Lattice Artifacts in Strongly interacting theories

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Strongly interacting theories

- Easily constructed with **Yang-Mills** fields coupled to **fermion fields**.

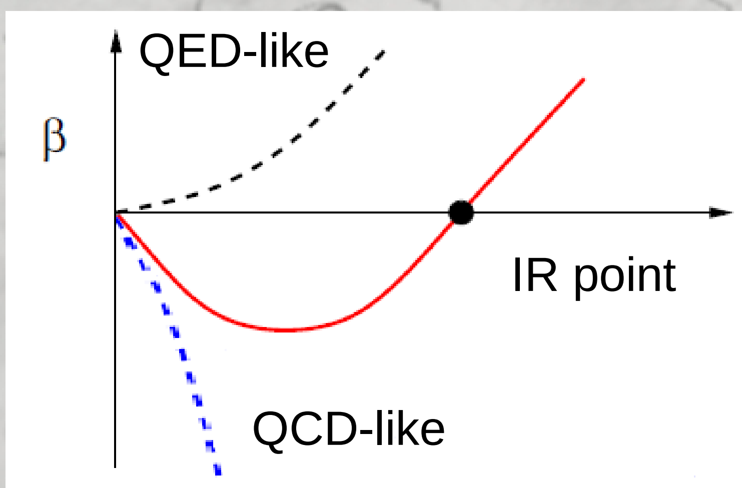
$$\mathcal{L} = \frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} + \sum_{j=1}^{N_f} \left(i\overline{\psi}_j \gamma_\mu D^\mu \psi_j \right)$$

- Starting with such Lagrangian, we specify a particular theory by fixing
  - The gauge group **SU(N)**
  - The number of flavours **Nf**.
  - The representation **R**.

- Some examples
  - Quantum Chromodynamics
  - Dynamical EW Symmetry breaking (**Technicolor**).
  - Unparticle Physics
  - ...

Strongly interacting theories

- Beta-function: $$\beta(g) = \mu \frac{\partial g}{\partial \mu}$$
- Which, in perturbation theory $$\beta(g) = -b_0 \frac{g^3}{(4\pi)^2} - b_1 \frac{g^5}{(4\pi)^4} + \ldots$$



$$b_0 = \frac{11}{3} N_c - \frac{4}{3} \mathrm{T_R} N_f$$

$$b_1 = \frac{34}{4} N_c^2 - \frac{20}{3} N_c \mathrm{T_R} N_f - 4 C_2(R) \mathrm{T_R} N_f$$

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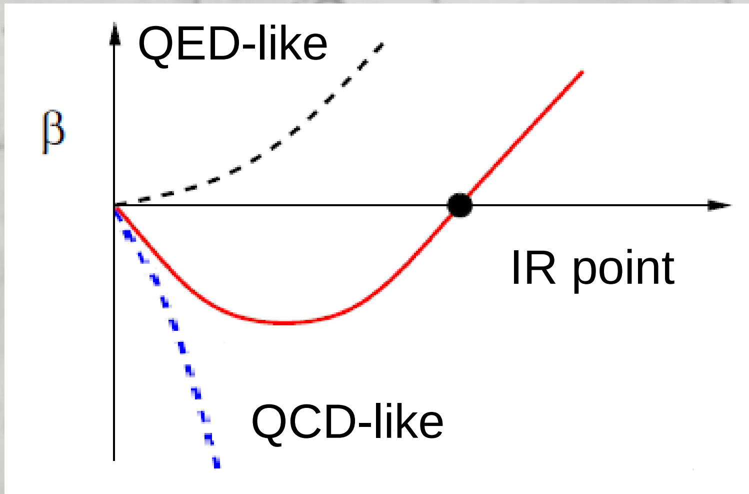
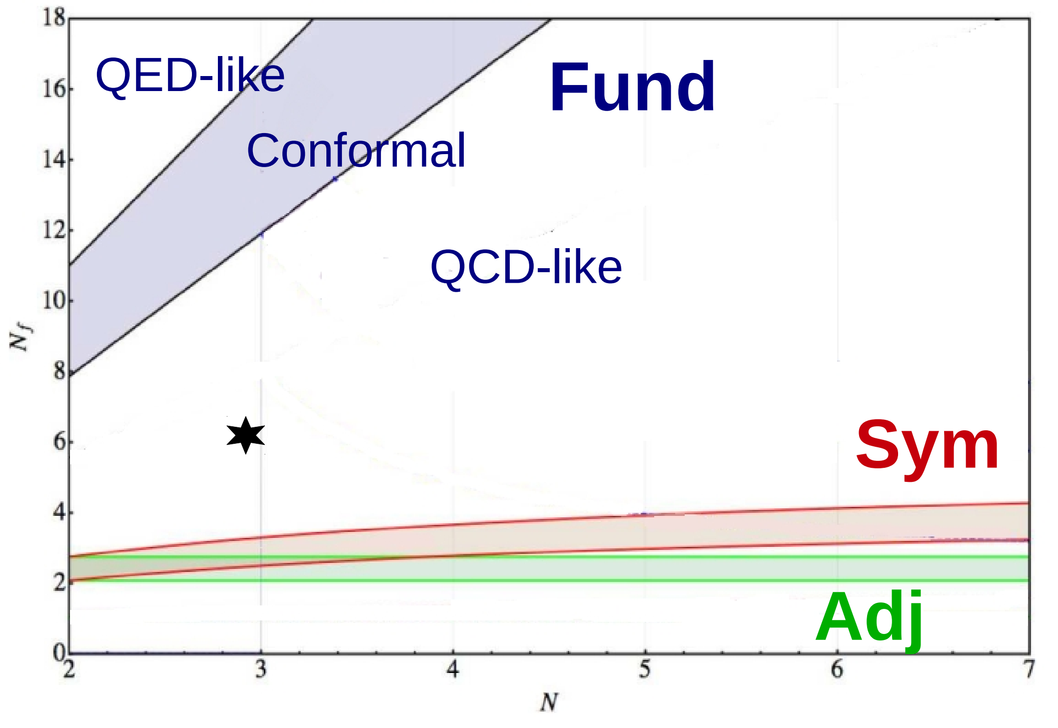




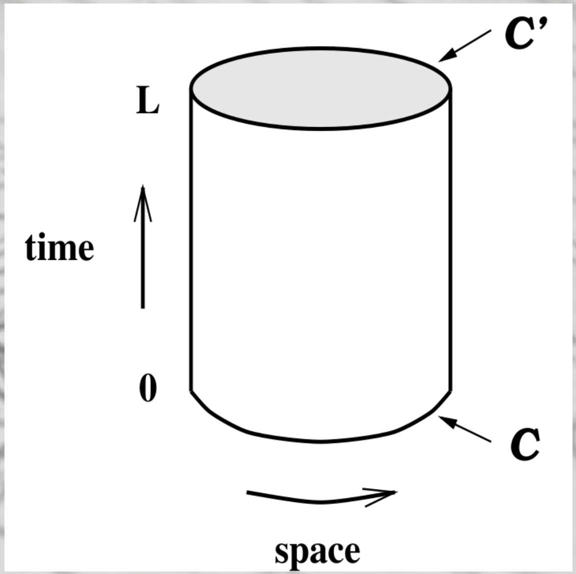
[Appelquist, Lane, Mahanta, Cohen, Georgi, Yamawaki, Schrock, Sannino,Tuominen, Dietrich]

The Schrodinger Functional

- Euclidean propagation amplitude from a field configuration in one boundary C to another field configuration in the oposite boundary C'.

