Noncommutativity of the momentum operator and the dynamics of translational moduli

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1-page summary

- Momentum operator of QFT usually commute: $[P^i, P^j] = 0$

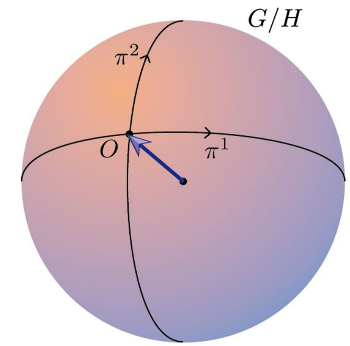
(in the presence of external magnetic field, $[P^i, P^j] = i B^{ij} Q$)

- I will discuss
  - Even if the magnetic field is zero, $[P^i, P^j] != 0$ in the presence of the topological excitation
  - The noncommutativity (central extensions)
    - reduces the number of independent translational zero mode of the topological excitation
    - induces a Magnus force

1. REVIEW OF THE LOW-ENERGY EFFECTIVE LAGRANGIAN

Effective Lagrangian

- Describes Nambu-Goldstone modes originated from the symmetry breaking *G*→*H*
- *G,H* are compact Lie groups.
- $\pi^a \ (a = 1, 2, \ldots, \dim(G/H))$ is a local coordinate of *G/H*.



- $\pi^a(\vec{x}, t)$ is a map from the base manifold *M*

to the target space *G/H.*

- The low-energy effective Lagrangian has the original symmetry *G.*

The general form of the nonrelativistic effective Lagrangian

- Non-linear sigma model
  - relativistic case

metric of *G/H*

$$\mathcal{L} = \frac{1}{2} g_{ab}(\pi) \partial_\mu \pi^a \partial^\mu \pi^b$$

  - nonrelativistic case

$$\mathcal{L}^{(\mathrm{metric})} = \frac{1}{2} \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - \frac{1}{2} g_{ab}(\pi) \nabla \pi^a \cdot \nabla \pi^b$$

$$\mathcal{L}^{(\mathrm{Berry})} = c_a(\pi) \dot{\pi}^a$$ (dominant in the low-energy limit)

Leutwyler PRD (1994)

- Maurer-Cartan form $\omega_a^i(\pi) T_i \mathrm{d}\pi^a = -i e^{-iT_a \pi^a} \mathrm{d} e^{iT_a \pi^a}$

$$g_{ab}(\pi) = g_{cd}(0) \omega_a^c(\pi) \omega_a^d(\pi)$$

$$c_a(\pi) = -\omega_a^i(\pi) e_i(0) \ , \quad f_{\rho i}{}^j e_j(0) = 0$$

HW,Brauner,Murayama, PRL (2013)

Consequence of the Berry phase term

$$c_a(\pi)\dot{\pi}^a = -\frac{1}{2}\rho_{ab}\pi^a\dot{\pi}^b + O(\pi^3)$$

$$i\rho_{ab} = \langle [Q_a, j_b^0(\vec{x},t)]\rangle = \lim_{\Omega\to\infty}\frac{1}{\Omega}\langle [Q_a, Q_b]\rangle$$

Partially symplectic structure
c.f. $L = p_i\dot{q}^i - H(q,p)$

- m(=rank ρ)-canonically conjugate pairs

$(\pi^1, \pi^2), (\pi^3, \pi^4), \dots, (\pi^{2m-1}, \pi^{2m})$ → reduces the number of independent modes

- Counting rule
  - type-A (unpaired) NGBs (usually linear dispersion)

    $n_A = \dim(G/H) - \text{rank}\,\rho$
  - type-B (paired) NGBs (usually quadratic dispersion)

    $n_B = (1/2)\text{rank}\,\rho$
  - The total number of gapless NGBs

    $n_A + n_B = \dim(G/H) - (1/2)\text{rank}\,\rho$

$$\rho = \left(\begin{array}{cccc|c} \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} & & O & & \\ & \begin{pmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{pmatrix} & & & O \\ & & \ddots & & \\ O & & & \begin{pmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{pmatrix} & \\ \hline & & O & & O \end{array}\right)$$

HW, Brauner, PRD (2012)
HW, Murayama, PRL (2013)

Geometry of G/H

- G/H may be seen as a fiber bundle with
  - the base space $B = G/K$ (symplectic)
  - the fiber $F = K/H$

  e.g. $G/H = U(2)/U(1) = S^3$.

