Broken spacetime symmetries, elastic variables, and Nambu-Goldstone modes

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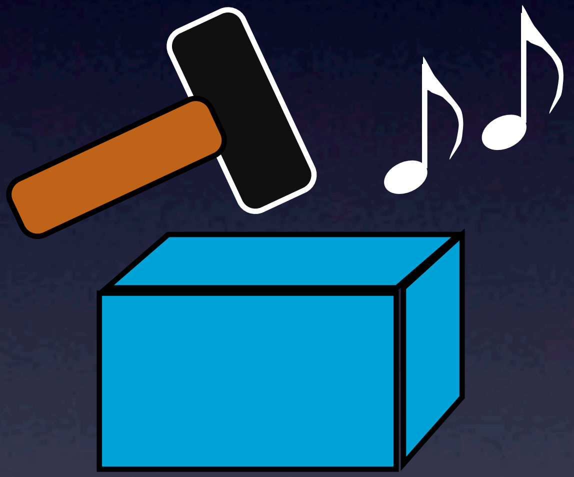
Based on YH, Phys. Rev. Lett. 110, 091601 (2013), 1203.1494 [hep-th],
Hayata, YH, 1312.0008 [hep-th]

Zero modes in nature

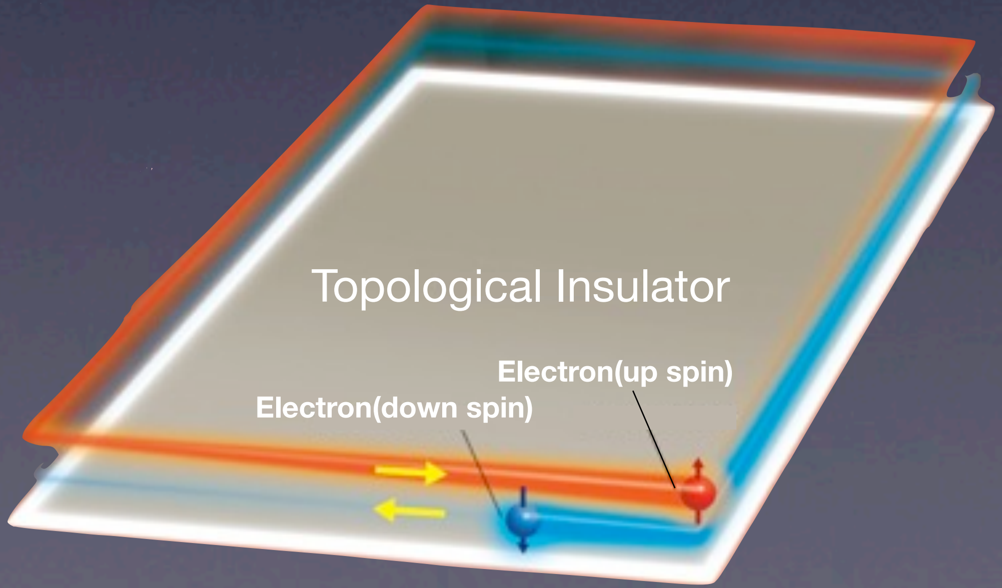
Light (Photon)

Gauge symmetry



Crystal Vibrations (Phonon)

Spontaneous symmetry
breaking of translation



Edge modes in topological insulator

Topology

Nambu-Goldstone theorem

Spontaneous breaking
of continuum symmetry

➡ gapless mode (NG mode)

- Relation between broken symmetries and NG modes?
- Dispersion of NG modes?

It is well studied in each case.

QCD, superfluid, ferromagnet,...

Elastic variables

Free energy:

$$F = g^{ab}(\partial \pi_a)(\partial \pi_b) + \cdots$$

NG modes

Gapless propagating mode

Dispersion relation:

$$\omega = ak^n + ibk^m$$

Two type of conserved charges

Translationally invariant

$$[P_\mu, Q_a] = 0$$

translational operator charge

Ex: Translationally invariant charges

Spacetime translation, chiral symmetry,

flavor symmetry, ...

Ex: Non-translationally invariant charges

Rotation, boost, conformal,

residual gauge symmetry in the covariant gauge,...

NG modes in QCD

Pion

SSB of chiral symmetry

$$SU(2)_L \times SU(2)_R \to SU(2)_V$$

$$N_{\mathrm{BS}} = 3, \quad N_{\mathrm{NG}} = 3$$

Dispersion: $\omega = k$ **Type-I**

NG modes in Kaon condensed CFL phase

Miransky, Shovkovy ('02) Schafer, Son, Stephanov, Toublan, and Verbaarschot ('01)

$$SU(2)_I \times U(1)_Y \to U(1)_{\mathrm{em}}$$

$$N_{\mathrm{BS}} = 3, \quad N_{\mathrm{NG}} = 2$$

Dispersion: $\omega = k^2$ **Type-II**

Example of NG mode in nonrelativistic systems

SSB of space-time symm.

Phonon in crystal

translation, rotation, Galilei

$$N_{\mathrm{BS}} = 9, \quad N_{\mathrm{NG}} = 3$$

SSB of internal symm.