  [Luescher et al. '92]

$$\mathcal{Z}[C,C'] = \int \mathcal{D}[A,\psi,\bar{\psi}]e^{-S[A,\psi,\bar{\psi}]}$$



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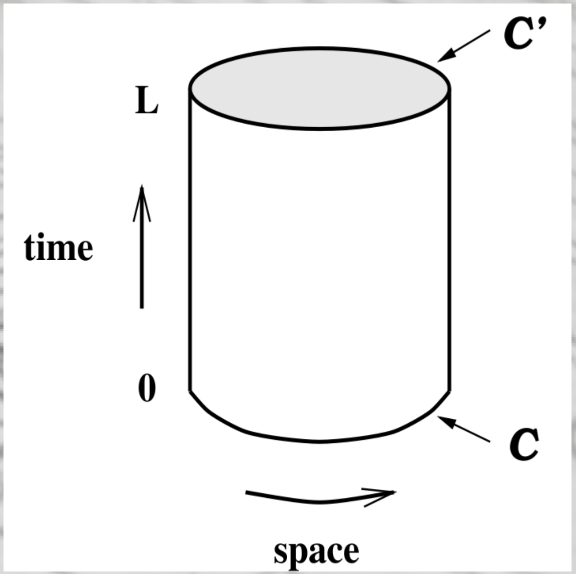

- Boundary Conditions:
  - Boundary gauge fields C and C' taken to be **abelian** and **spatialy constant**.

$$C_k = \frac{i}{L}\mathrm{diag}(\phi_{1k}, ..., \phi_{Nk})$$

- 2 Fermion regularizations:

SF [Sint '94]

$$\begin{aligned} P_+\psi|_{x_0=0} &= P_-\psi|_{x_T=0} = 0 \\ \overline{\psi}P_-|_{x_0=0} &= \overline{\psi}P_+|_{x_T=0} = 0 \end{aligned}$$

χSF [Sint '05-'10]

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The Schrodinger Functional

- C and C', induce a background abelian chromoelectric field B.
- The effective action of B

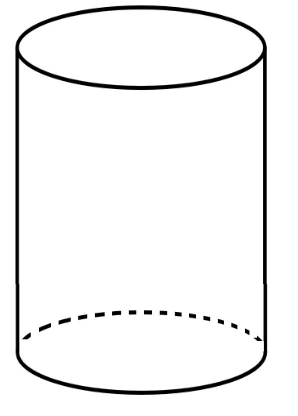
$$\Gamma[B] = -\ln \mathcal{Z}[C, C']$$

$$\Gamma[B] \xrightarrow{g_0 \to 0} \frac{1}{g_0^2}\Gamma_0[B] + \Gamma_1[B] + g_0^2\Gamma_2[B] + \ldots$$

- If **mq=0** and **T=L**, then **L** is the only scale.
  B depends on a parameter **B(η)**.

- Define a **renormalized coupling g(L)**

$$\bar{g}^2(L) = \frac{\partial\Gamma_0/\partial\eta|_{\eta=0}}{\partial\Gamma/\partial\eta|_{\eta=0}}$$



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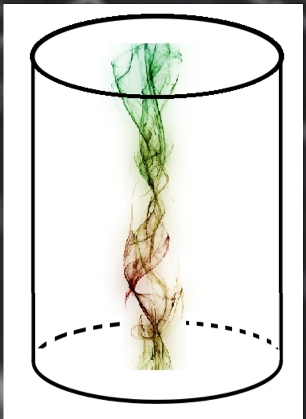

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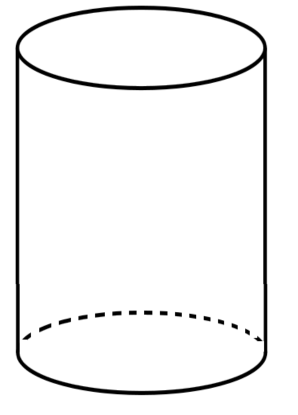

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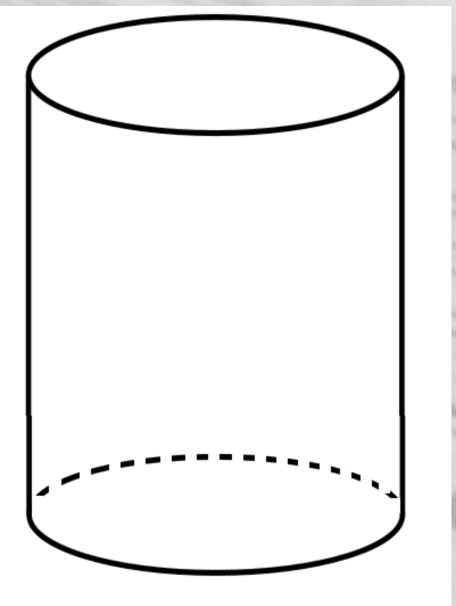

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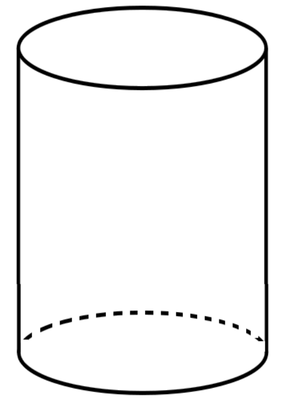

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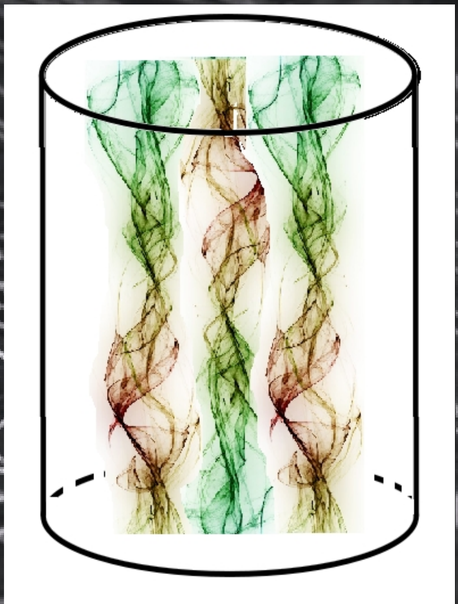

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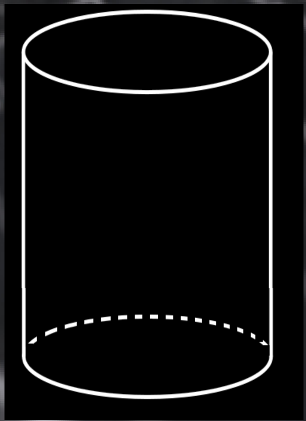

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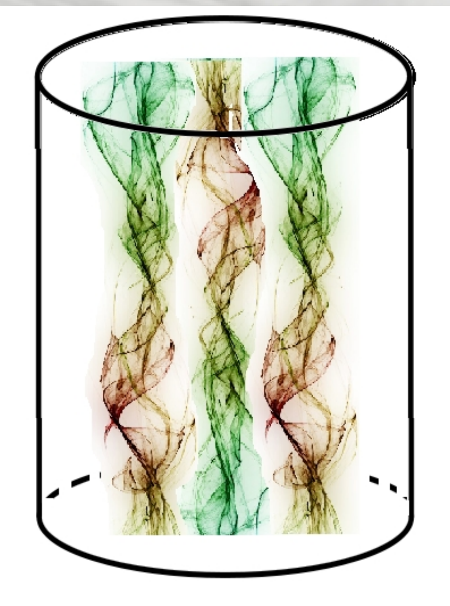

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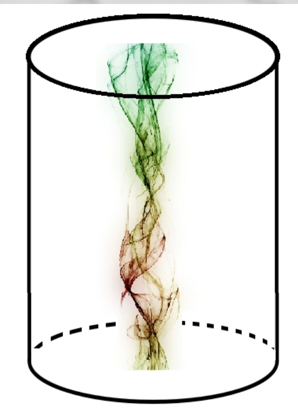

Monitoring cutoff effects

- The **Step Scaling Function**, an integrated version of the beta function

$$\sigma(s,u) \equiv \bar{g}^2(sL)\Big|_{u=\bar{g}^2(L)}$$

With a lattice counterpart:

$$\Sigma(s=2,u,L/a) = \bar{g}^2(2L)|_{u=\bar{g}^2(L)}$$

- In perturbation theory:

$$\sigma(u) \xrightarrow{g_0\to 0} u + \sigma_1 u^2 + O(u^3)$$

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Cutoff effects of **100%** are equivalent to missing **Nf** by a factor **2** !!!!

$$N_f \Sigma_{1,1} = (1 + \delta_{1,1})\, N_f \sigma_{1,1} \approx 2 N_f \sigma_{1,1}$$

$$\delta_{1,1} = 100\%$$

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Towards an optimal regularization.