  Miranski,Shovkovy PRL 2002
  Schafer et al, Phys.Lett.B 2002

    - $B = U(2) / U(1) \times U(1) = S^2$ (symplectic manifold; type-B)
    - $F = U(1) \times U(1) / U(1) = S^1$ (type-A NGBs)

- When $H^2(G/H)$ is nonzero, the Berry phase term may be seen as the Wess-Zumino term

The ϑ-term for the time direction (1D) + an extended direction (1D) = 2D

Stability condition

situation	$T = 0$	$T \neq 0$
only type-A	z	$2z$
only type-B	0	z

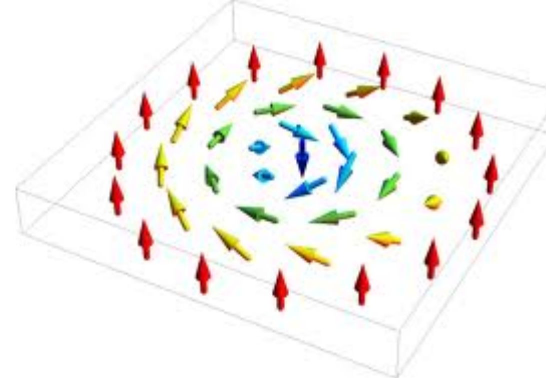
← SSB at 1+1 D is possible!

The lower bound of the space dimension *d* for $\omega = k^z$.
If *d* equals to or smaller than this value, the symmetry is restored.
(can be an algebraic (power law) long-range order)

mixed case is nontrival and have to be discussed case by case.

2. REVIEW OF TOPOLOGICAL EXCITATIONS

Skyrmions



- A *static* topologically nontrivial field configurations classified of the map $M \rightarrow G/H$
- When $M = S^n$, this is classified by $\pi_n(G/H)$.

e.g., $G/H = S^2$ in 2 dimensions: skyrmions in magnet
$G/H = S^3 = SU(2)$ in 3 dimensions: usual skyrmions

- When M is other manifold, we have to deal with it one by one.

e.g., $M = G/H = T^n$ (torus)

$\pi_2(G/H) = 0$ but the map $M \rightarrow G/H$ is characterized by two winding numbers N_1, N_2.

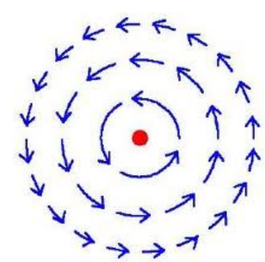
Topological defects

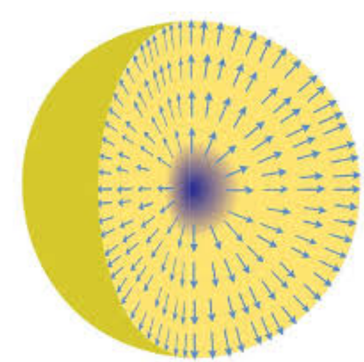
- Take a sphere S^n surrounding the defect and calculate the winding number $\pi_n(G/H)$ of the map $S^n \rightarrow G/H$.

e.g.,

magnetic monopoles in magnets ($G/H=S^2$)

vortices in superfluids ($G/H=S^1$)

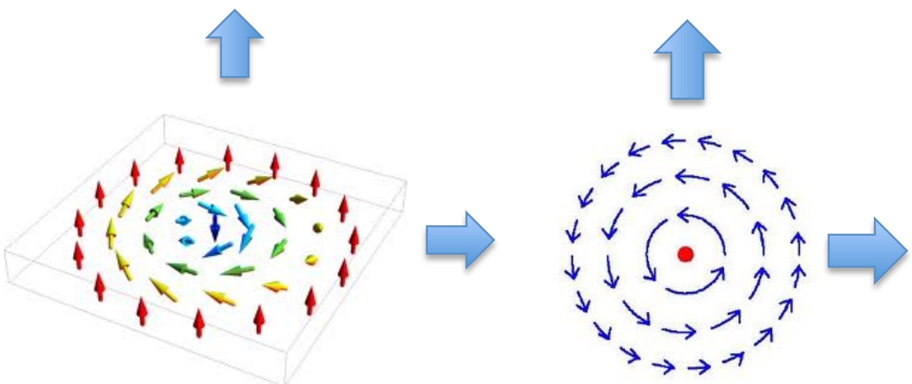




Translational moduli

for 2D skyrmions (dim(M)=2) and vortices ($\pi_1(G/H)$!=0)