Spin waves in ferromagnet

SSB of rotation O(3)→O(2)

$$N_{\mathrm{BS}} = 2, \quad N_{\mathrm{NG}} = 1$$

Spontaneous breaking of translationally invariant charges

Nambu-Goldstone theorem

(Lorentz invariant system)

Nambu('60), Goldstone(61), Nambu, Jona-Lasinio('61), Goldstone, Salam, Weinberg('62).

$$N_{NG} = N_{BS}$$

# of NG modes

# of broken symmetries

Dispersion relation

$$\omega = k$$

Generalization

Nielsen - Chadha ('76)

$$N_{\text{type-I}} + 2N_{\text{type-II}} \geq N_{\text{BS}}$$

Type-I: $\omega \propto k^{2n+1}$ **Type-II:** $\omega \propto k^{2n}$

Schafer, Son, Stephanov, Toublan, and Verbaarschot ('01)

$$\langle [Q_a, Q_b] \rangle = 0 \quad \Rightarrow \quad N_{\text{NG}} = N_{\text{BS}}$$

Watanabe - Brauner ('11)

$$N_{\text{BS}} - N_{\text{NG}} \leq \frac{1}{2} \text{rank} \langle [Q_a, Q_b] \rangle$$

Example of Type-II modes

	N_{BS}	$N_{\text{type-I}}$	$N_{\text{tyep-II}}$	$\frac{1}{2}\text{rank}\langle[Q_a, Q_b]\rangle$	$N_{\text{type-I}} + 2N_{\text{type-II}}$
Spin wave in ferromanget O(3)→O(2)	2	0	1	1	2
NG modes in Kaon condensed CFL $SU(2) \times SU(1)_Y \to U(1)_{em}$	3	1	1	1	3
Kelvin waves in vortex superfluid translation P_x, P_y	2	0	1	1	2

Known examples satisfy

$$N_{\text{type-I}} + 2N_{\text{type-II}} = N_{\text{BS}} \qquad N_{\text{BS}} - N_{\text{NG}} = \frac{1}{2}\text{rank}\langle[Q_a, Q_b]\rangle$$

Recent development

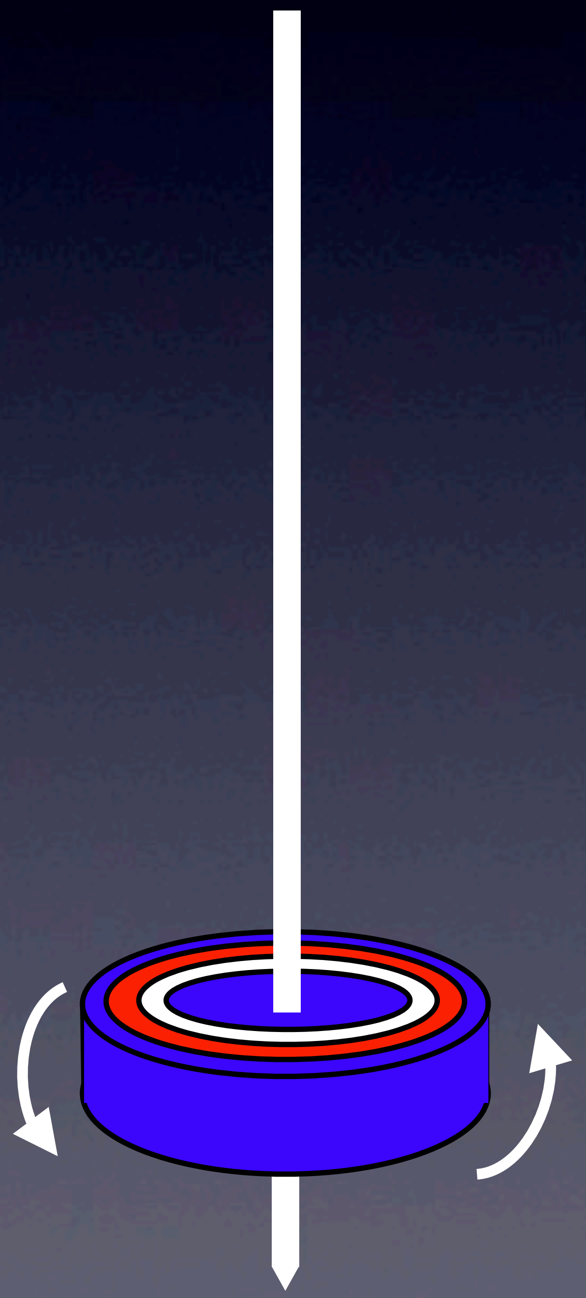
Watanabe, Murayama ('12), YH ('12)

- $N_{\mathrm{BS}} - N_{\mathrm{NG}} = \frac{1}{2}\mathrm{rank}\langle[Q_a, Q_b]\rangle$
- $N_{\text{type-I}} + 2N_{\text{type-II}} = N_{\mathrm{BS}}$
- $N_{\text{type-II}} = \frac{1}{2}\mathrm{rank}\langle[Q_a, Q_b]\rangle$