- Spatial bc for fermion fields are periodic up to a phase **θ**.

$$\psi(x+L\hat{k}) = e^{i\theta/L}\psi(x),$$
$$\overline{\psi}(x+L\hat{k}) = e^{-i\theta/L}\overline{\psi}(x),$$

- Condition number:

$$\kappa(\Delta_2) = \left(\frac{\lambda_{max}}{\lambda_0}\right)^{-1/2}$$

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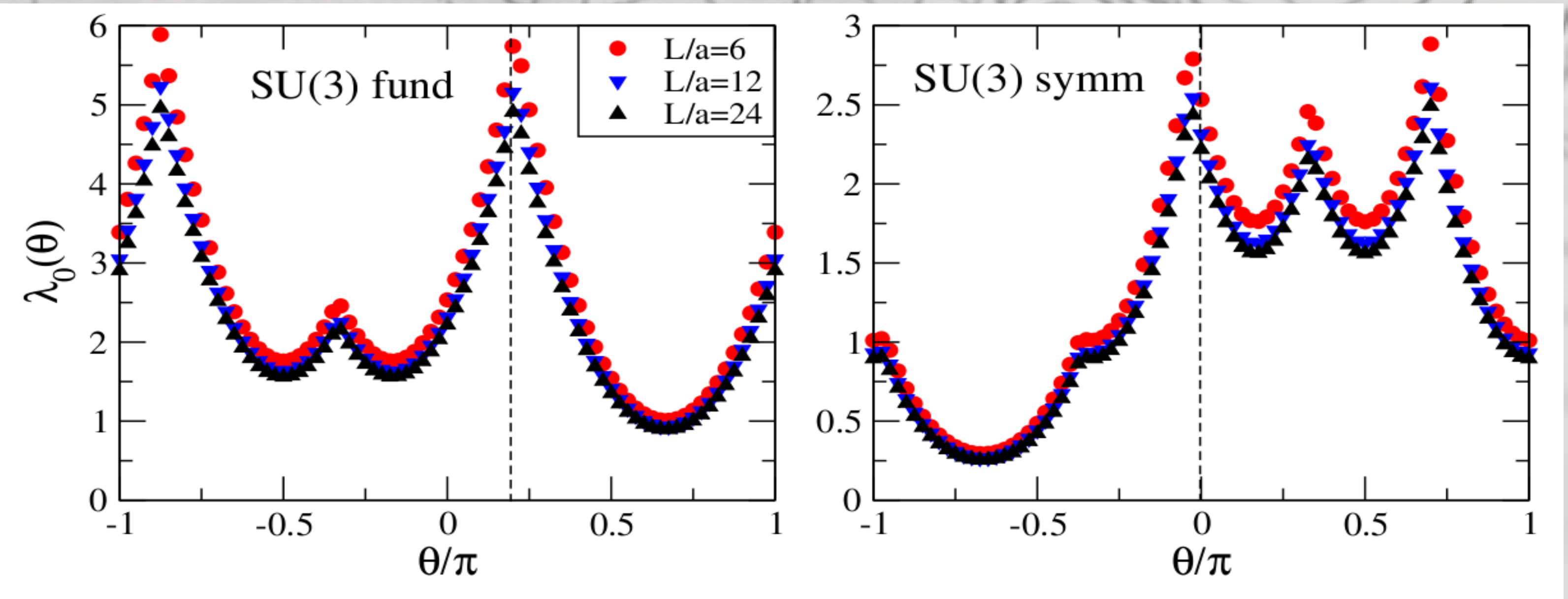

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- **SF**: O(a) effects coming from bulk and boundaries.

    - Boundary counterterms: c_t, $\tilde{c}_t$.
    - Bulk counterterm, clover term: C_{sw}

- **χSF**: Only boundary counterterms : c_t, d_s, z_f (dim(z_f)= 3)

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SF

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\begin{aligned}
c_{SW} &= c_{SW}^{(0)} + c_{SW}^{(1)} g_0^2 + O(g_0^4) \\
c_t &= c_t^{(0)} + c_t^{(1)} g_0^2 + O(g_0^4) \\
\tilde{c}_t &= \tilde{c}_t^{(0)} + \tilde{c}_t^{(1)} g_0^2 + O(g_0^4)
\end{aligned}
$$

χSF

$$
\begin{aligned}
c_t &= c_t^{(0)} + c_t^{(1)} g_0^2 + O(g_0^4) \\
z_f &= z_f^{(0)} + z_f^{(1)} g_0^2 + O(g_0^4) \\
d_s &= d_s^{(0)} + d_s^{(1)} g_0^2 + O(g_0^4)
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$$c_t = c_t^{(0)} + c_t^{(1)} g_0^2 + O(g_0^4)$$
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- Tree level values

$$c_t^{(0)} = 1, \tilde{c}_t^{(0)} = 1, C_{SW}^{(0)} = 1$$

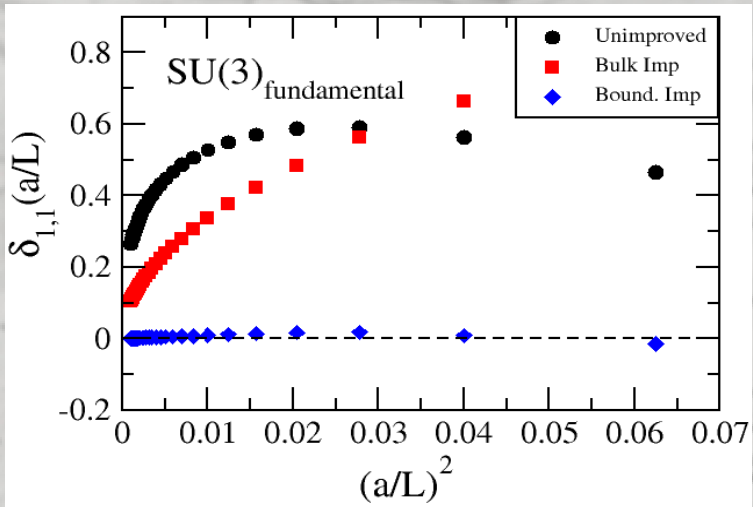
$$z_f^{(0)} = 1, d_s^{(0)} = 1/2$$

$$p_1(L/a)$$

Cutoff effects:

Fundamental

SF



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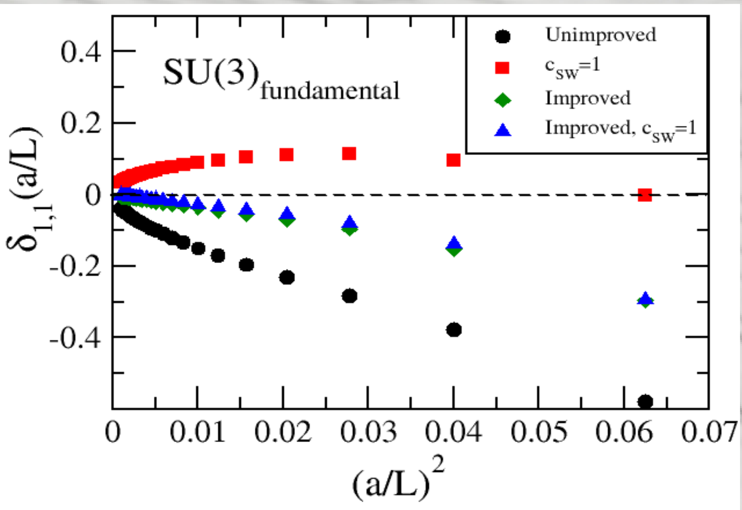