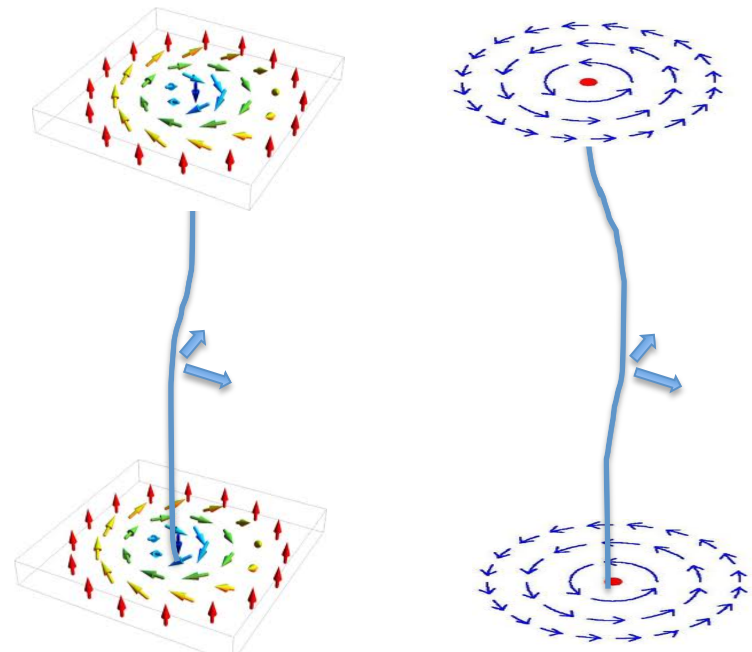
In 2+1 dimensions



translational zero modes

To describe the quantum mechanics of the translational zero modes, we introduce the position variables $u^x(t)$, $u^y(t)$

In 3+1 dimensions



Nambu-Goldstone excitation localized on the string. The wave vector is along the string direction.

$u^x(t,z)$, $u^y(t,z)$

Effective Lagrangian of the zero modes

Start from a static field configuration $\pi^a(x)$.
Replace by $\pi^a(x - u(t))$, put this into the Lagrangian,
and expand it to the quadratic order in u.

$$L = \frac{M}{2}(\dot{u}_x^2 + \dot{u}_y^2) + \frac{B}{2}(u_y \dot{u}_x - u_x \dot{u}_y)$$

- One can show that B is proportional to the winding number of the texture $\pi^a(x)$; e.g., $B = 2\pi n_0 N$ for a superfluid vortex, and $B = 4\pi s N$ for a skyrmion in magnets.
- The Lagrangian is identical to the single-particle quantum mechanics under the magnetic field B. The effective Lorentz force is the famous Magnus force.
- Invariant (up to a total derivative) under the sift: $u_a \rightarrow u_a + c_a$
- The momentum operators satisfy $[P_x, P_y] = i B$.

M. Stone
PRD (1996)

→ The same commutation relation must be seen at a more microscopic level

3. NONCOMMUTATIVITY OF THE MOMENTUM OPERATOR

Momentum operator and its commutation relation

- Noether theorem

$$T^{0i} = \frac{\partial \mathcal{L}}{\partial \dot{\pi}^a} \delta \pi^a = p_a \partial_i \pi^a$$

- Canonical commutation relation

$$[\pi^a(\vec{x}, t), p_b(\vec{x}', t)] = i \delta^a_b \delta^d(\vec{x} - \vec{x}')$$

- Generator of the transformation

$$[P^i, T^{0j}(x)] = -i \partial_i T^{0j}(x)$$

$$[P^i, P^j] = 0 \quad \text{if} \quad T^{0j}(x) \to 0 \quad \text{as} \quad |\vec{x}| \to \infty$$

For skyrmions

From more careful calculation

$$[P^i, T^{0j}(x)] = -i\partial_i T^{0j}(x) + \underline{ip_a(x)(\partial_i\partial_j - \partial_j\partial_i)\pi^a(x)}$$

Therefore,

$$[P^i, P^j] = -i\int \mathrm{d}^d x\,[\partial_i p_a(x)\partial_j\pi^a(x) - \partial_j p_a(x)\partial_i\pi^a(x)]$$