Intuitive example for type-II NG modes

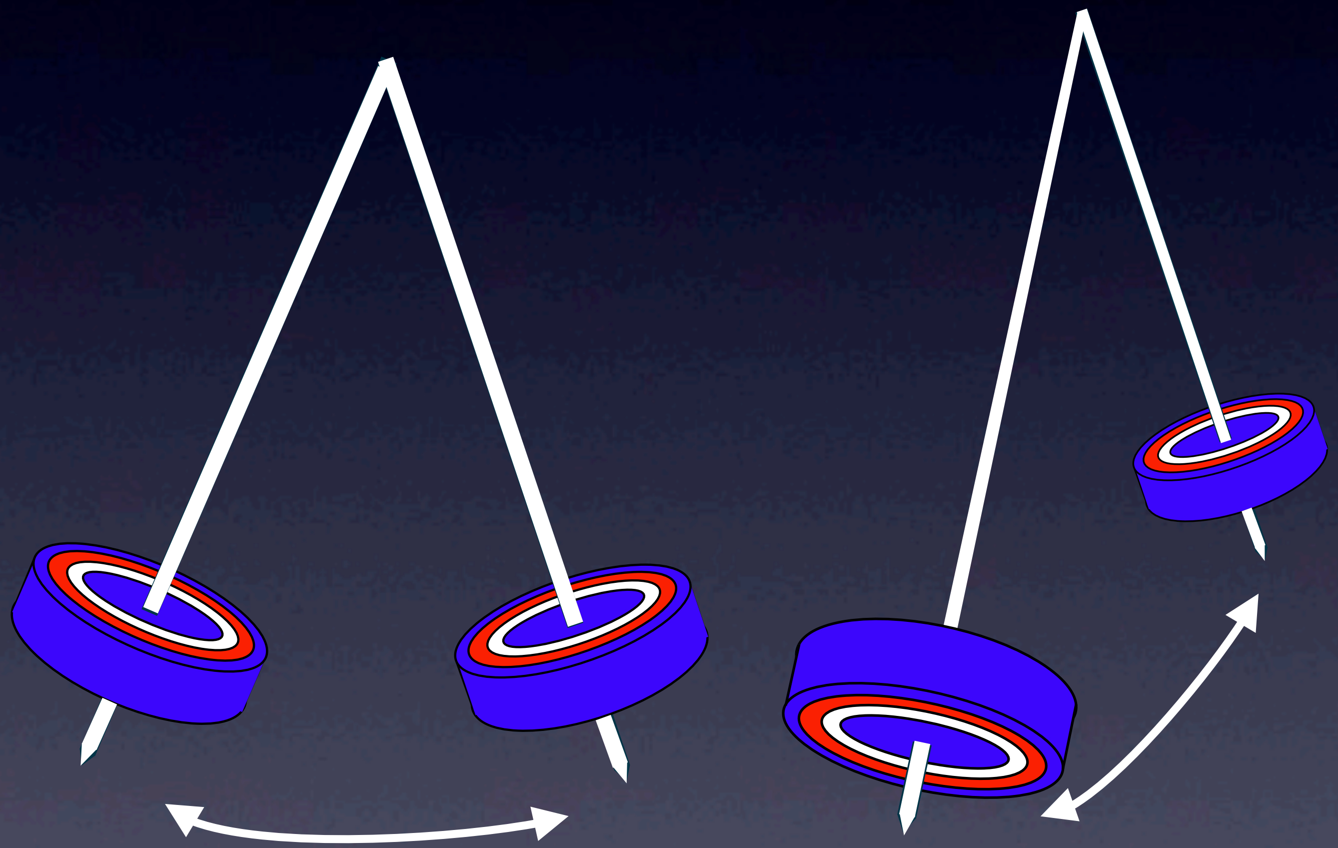
Pendulum with a spinning top

- Rotation symmetry is explicitly broken by a weak gravity.
- Rotation along with z axis is unbroken.
- Rotation along with x or y is broken.
- The number of broken symmetry is two.



Intuitive example for type-II NG modes

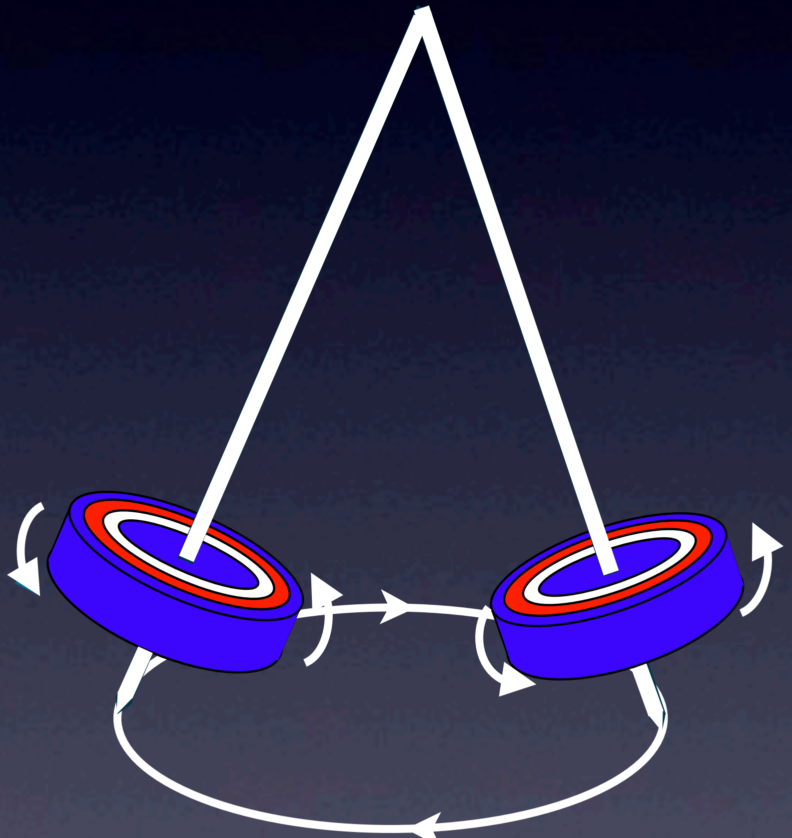
Pendulum has two oscillation motions



if the top is not spinning.

Intuitive example for type-II NG modes

If the top is spinning,



the only one rotation motion (Precession) exists.

In this case, $\{L_x, L_y\}_P = L_z \neq 0$

Spontaneous breaking of non-translationally invariant charges

Spontaneous breaking of non-translationally invariant charges

Low - Manohar's argument

Low, and Manohar ('02)