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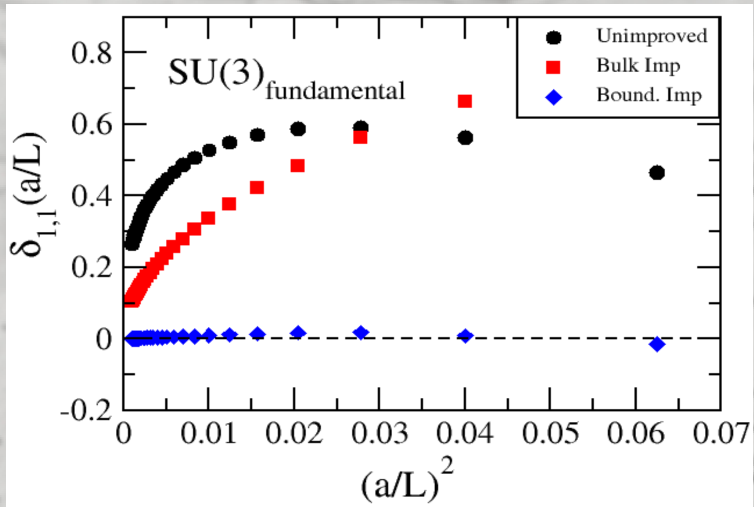

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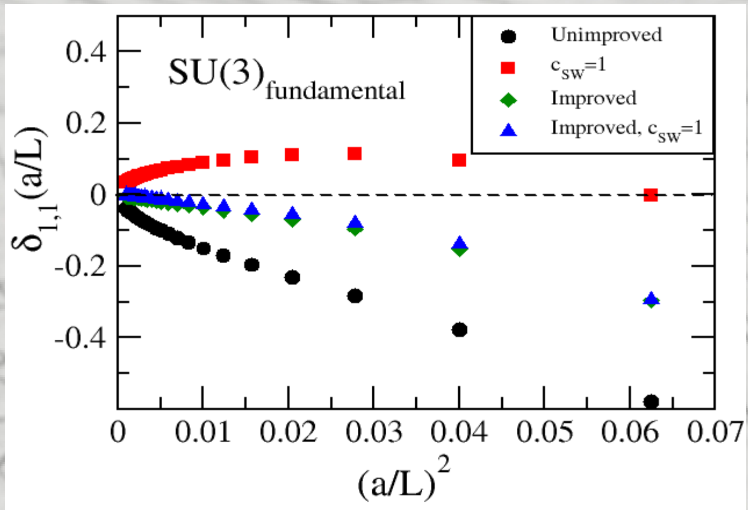
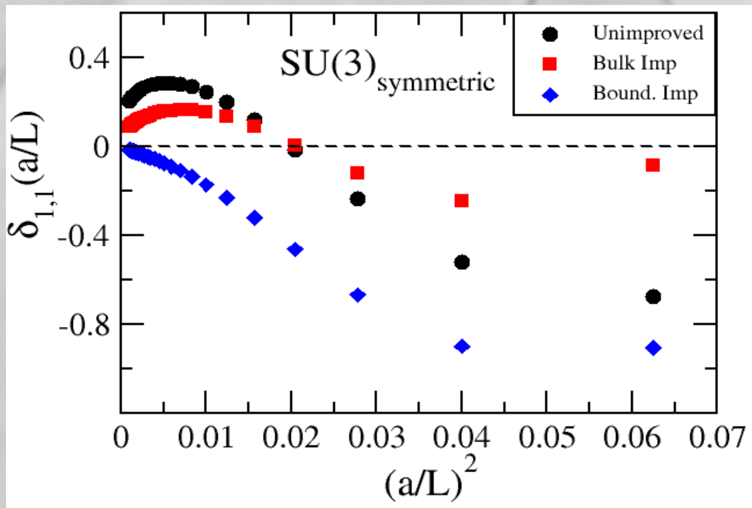
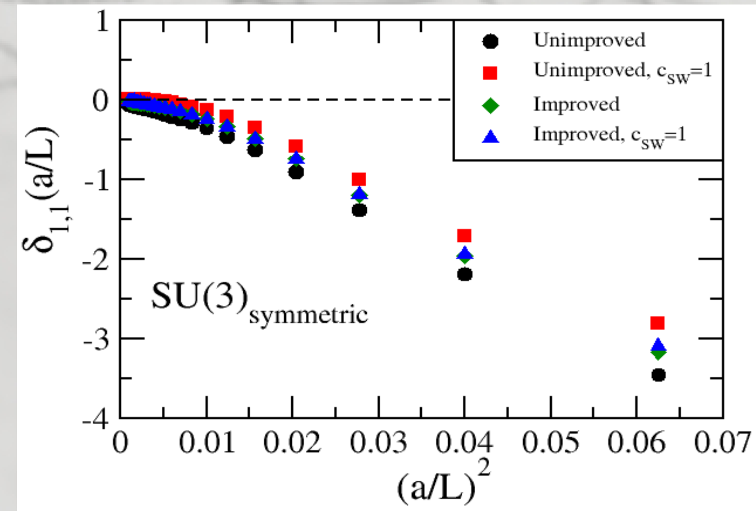


Symmetric





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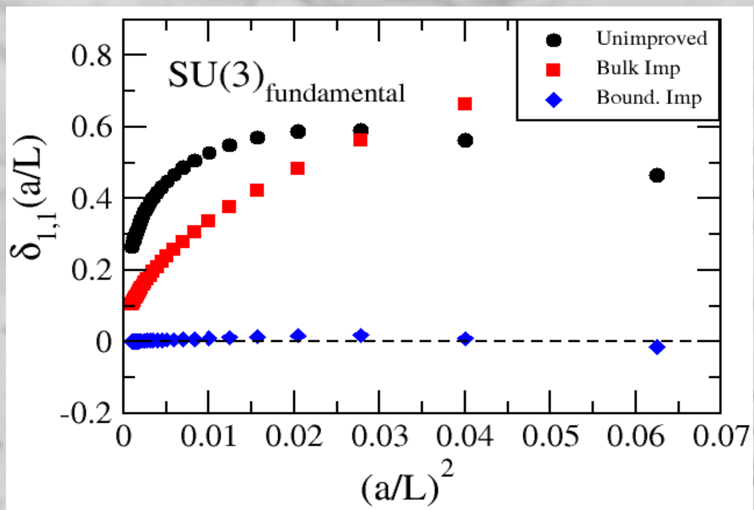
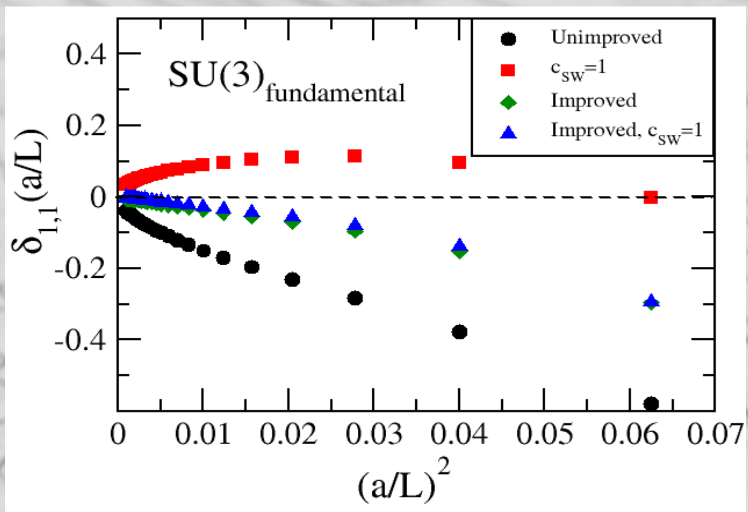
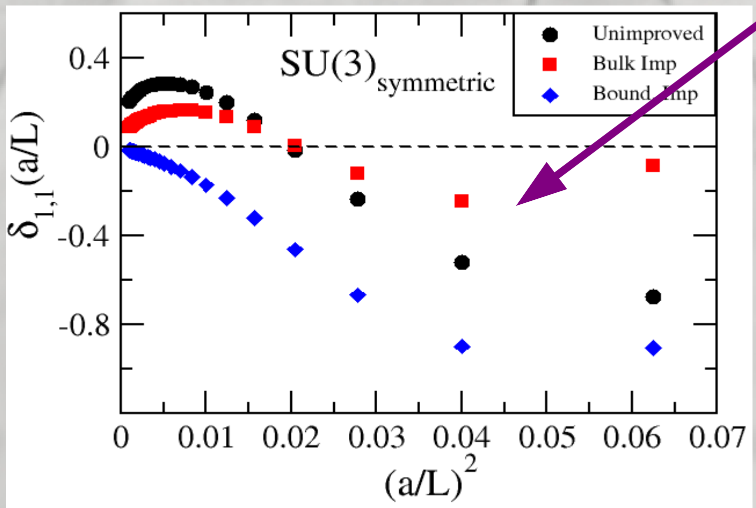
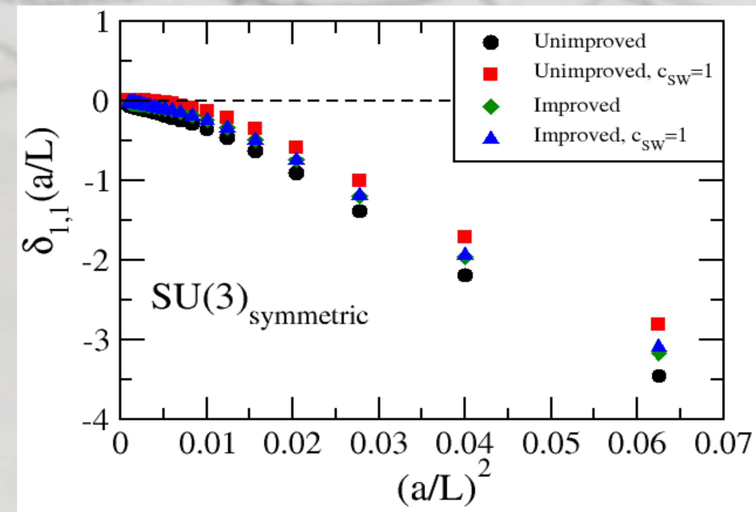