For the Lagrangian $\mathcal{L}^{(\mathrm{Berry})} + \mathcal{L}^{(\mathrm{metric})}$,

$$\mathcal{L}^{(\mathrm{metric})} = \frac{1}{2}\bar{g}_{ab}(\pi)\dot{\pi}^a\dot{\pi}^b - \frac{1}{2}g_{ab}(\pi)\nabla\pi^a\cdot\nabla\pi^b$$

$$\mathcal{L}^{(\mathrm{Berry})} = c_a(\pi)\dot{\pi}^a$$

and for a static field configurations,

(= 0 when $H^2(G/H)$ is trivial)

$$[P^i, P^j] = -i\int \mathrm{d}^d x\frac{1}{2}\left[\frac{\partial c_b(\pi)}{\partial\pi^a} - \frac{\partial c_a(\pi)}{\partial\pi^b}\right](\partial_i\pi^a\partial_j\pi^b - \partial_i\pi^b\partial_j\pi^a)$$

(when $H^2(G/H)$ is nontrivial)

$$= -i\epsilon_{ij}\int \pi^*\omega = -iN\epsilon_{ij}\int_{C_2}\omega \quad (\text{for } d=2;\ \omega\equiv \mathrm{d}[c_b(\pi)\mathrm{d}\pi^b])$$

two-cycle C_2: 2D manifold (in G/H) without boundary

Simple example: skyrmions in ferromagnet

$G/H = SO(3) / SO(2) = S^2$; S^2 itself the nontrivial two-cycle C_2

$$\mathcal{L} = s\frac{n_y \dot{n}_x - n_x \dot{n}_y}{1+n_z} + \frac{\bar{g}_0}{2}\dot{\vec{n}}^2 - \frac{g_0}{2}(\nabla\vec{n})^2$$

spin density $s = S/a^d$

$$[P^x, P^y] = 4\pi i s \int d^2x \, \frac{1}{4\pi}\vec{n}\cdot\partial_x\vec{n}\times\partial_y\vec{n}$$
$$= 4\pi i s \int n^*\omega = 4\pi i s N \int_{S^2} \omega$$

$$\omega = \frac{1}{4\pi}\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi$$

Other examples with nontrivial $H^2(G/H)$:

- $G/H = CP^n$ (i.e., the CP^n model)

(n = 1 case is identical to the above magnet example)

- $G/H = T^2$ [$\pi_2(G/H) = H^2(G/H)$ when $\pi_1(G/H)=\pi_0(G/H)=0$]
- G/T where G is semisimple and T is the maximal torus ($U(1)$s) of G

Vortices in superfluid

$$[P^i, T^{0j}(x)] = -i\partial_i T^{0j}(x) + \underline{ip_a(x)(\partial_i\partial_j - \partial_j\partial_i)\pi^a(x)}$$

$$\pi(x) = \theta(x) \quad , \qquad p(x) = n(x) - n_0$$

$$(\partial_x\partial_y - \partial_y\partial_x)\theta(\vec{x}) = 2\pi N\delta^2(\vec{x})$$

$$\lim_{|\vec{x}|\to\infty} n(x) = n_0$$

$$n(\vec{x} = 0) = 0$$

(expected result in the dual picture)

$$\Rightarrow \quad [P^x, P^y] = -2\pi i n_0 N$$

Thousless et at, PRL (1996)

For a relativistic superfluid $n_0 = 0$ and thus they commute.

This calculation can be easily generalized for a general $\pi_1(G/H) = Z^m$.
The condition for a nonzero commutator is $p_a(\vec{x} = 0) \neq \lim_{|\vec{x}|\to\infty} p_a(\vec{x})$

Quantization of the central extension

$$L = \frac{M}{2}(\dot{u}_x^2 + \dot{u}_y^2) + \frac{B}{2}(u_y \dot{u}_x - u_x \dot{u}_y)$$

B x (Area) is quantized to an integer multiple of 2*π.*
Indeed,

total # of particles

B x (Area) = 2π*N* n_0 x (Area) for a superfluid vortex.

B x (Area) = 2π*N* 2s x (Area) for a skyrmion in magnets.

total spins. s = S/a^2

For skyrmions in general, the quantization follows from the requirement on the Wess-Zumino term.

Summary

- We have shown that the momentum operator may not commute in the presence of skyrmions ($H^2(G/H)!=0$) and vortices ($\pi_1(G/H) = Z^m$)
- The quantum mechanics of the zero modes is

$$L = \frac{M}{2}(\dot{u}_x^2 + \dot{u}_y^2) + \frac{B}{2}(u_y\dot{u}_x - u_x\dot{u}_y)$$

where $[P^x, P^y]=i B$.

B x (Area) is quantized to an integer multiple of 2π.