Ex.: String

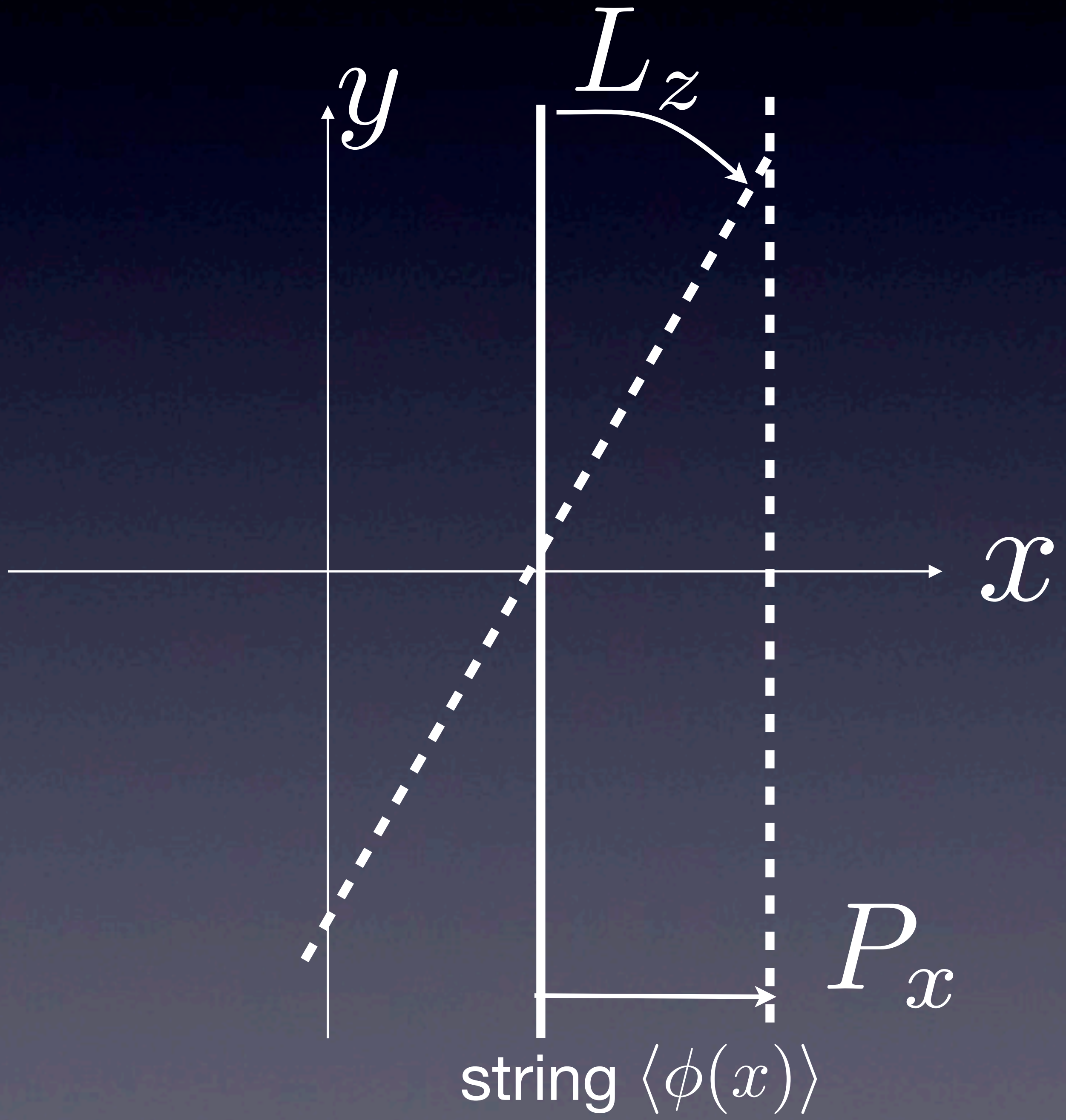
order parameter: $\langle\phi(x)\rangle$

trans.:$\langle[P_x,\phi]\rangle = i\partial_x\langle\phi\rangle \neq 0$

rot.: $\langle[L_z,\phi]\rangle = -iy\partial_x\langle\phi\rangle \neq 0$

Two broken symm.,

but one NG mode.



This talk

We clarify

relation between broken symmetries and elastic variables.

the dispersion relation for broken (non-)translationally invariant symmetries at finite T.

Assumption

Translation is not completely broken at least one direction.

$$\chi_{ij}(k) \sim \frac{1}{k^2 + m^2}$$

m=0 for elastic variables.

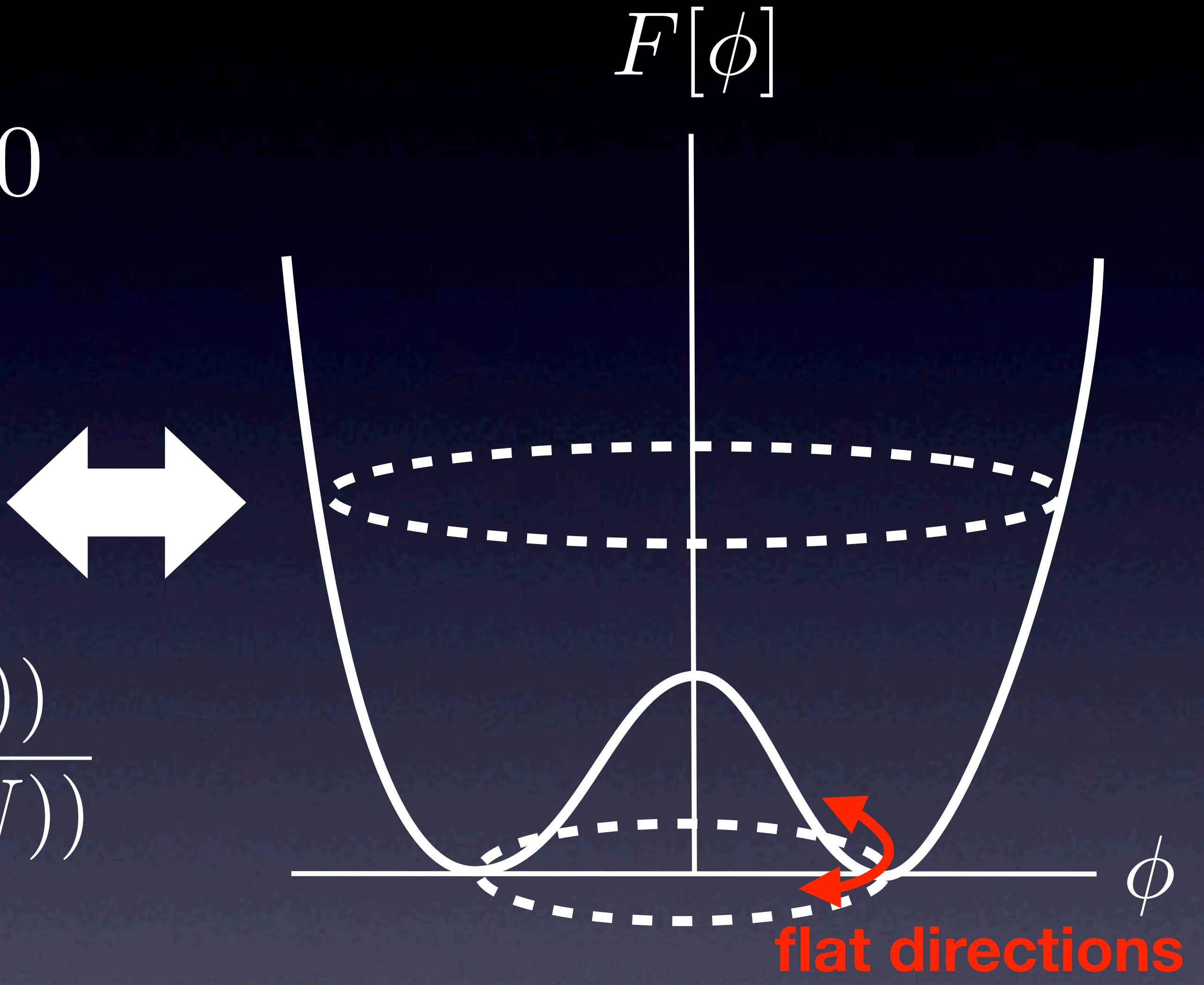
Elastic variable and Spontaneous symmetry breaking