Symmetric

Large higher order cutoff effects!!!!





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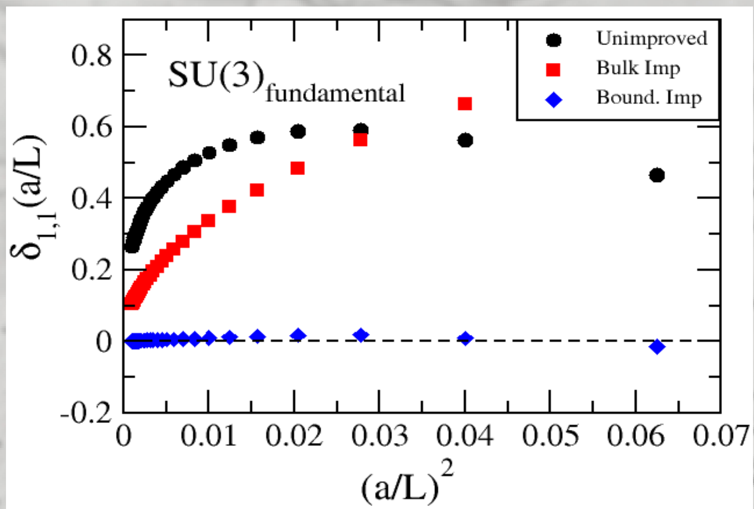

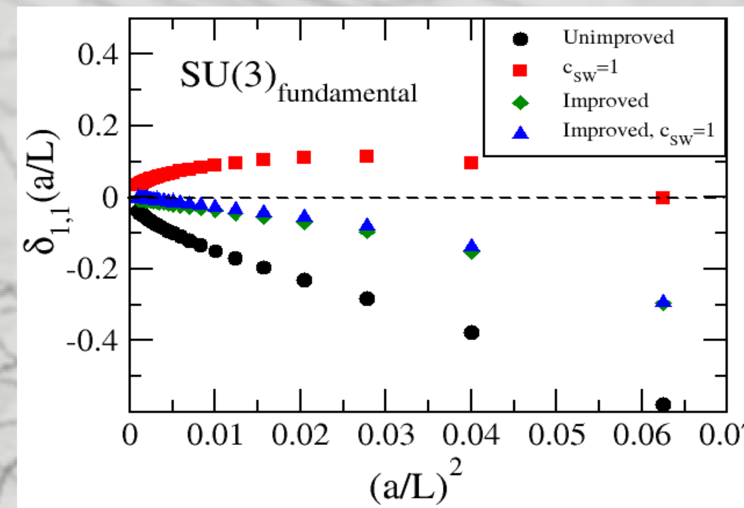

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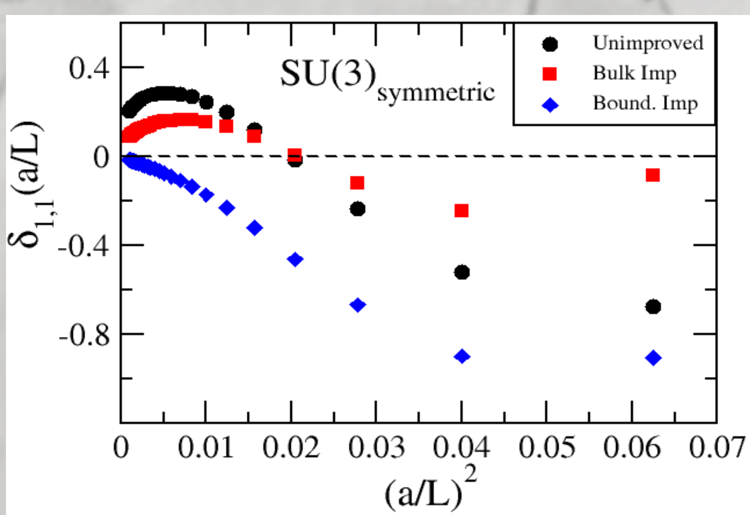
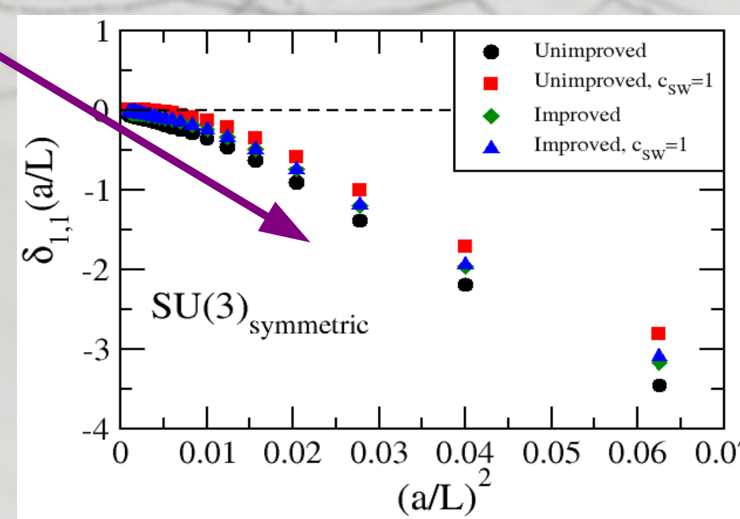




Cutoff effects:

