat{e}^2 \partial_t$$



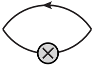
$$-(e_u^2 + e_d^2)\hat{e}^2 \partial_t$$

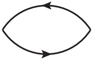


$+$

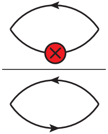


$- C_m$





$$-\ (\Delta k_u + \Delta k_d)\partial_t$$

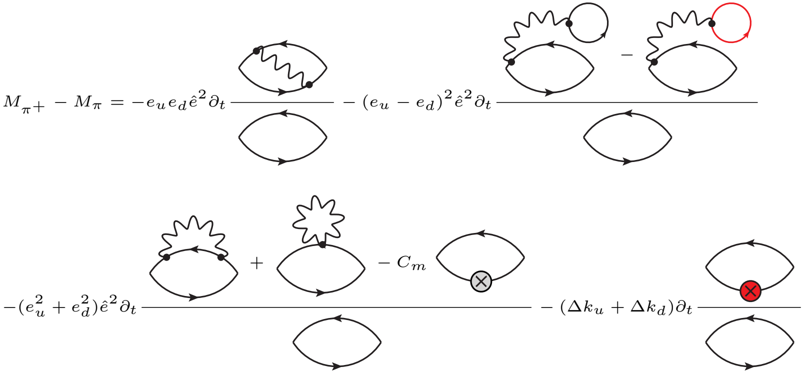


$+$[isosymmetric vac. pol.]

the fermion disconnected diagram in the first line vanishes by parity in the continuum and will be neglected in the following

the physics: a first look at the pions

by expanding the two-point function of an interpolating operator having the quantum numbers of the charged pions, we get



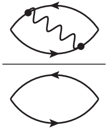
$+$[isosymmetric vac. pol.]

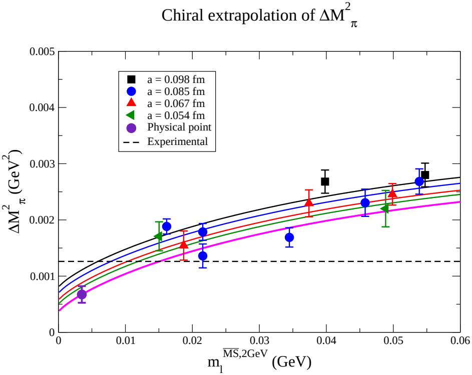
the tadpole diagrams in the first line are a (OZI violating) “dynamical QED effect” and are presumably not negligible. we have implemented them in the code and we are running with several stochastic sources per gauge configurations (we get a signal!) but we shall neglect the corresponding contribution in the following . . .

by taking the difference of the formulae shown in the previous two slides we get

$$M_{\pi^+} - M_{\pi^0} =$$

$$\frac{(e_u - e_d)^2 \hat{e}^2}{2} \, \partial_t$$





chiral perturbation theory formulae are known

M.Hayakawa, S.Uno Prog.Theor.Phys. 120 (2008)

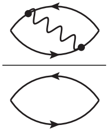
$$M^2_{\pi^+} - M^2_{\pi^0} =$$

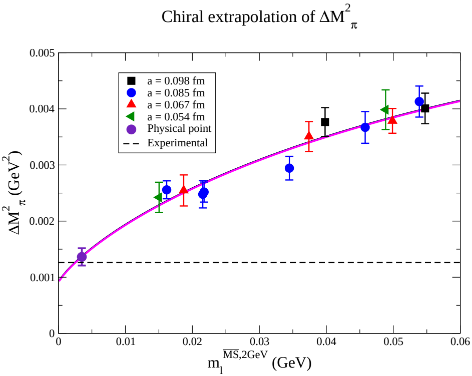
$$4\pi \hat{e}^2 f_0^2 \left\{ 4\frac{C}{f_0^4} - \left(3 + 16\frac{C}{f_0^4}\right) \frac{M_\pi^2}{(4\pi f_0^2)} \left[K(\mu) + \log\left(\frac{M_\pi^2}{\mu^2}\right) \right] \right\}$$

by taking into account finite volume corrections as calculated in χpt+QED, we get

$$M_{\pi^+} - M_{\pi^0} =$$

$$\frac{(e_u - e_d)^2 \hat{e}^2}{2} \, \partial_t$$





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$$M^2_{\pi^+} - M^2_{\pi^0} =$$

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outlooks

- first results obtained by applying our method look very promising
- the method is general and can be applied to many observables, even at second order
- we shall also refine our results in the case of nucleon masses and form factors
- finish the computation of QED effects by quantifying all the systematics
- first small steps toward the calculation of other observables that are relevant for phenomenological applications (long distance effects, etc.)
- U-spin corrections?

$$\langle \text{surface of the unitarity triangle} \rangle \propto (m_s^2 - m_d^2)\,(m_b^2 - m_d^2)\,(m_b^2 - m_s^2)$$