$$\langle [Q_a, \phi_i] \rangle \equiv \mathrm{tr}\rho\, [Q_a, \phi_i] \neq 0$$

$$a = 1, \cdots, N_{\mathrm{BS}}$$

Vacuum: $\rho = |\Omega\rangle\langle\Omega|$

In medium: $\rho = \dfrac{\exp(-\beta(H-\mu N))}{\mathrm{tr}\exp(-\beta(H-\mu N))}$



$\langle [\phi_i, Q_a] \rangle$ →**elastic variable**

Free energy: $F = \dfrac{1}{2}(\nabla\pi)^2 + \cdots$

Suppose the classical action is invariant under

$$\phi_i \to \phi_i + \epsilon^a [iQ_a, \phi_i]$$

Free energy $F[\phi]$ **satisfies**

$$\int d^d x \frac{\delta F[\phi]}{\delta \phi_i(\boldsymbol{x})} h_{ai}(\boldsymbol{x}) = 0$$

$$h_{ai}(\boldsymbol{x}) \equiv \langle [iQ_a, \phi_i(\boldsymbol{x})] \rangle$$

➡ $$\int d^d x \frac{\delta^2 F[\phi]}{\delta \phi_j(\boldsymbol{y}) \delta \phi_i(\boldsymbol{x})} h_{ai}(\boldsymbol{x}) = 0$$

Inverse susceptibility: $\chi^{-1}_{ij}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\delta^2 F[\phi]}{\delta \phi_i(\boldsymbol{x}) \delta \phi_j(\boldsymbol{y})}$ has zero.

# of independent zero modes
= # of independent eigenvectors
(elastic variables)

Eigenvector should be chosen as eigenvector for translation

$$h_{ai}(\boldsymbol{x}) \equiv \langle [iQ_a, \phi_i(\boldsymbol{x})] \rangle$$

$$T_{\boldsymbol{x}} Q_a T_{\boldsymbol{x}}^{\dagger} = c_a{}^b(\boldsymbol{x}) Q_b$$

Linear combination transforms

$$f^a h_{ai}(\boldsymbol{x}) = f^a c_a{}^b(\boldsymbol{R}) h_{bi}(\boldsymbol{x})$$

$$f^a(\delta_a{}^b - c_a{}^b(\boldsymbol{R})) \equiv f^a A_a^b = 0$$

of independent elastic variables

$$N_{\mathrm{EV}} = \dim \ker A$$

For translational invariant charges $A = 0$

$$N_{\mathrm{EV}} = N_{\mathrm{BS}}$$

For Lorentz invariant system

Goldstone, Salam, Weinberg ('62)

$$D^{-1}_{ji}(k=0)h^{ai}=0$$

$$h_{ai}(\boldsymbol{x})\equiv\langle[iQ_a,\phi_i(\boldsymbol{x})]\rangle$$

Lorentz invariance: $k=0 \Rightarrow k_\mu^2=0$

Dispersion: $\omega=|\boldsymbol{k}|$

$$N_{\text{NG}}=N_{\text{EV}}=N_{\text{BS}}$$

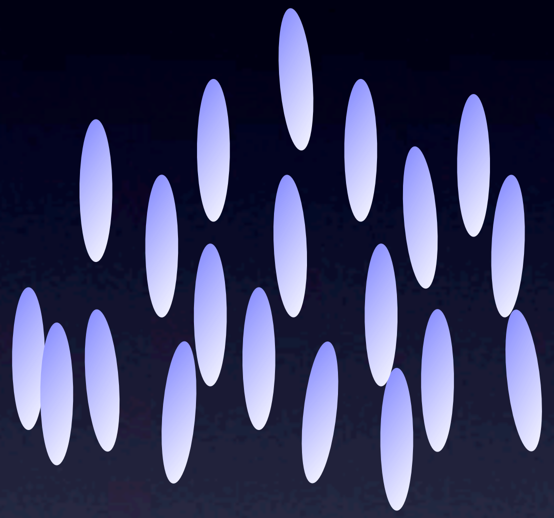
Ex1: Liquid crystal

Rotation symmetry:

$$\langle [T_{\boldsymbol{R}} L_{iz} T_{\boldsymbol{R}}^{\dagger}, \phi_k(\boldsymbol{x})] \rangle = \langle [L_{iz}, \phi_k(\boldsymbol{x})] \rangle - R_i \langle [P_z, \phi_k(\boldsymbol{x})] \rangle + R_z \langle [P_i, \phi_k(\boldsymbol{x})] \rangle$$