- Symmetric (or adjoint) fields see a field

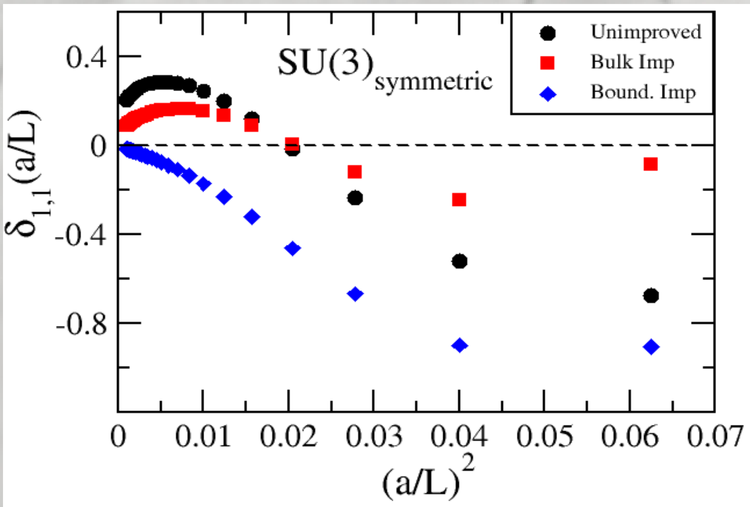
$$C_k = \frac{i}{L} \mathrm{diag}\left(\phi^R_{k1}, \phi^R_{k2}, \ldots, \phi^R_{kN}\right)$$

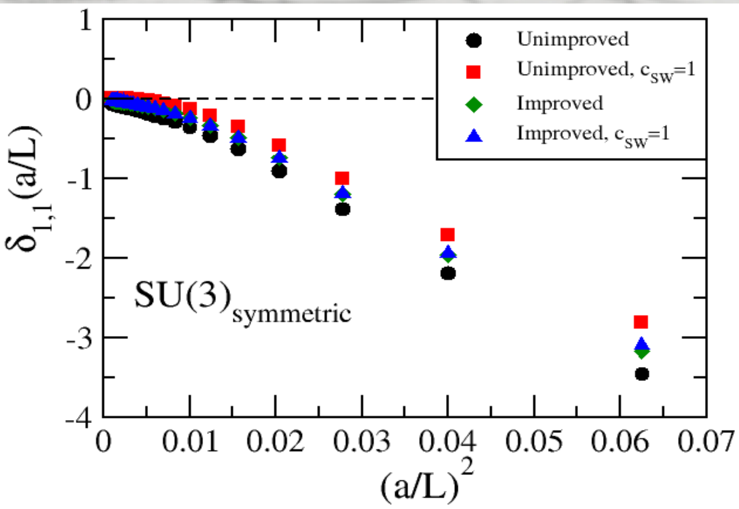
$$\begin{aligned} \phi^S_1 &= 2\phi_1 \\ \phi^S_2 &= \phi_1 + \phi_2 \\ \phi^S_3 &= \phi_1 + \phi_3 \\ \phi^S_4 &= 2\phi_2 \\ \phi^S_5 &= \phi_2 + \phi_3 \\ \phi^S_6 &= 2\phi_3 \end{aligned}$$

- The Background Field is made weaker:

$$\phi^S_i \to \phi^S_i / 2$$

Symmetric





Cutoff effects:

• Symmetric (or adjoint) fields see a field

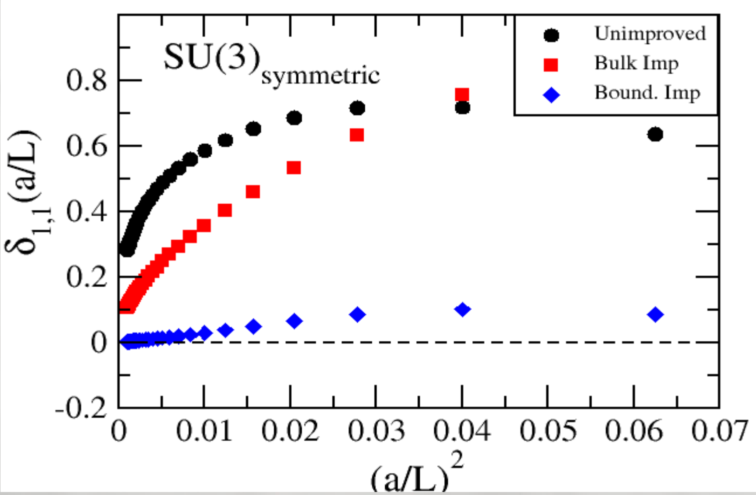
$$C_k = \frac{i}{L}\mathrm{diag}\left(\phi^R_{k1}, \phi^R_{k2}, \ldots, \phi^R_{kN}\right)$$

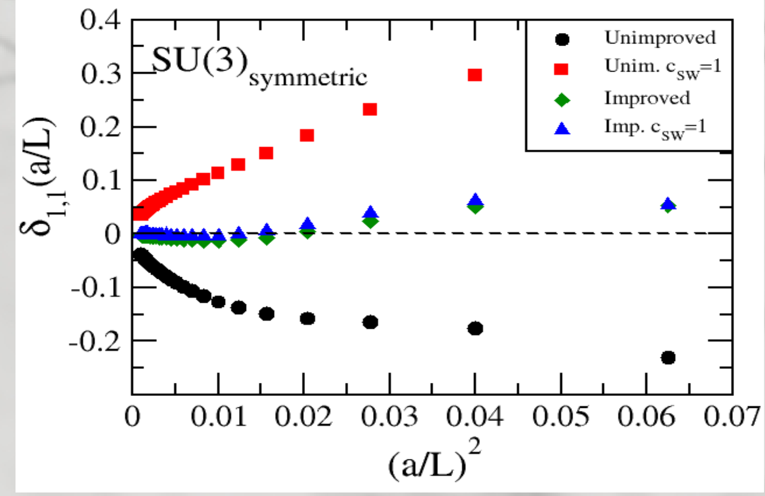
$$\begin{aligned} \phi^S_1 &= 2\phi_1 \\ \phi^S_2 &= \phi_1 + \phi_2 \\ \phi^S_3 &= \phi_1 + \phi_3 \\ \phi^S_4 &= 2\phi_2 \\ \phi^S_5 &= \phi_2 + \phi_3 \\ \phi^S_6 &= 2\phi_3 \end{aligned}$$

• The Background Field is made weaker:

$$\phi^S_i \rightarrow \phi^S_i/2$$

Symmetric BF → BF/2 [Sint, V., Lattice 2011]



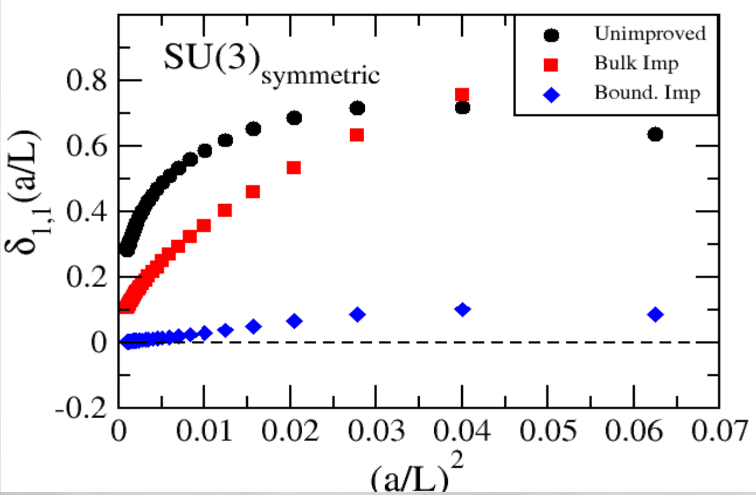


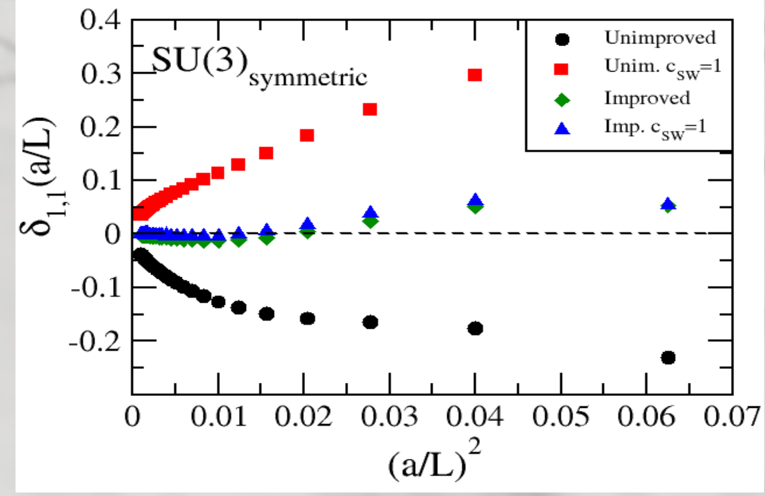
Cutoff effects:

Problems alleviated, but...

