TDiff in the Dark: Gravity with broken diffeomorphisms in the matter sector

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[D. Jaramillo-Garrido, A. L. Maroto, and P. Martín-Moruno. arXiv:2307.14861]

Outline

I. Introduction

Background and motivation

- Theory of General Relativity (Einstein, 1915).
- Precision Cosmology → current **accelerated expansion** of the Universe.
  - Within the framework of GR: cosmological constant, dark energy.
  - **Modification** of the theory on cosmological scales.
- Interest has grown in theories with a **broken Diff invariance**.
  - Unimodular gravity (UG) is the most popular.
  - We will consider the breaking of Diff invariance down to TDiff in the **matter sector**.

[R. Carballo-Rubio, L.J. Garay, and G. García-Moreno. arXiv:2207.08499]
[E. Álvarez, D. Blas, J. Garriga, and E. Verdaguer. arXiv:hep-th/0606019]

Transverse Diffeomorphisms (TDiffs)

- TDiffs: $x \to \hat{x}(x)$ / $J = \det\left(\frac{\partial \hat{x}}{\partial x}\right) = 1$

- Infinitesimally: $x^\mu \to \hat{x}^\mu = x^\mu + \xi^\mu(x)$ / $\partial_\mu \xi^\mu = 0$ ("transverse")

- The determinant $g = \det(g_{\mu\nu})$ is a **TDiff scalar**, since $\hat{g}(\hat{x}) = J^2\, g(x)$

- New possibilities for **couplings**! $d^4x \sqrt{|g|} \longrightarrow d^4x\, f(g)$

Main ideas of the work

- We consider a scalar field model which couples to gravity via **arbitrary functions of the determinant** → the symmetry is TDiff.
- Conservation of the Energy-Momentum Tensor is automatic in Diff theories, but not in TDiff → it imposes new **constraints** on the metric.
- Main aim: to perform a **general study** (without assuming a particular geometry) which allows us to gain intuition on posible **new phenomenology** within this framework.

II. Preliminary concepts

TDiff scalar field model

$$\hbar = c = 1\,, \quad (+,-,-,-)$$

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x\,\sqrt{|g|}R$$

$$S = S_{EH} + S_m$$

$$S_m = \int d^4x\,\tilde{\mathcal{L}}_m = \int d^4x\,\left\{\underbrace{\frac{f_k(g)}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi}_{\text{kinetic}} - \underbrace{f_v(g)V(\psi)}_{\text{potential}}\right\}$$

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}}\left\{\frac{f_k(g)}{2}\partial_\mu\psi\partial_\nu\psi + g\left[f_v'(g)V(\psi) - \frac{1}{2}f_k'(g)\partial_\alpha\psi\partial^\alpha\psi\right]g_{\mu\nu}\right\}$$

[E. Álvarez, A. F. Faedo, and J. J. López-Villarejo. arXiv:0904.3298] [A. L . Maroto. arXiv:2301.05713]

Equations of Motion

- Einstein equations for the gravitational field:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- Euler-Lagrange EoM for the scalar field ψ :

$$\partial_\mu \left(f_k(g) \partial^\mu \psi \right) + f_v(g) V'(\psi) = 0$$

[A. L . Maroto. arXiv:2301.05713]

Potential domination

- In this regime we neglect the kinetic term:

Action: $$S_m \simeq \int d^4x \, [-f_v(g)\, V(\psi)]$$

EoM: $$V' = 0 \implies V(\psi) = V(\psi_0) \equiv V_0 = \text{const.}$$

EMT: $$T_{\mu\nu} = \frac{2g}{\sqrt{|g|}} \, f_v' V \, g_{\mu\nu}$$

- Interest: dark energy models, slowly-varying fields (slow-roll inflation)…

Kinetic domination

- In this regime we neglect the potential term:

Action: $$S_m \simeq \int d^4x \, \frac{f_k(g)}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi$$

EoM: $$\partial_\mu \left(f_k \partial^\mu \psi \right) = 0$$

EMT: $$T_{\mu\nu} = \frac{1}{\sqrt{|g|}} \left[f_k \partial_\mu \psi \partial_\nu \psi - g f_k' (\partial \psi)^2 g_{\mu\nu} \right]$$

- Interest: dynamical dark energy models, rapidly-varying fields (fast-roll)…

III. The perfect fluid approach

Model $\Leftrightarrow$ perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}$$

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}}\left\{\frac{f_k(g)}{2}\partial_\mu\psi\partial_\nu\psi + g\left[f_v'(g)V(\psi) - \frac{1}{2}f_k'(g)\partial_\alpha\psi\partial^\alpha\psi\right]g_{\mu\nu}\right\}$$