Nematic phase

$$\underset{i=x,y}{\langle [L_{iz}, \phi_k(\boldsymbol{x})] \rangle \neq 0} \quad \langle [P_i, \phi_k(\boldsymbol{x})] \rangle = \langle [P_z, \phi_k(\boldsymbol{x})] \rangle = 0$$



$$A = \begin{array}{cc} L_{xz} & L_{yz} \\ \end{array} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad N_{\mathrm{EV}} = \dim\ker A = N_{\mathrm{BS}} = 2$$

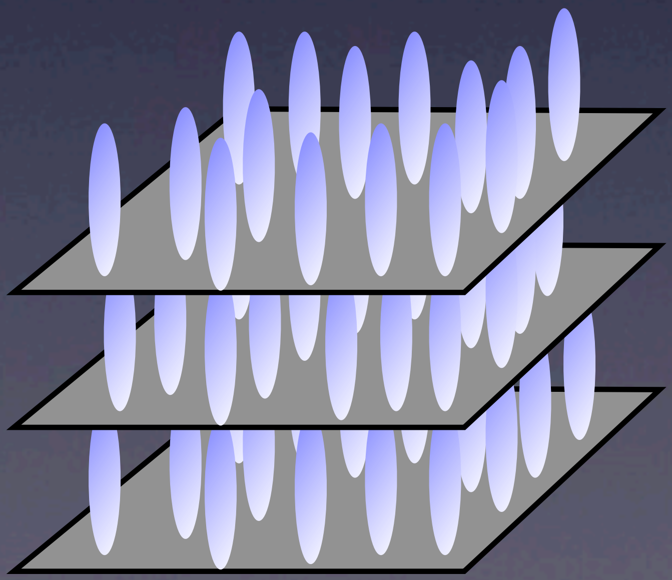
Smectic-A phase

$$\langle [L_{iz}, \phi_k(\boldsymbol{x})] \rangle \neq 0$$

$$\langle [P_z, \phi_k(\boldsymbol{x})] \rangle \neq 0$$

$$\langle [P_i, \phi_k(\boldsymbol{x})] \rangle = 0$$

$$\Rightarrow \quad A = \left(\begin{array}{ccc} L_{xz} & L_{yz} & P_z \\ 0 & 0 & R_x \\ 0 & 0 & R_y \\ 0 & 0 & 0 \end{array}\right)$$



$$N_{\mathrm{EV}} = \dim\ker A = 1$$

Ex2: Conformal field theory

Scalar condensate: $\langle \phi(x) \rangle = c$

Symmetry breaking $ISO(2,3) \rightarrow ISO(1,3)$

Dilatation $\langle [D, \phi(x)] \rangle = 3c$

Special conformal $\langle [K_\mu, \phi(x)] \rangle = 6cx_\mu$

Translation $\langle [P_\nu, [K_\mu, \phi(x)]] \rangle = -\eta_{\mu\nu} \langle [D, \phi(x)] \rangle$

➡

$$A = \begin{array}{c} \begin{array}{cc} \scriptstyle K_\mu & \scriptstyle D \end{array} \\ \begin{pmatrix} 0 & R_\mu \\ 0 & 0 \end{pmatrix} \end{array}$$

$$N_{\mathrm{EV}} = N_{\mathrm{NG}} = \mathrm{dimker} A = 1$$

Independent NG mode is only one.

Ex3: QED

Ferrari, Picasso ('71), Hata ('82), Kugo, Terao, Uehara ('85)

Covariant gauge $\mathcal{L}_{\mathrm{GF}} = B\partial^\mu A_\mu + \frac{1}{2}\alpha B^2$

Gauge parameter $\theta(x) = a + b_\mu x^\mu$

➡ charges Q, Q_μ Q$_\mu$ is always broken: $\langle [Q_\mu, A_\nu] \rangle = \delta_{\mu\nu}$

Under translation: $[P_\nu, Q_\mu] = i\eta_{\nu\mu} Q$

Coulomb phase: *Q* is unbroken.

$A = 0$ ➡ $N_{\mathrm{EV}} = N_{\mathrm{NG}} = 4$

NG boson: Photon (2, physical)

scalar and longitudinal parts (unphysical)

Higgs phase: *Q* is broken.

$$A = \begin{array}{cc} Q_\mu & Q \\ \end{array} \begin{pmatrix} 0 & R_\mu \\ 0 & 0 \end{pmatrix}$$

$\langle [Q, \phi] \rangle = v$ NG higgs (unphysical)

$\langle [Q_\mu, \phi] \rangle = x_\mu v$

➡ $N_{\mathrm{EV}} = N_{\mathrm{NG}} = 1$

Nambu-Goldstone modes and their dispersion

for spontaneous breaking of translationally invariant charges

Slow variables

Conserved charge density

$$\partial_t n_a = -\partial_i j_a^i$$

Elastic variables

$$F = \frac{g^{ab}}{2}(\partial_i \pi_a)(\partial^i \pi_b) + \cdots$$

(We assume no other slow variable exists.)