- Modified BF → pure gauge part must be recomputed.
- Non-symmetric fields induce big statistical fluctuations.

Symmetric BF → BF/2 [Sint, V., Lattice 2011]





Removing higher order effects:

- The parameter η is added to define the coupling

$$C_k = \frac{i}{L}\mathrm{diag}(\phi_{1k}, \ldots, \phi_{Nk}) + \frac{i}{L}\eta\lambda_8|_{\eta=0}$$

- The BF can depend on an extra parameter ν.

$$\eta\lambda_8 \longrightarrow \eta\left(\lambda_8 + \nu\lambda_3\right)$$

- A whole familly of renormalized couplings can be defined

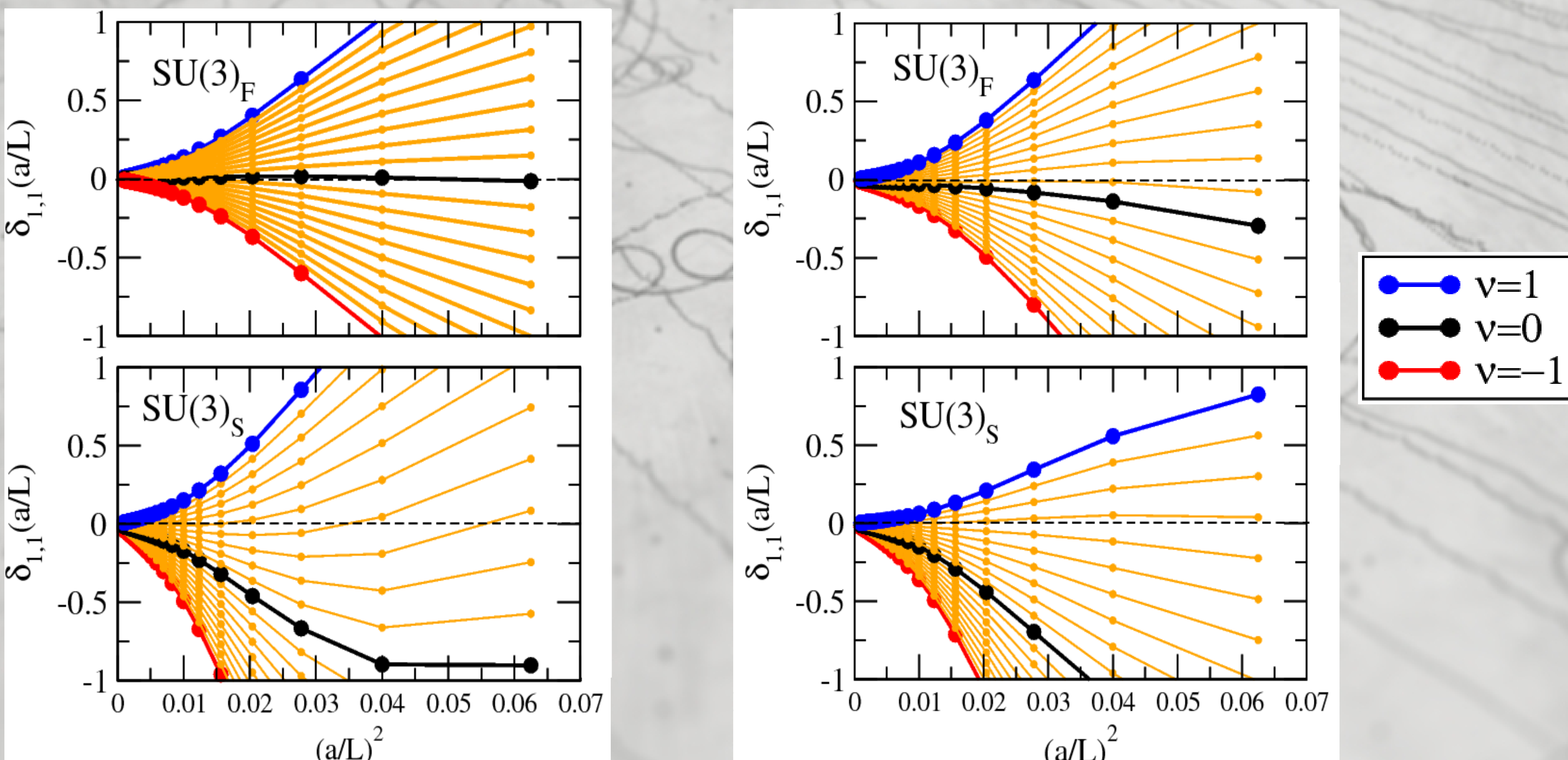
$$\frac{1}{\overline{g}_\nu^2(L)} = \frac{1}{\overline{g}^2(L)} - \nu\overline{v}(L)$$

$$\overline{v}(L) = \frac{1}{\kappa}\frac{\partial}{\partial\nu}\left\{\frac{\partial\Gamma}{\partial\eta}\Big|_{\eta=0}\right\}\Big|_{\nu=0}$$

Removing higher order effects:

- Modifying **ν** all the familly of couplings can be explored

SF **χSF**



Removing higher order effects:

- Modifying ν all the familly of couplings can be explored

- Cutoff effects **dramatically** reduced.
- The **BF** is **not modified** (no extra calculations needed).
- The quantities **g** and ν can be computed once, and the optimal ν chosen a posteriori.

Removing higher order effects:

- Modifying **ν** all the familly of couplings can be explored

- Cutoff effects **dramatically** reduced.
- The **BF** is **not modified** (no extra calculations needed).
- The quantities **g** and **ν** can be computed once, and the optimal **ν** chosen a posteriori.

- **SU(2):** Only one **abelian** direction.
- Consider **non-abelian** directions in the algebra.

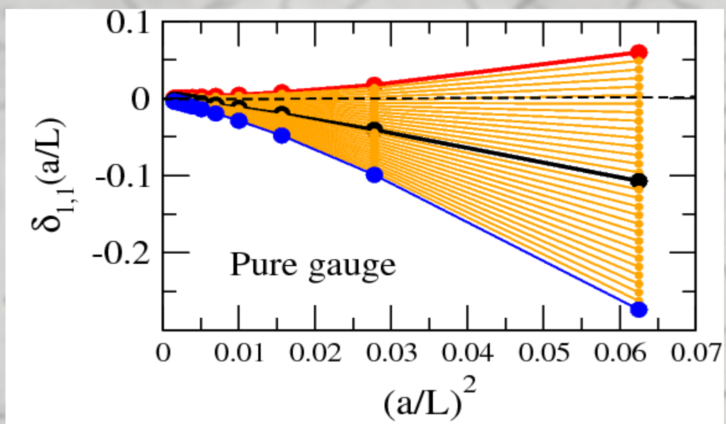
$$\eta\tau_3 \longrightarrow \eta\left(\tau_3 + \sum_{i=1}^{2}\nu_i\tau_i\right)$$