- **Assumption:** $\partial_\mu\psi$ is a timelike vector.
- Our model may be equivalently described as a perfect fluid with:

$$u^\mu \equiv \frac{\partial^\mu\psi}{\sqrt{(\partial\psi)^2}}$$

$$p = \frac{-2g}{\sqrt{|g|}}\left[f_v'V - \frac{1}{2}f_k'(\partial\psi)^2\right]$$

$$\rho = \frac{2}{\sqrt{|g|}}\left\{\frac{1}{2}f_k(\partial\psi)^2 + g\left[f_v'V - \frac{1}{2}f_k'(\partial\psi)^2\right]\right\}$$

Potential domination in the perfect fluid

- In the potential domination limit:

$$T_{\mu\nu} = \frac{2g}{\sqrt{|g|}} f_v' V g_{\mu\nu} \qquad \longrightarrow \qquad p = -\rho = \frac{-2g}{\sqrt{|g|}} f_v' V$$

- Simple Equation of State (EoS):

$$p = w\rho \,, \quad \boxed{w = -1}$$

Kinetic domination in the perfect fluid

- In the kinetic limit: $$\rho = \frac{(\partial\psi)^2}{\sqrt{|g|}}\left(f_k - g f_k'\right) \,, \quad p = \frac{(\partial\psi)^2}{\sqrt{|g|}} g f_k'$$

- EoS: $$\boxed{w = \frac{p}{\rho} = \frac{g f_k'}{f_k - g f_k'} \equiv \frac{F}{1-F} \,,} \qquad \boxed{F \equiv \frac{g f_k'}{f_k}}$$

- In GR we have stiff matter:

$$f_k \propto \sqrt{|g|} \Leftrightarrow F = 1/2 \Leftrightarrow w = 1 \quad \longrightarrow \quad \boxed{p = \rho}$$

IV. EMT conservation

EMT conservation in a TDiff theory

- In GR, Diff invariance $\Longrightarrow$ EMT conservation on the solutions to the EoM.
- TDiff: less symmetry, additional constraint on allowed transformations.
- We still use S_{EH} and the Einstein field equations.
- EMT conservation is a consequence of **Bianchi identities**, since $\nabla_\mu G^{\mu\nu} = 0$
- Solutions fulfil $\boxed{\nabla_\mu T^{\mu\nu} = 0}$ → **condition on the metric!**

Gauge degrees of freedom

SYMMETRY	TRANSFORMATIONS	GAUGE D.O.F.
Diff	ξ^μ	4
TDiff	ξ^μ s.t. $\partial_\mu \xi^\mu = 0$	4 – 1 = 3

- We can fix 3 metric components in a TDiff theory, the 4th one is **physical**.
- The additional constraint to find it comes from EMT conservation.

Review of EMT conservation

- EMT conservation: $$\nabla_\mu T^{\mu\nu} = 0$$

u^μ (for a perfect fluid) $h^\mu{}_\nu = \delta^\mu_\nu - u^\mu u_\nu$

$$\dot{\rho} + (\rho + p)\nabla_\mu u^\mu = 0$$

$$(\rho + p)\dot{u}^\mu - (g^{\mu\nu} - u^\mu u^\nu)\nabla_\nu p = 0$$

Potential domination: EMT conservation

- The **EMT conservation** gives:

$$\left(\frac{1}{2}f_v' + g f_v''\right)\partial_\mu g = 0$$

→ $$f_v(g) = A\sqrt{|g|} + B$$ (GR + const.)

→ $$\partial_\mu g = 0 \implies g = \text{const.}$$ (our **constraint**!)

- Recall the energy density and pressure in the potential limit:

$$p = -\rho = \frac{-2g}{\sqrt{|g|}} f_v' V = \text{const.}$$ → **cosmological constant**

Kinetic domination: EoM

- Recall the kinetic EoM: $\partial_\mu \left(f_k \partial^\mu \psi \right) = 0$
- Their **solutions** satisfy:

$$(\partial\psi)^2 = \frac{C_\psi(x)}{\left(\frac{\delta V\, f_k}{\sqrt{|g|}}\right)^2}$$

$$\dot{C}_\psi(x) = u^\mu \partial_\mu C_\psi(x) = 0$$

$$\nabla_\mu u^\mu = u^\mu \partial_\mu (\ln \delta V)$$

Kinetic domination: metric constraint

- **Kinetic EoM + longitudinal EMT conservation**.
- Assuming $\rho \neq 0$, we obtain the following relation:

$$(2F - 1)\frac{g}{f_k} = C_g(x)\delta V^2$$

$$F \equiv \frac{g f_k'}{f_k}$$

$$\dot{C}_g(x) = u^\mu \partial_\mu C_g(x) = 0$$

↓

This is the sought-for **constraint** on the metric!