NG modes

Canonical pairs π_i, n_a

$$\{n_a, \pi^i\}_{\mathrm{P}} = -i\langle[Q_a, \pi^i]\rangle$$

cf. Nambu ('04)

of canonical pair = NNG

Type-A NG mode

n_a's are canonically independent

Type-B NG mode

n_a are also elastic variables

the number of canonical pairs=

$$N_{\text{type-B}} = \frac{1}{2}\mathrm{rank}[Q_a, Q_b]$$

The number of Type-B pairs

$$N_{\text{type-B}} = \frac{1}{2}\text{rank}\langle[Q_a, Q_b]\rangle$$

The number of Type-A pairs

$$N_{\text{type-A}} = N_{\text{BS}} - 2N_{\text{type-B}}$$

- $N_{\text{type-A}} + 2N_{\text{type-B}} = N_{\text{BS}}$
- $N_{\text{BS}} - N_{\text{NG}} = \frac{1}{2}\text{rank}\langle[Q_a, Q_b]\rangle$

Generalized Langevin equation

formal derivation: Mori ('65)

$$\partial_t A_n = \{A_n, F[A]\}_P - \Gamma_{nm} \frac{\partial F}{\partial A_m} + \xi$$

Free energy: $F[A]$

Poisson bracket: $\{A_n, A_m\}_P \equiv -i\langle [A_n, A_m] \rangle$

Dissipation term: Γ

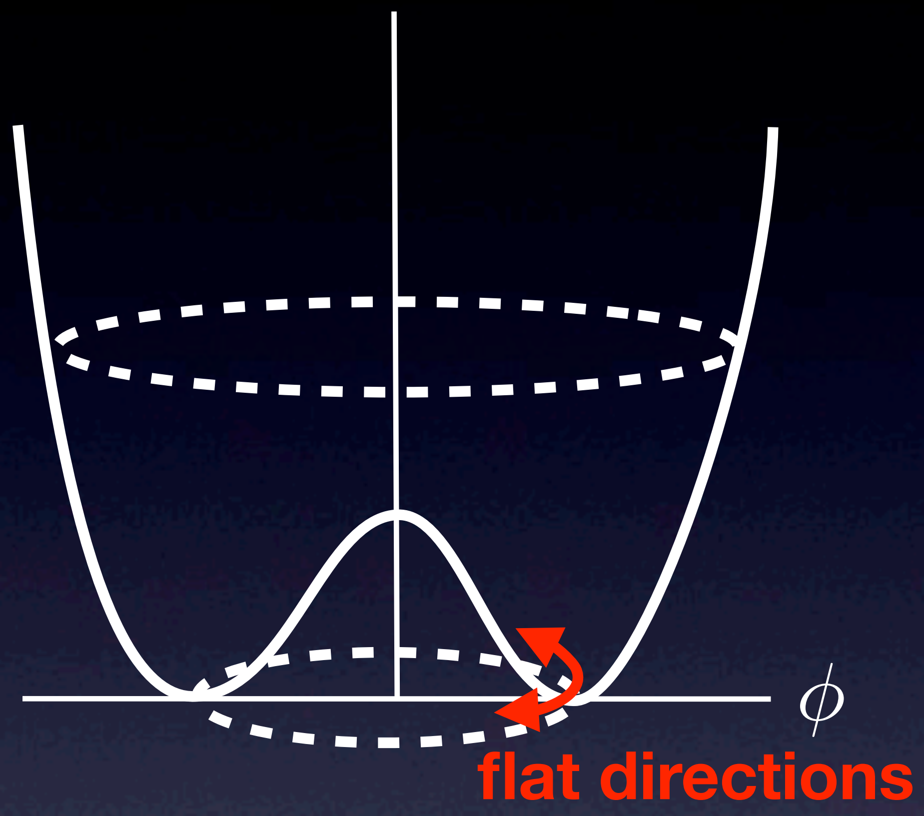
Noise term: ξ

First, we neglect dissipation effect, i.e. $\Gamma=0$.

Type-A NG mode

$$F = \frac{\chi_n^{-1}}{2} n^2 + \frac{1}{2}(\nabla \pi)^2 + \cdots$$

$$\langle [iQ, \pi] \rangle = c, \langle [Q, n] \rangle = 0$$



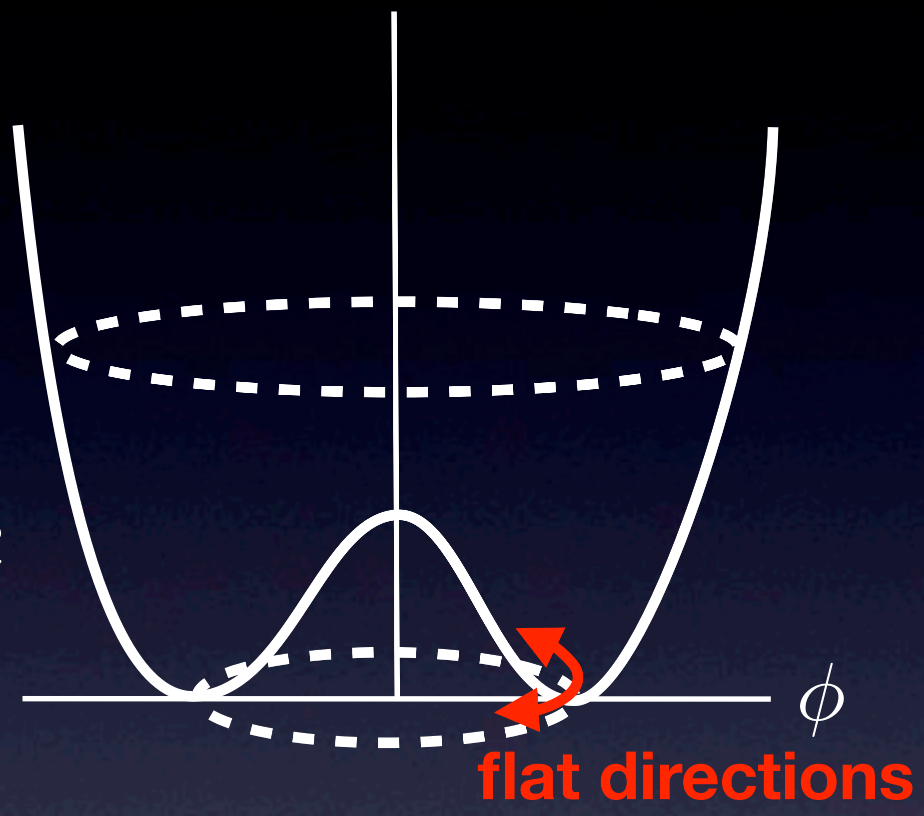
$$\omega^2 = k^2 \chi_n^{-1} c^2 \Rightarrow \mathrm{Re}\,\omega \sim k$$ **Type-I**