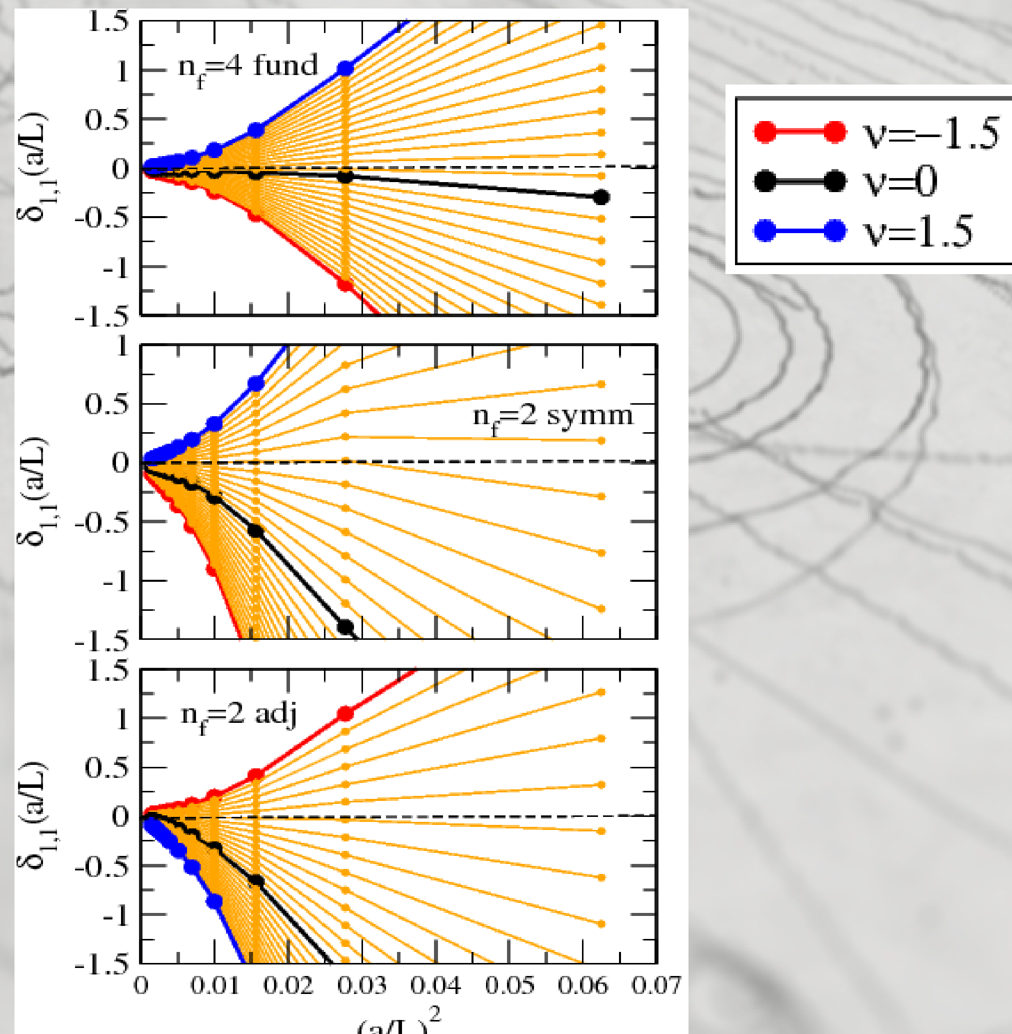
$$\frac{1}{\overline{g}^2_{\vec{V}}(L)} = \frac{1}{\overline{g}^2(L)} - \sum_{i=1}^{2}\nu_i\overline{v}_i(L)$$

Removing higher order effects:

- The full theory, for fundamental nf=4, and adjoint and symmetric nf=2.

χSF





Conclusions:

- We want to study the coupling **g** of strongly interacting theories.

- We put them in the **lattice**.

- **Cutoff effects** are the **enemy**. They must be **terminated**.

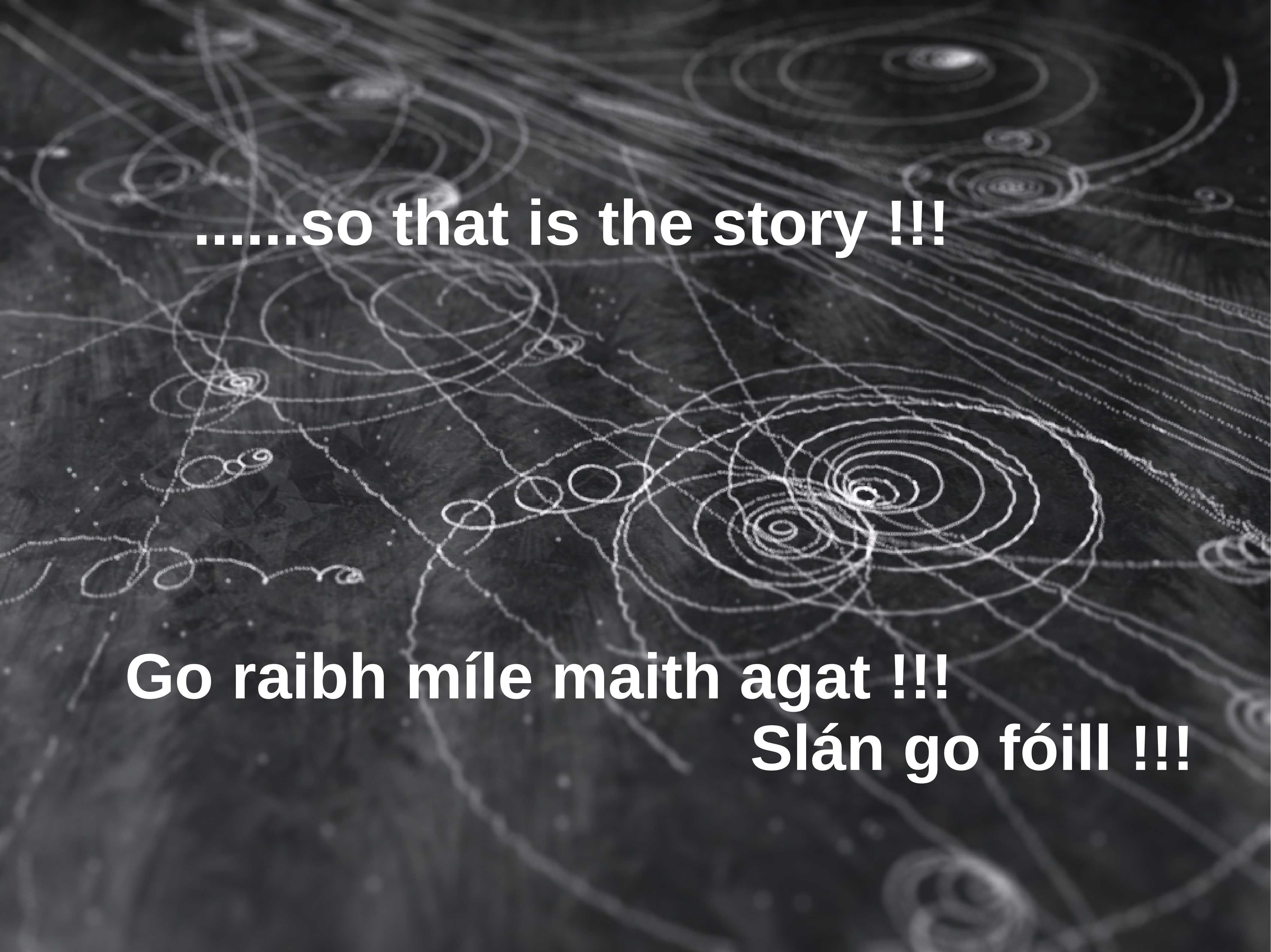
- They are **large**, hence we develop alternative strategies.

- We **remove** them

    - At **O(a)** through **Symanzik's improvement**.
    - At **higher orders** through a **redefinition of the coupling**.

- The **χSF** regularization works particularly well.

- The **condition number** is **minimized** by a choice of spatial BC.



......so that is the story !!!

Go raibh míle maith agat !!!

Slán go fóill !!!