Type-A = Type-I

Type-B NG mode

$$F = a\frac{1}{2}(\nabla n_1)^2 + b\frac{1}{2}(\nabla n_2)^2$$

$$\langle [Q_1, n_2] \rangle = c, \langle [Q_2, n_1] \rangle = -c$$



$$\omega^2 = k^4 abc^2 \quad \Rightarrow \quad \mathrm{Re}\,\omega \sim k^2$$ **Type-II**

Type-B = Type-II

Dissipation effect

Type-A

$$\partial_t A_n = \{A_n, F[A]\}_P - \Gamma_{nm}\frac{\partial F}{\partial A_m} + \xi$$

$$F = \frac{\chi_n^{-1}}{2}n^2 + \frac{1}{2}(\nabla\pi)^2 + \cdots$$

Kubo formula

$$\Gamma_{\pi\pi} = \int_0^\infty dt\int_0^\beta d\tau\int d^3x\langle\partial_t\pi(t-i\tau,\boldsymbol{x})\partial_t\pi(0,\boldsymbol{0})\rangle$$

$$\Gamma_{nn} = k^2\int_0^\infty dt\int_0^\beta d\tau\int d^3x\langle j^i(t-i\tau,\boldsymbol{x})j^i(0,\boldsymbol{0})\rangle$$

$$\Gamma_{\pi\pi}\frac{\partial F}{\partial\pi}\sim k^2 \qquad \Gamma_{nn}\frac{\partial F}{\partial n}\sim k^2 \qquad \Rightarrow \qquad \mathrm{Im}\,\omega\sim k^2$$

Dissipation effect

Type-B

$$\partial_t A_n = \{A_n, F[A]\}_P - \Gamma_{nm}\frac{\partial F}{\partial A_m} + \xi$$

$$F = a\frac{1}{2}(\nabla n_1)^2 + b\frac{1}{2}(\nabla n_2)^2$$

Kubo formula

$$\Gamma_{n_a n_b} = k^2 \int_0^\infty dt \int_0^\beta d\tau \int d^3x \langle j_a^i(t - i\tau, \boldsymbol{x}) j_b^j(0, \mathbf{0})\rangle$$

$$\Gamma_{n_a n_b}\frac{\partial F}{\partial n_b} \sim k^4 \quad \Rightarrow \quad \mathrm{Im}\,\omega \sim k^4$$

Summary: Dispersion relation

Type-A (Type-I)

$$\omega = ak + ibk^2$$

Type-B (Type-II)

$$\omega = ak^2 + ibk^4$$

cf. Holographic model suggests $\omega = ak^2 + ick^2$

Amadoa, Daniel Arean, Jimenez-Albac, Landsteinerc, Melgarc, Salazar Landead ('13)

NG modes for spontaneous breaking of non-translationally invariant charges

Dispersion relation

Ex) Liquid crystal

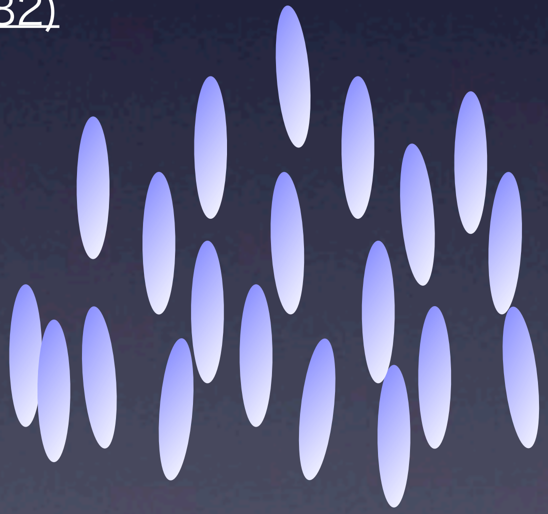
Nematic phase: rotation O(3)→O(2)

$N_{\mathrm{BS}} = N_{\mathrm{EV}} = 2$ $L_i(x) = \epsilon_{ijk} x^j T^{0k}(x)$ $i = 1, 2$

Dispersion relation: $\omega = ak^2 + ibk^2$ Hosino, Nakano('82)

Real and imaginary parts are the same order (damped oscillation)

In case $a = 0$, (overdamping)



Ex) Capillary wave (ripplon)

$$\omega \sim k^{3/2}$$

Summary

For translationally invariant charges

- Independent elastic variable=N_{BS}
- $N_{\mathrm{BS}} - N_{\mathrm{NG}} = \frac{1}{2}\mathrm{rank}\langle[Q_a, Q_b]\rangle$
- $N_{\text{type-I}} + 2N_{\text{type-II}} = N_{\mathrm{BS}}$
- $N_{\text{type-II}} = \frac{1}{2}\mathrm{rank}\langle[Q_a, Q_b]\rangle$

Type-A (Type-I): $\omega = ak + ibk^2$

Type-B (Type-II): $\omega = ak^2 + ibk^4$

Summary

For non-translationally invariant charges

Elastic variables:

$$T_{\boldsymbol{x}} Q_a T_x^\dagger = c_a{}^b(\boldsymbol{x}) Q_b.$$

$$N_{\mathrm{EV}} = \mathrm{dimker}(\delta_a{}^b - c_a{}^b(\boldsymbol{x}))$$

a,b: index of broken charges

NG mode:

Dispersion seems to depend on conserved charges.
Sometimes, it is overdamping at finite T.
What is the general rule?