Transverse results & a simplifying relation

- **Constraint + transverse EMT conservation** → $\vec{\nabla}\left(C_g C_\psi\right) = 0$
- The functions C_g and C_ψ are inversely proportional:

$$\left.\begin{aligned} \dot{C}_\psi = \dot{C}_g = 0 \\ \vec{\nabla}\left(C_g C_\psi\right) = 0 \end{aligned}\right\} \Longrightarrow \boxed{C_g C_\psi = \text{const.} \equiv c_\rho}$$

- The constraint depends on the solutions to the EoM through the relation between C_g and C_ψ, but **we only have one independent constraint.**

Energy density & pressure in a TDiff theory

- We may find a simple expression for ρ in the **kinetic domination** regime of a TDiff theory:

$$\rho = \frac{c_\rho}{(1-w)\sqrt{|g|}}$$

(valid for all geometries!)

- **Not valid for Diff** $(w = 1)$.
- Since $w = w(g)$, the energy density and the pressure only depend on the metric determinant $\rightarrow$ **adiabatic perturbations**!

Speed of sound (adiabatic perturbations)

- In the kinetic regime, the perturbations of the fluid are **adiabatic**.
- The **speed of sound** of these adiabatic perturbations is:

$$\delta p = c_s^2\, \delta\rho \longrightarrow \boxed{c_s^2 = w + w'\,\frac{\rho}{\rho'}}$$

$$c_s^2 = -\frac{g f_k (f_k' + 2 g f_k'')}{f_k^2 + (2 g f_k')^2 - g f_k (5 f_k' + 2 g f_k'')}$$

V. Dark sector applications

TDiff Dark Matter

- In a **power-law** model:

$$f_k(g) = C|g|^\alpha \longrightarrow w = \frac{\alpha}{1-\alpha} = \text{const.}$$

(arbitrary!)

$$c_s^2 = w$$

- ECs:

$$\begin{aligned} \text{NEC:} \quad & C > 0\,, \\ \text{WEC:} \quad & C > 0\,, \quad \alpha \leq 1\,, \\ \text{SEC:} \quad & C > 0\,, \quad \alpha \geq -1/2\,, \\ \text{DEC:} \quad & C > 0\,, \quad \alpha \leq 1/2\,. \end{aligned}$$

e.g.: $\alpha = 0$

↓

TDiff DM model

[A. L . Maroto. arXiv:2301.05713]

Gravitational domains

- In an **exponential** model:

$$f_k(g) = Ce^{\beta g} \longrightarrow w = \frac{\beta g}{1-\beta g}$$

- ECs:

$$\begin{aligned} \text{NEC:} &\quad C > 0\,, \\ \text{WEC:} &\quad C > 0\,, \quad \beta g \leq 1\,, \\ \text{SEC:} &\quad C > 0\,, \quad \beta g \geq -1/2\,, \\ \text{DEC:} &\quad C > 0\,, \quad \beta g \leq 1/2\,. \end{aligned}$$

e.g.: $\beta g < 0$

↓

TDiff DM/DE model

[A. L . Maroto. arXiv:2301.05713] [David Alonso-López, J. de Cruz, and A. L. Maroto. *In preparation.*]

VI. Conclusions

Summary of the results obtained

- Consequences of **breaking** the symmetry (Diff → TDiff) in the **matter sector**.
- (Minimally coupled) scalar field, described as a perfect fluid.
- Study in **general contexts**: we have not assumed a geometry.
- EMT conservation imposes an additional (physical) **constraint** on the metric.
- Simple expression for the energy density → useful in **perturbation theory**.
- Particular cases → possibilities for DM & DE models → **TDiff dark sector**.

Thank you for your attention!

(backup)

Energy-Momentum Tensor (EMT)

- EMT definition: $$T^{\mu\nu} = \frac{-2}{\sqrt{|g|}}\frac{\delta S_m}{\delta g_{\mu\nu}}\,, \quad T_{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S_m}{\delta g^{\mu\nu}}$$

- For a perfect fluid: $$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}$$

Transverse vectors

- Triplet of transverse vectors:

$$\vec{w}^{\mu} \equiv (w_1^{\mu}, w_2^{\mu}, w_3^{\mu}) \ , \quad \vec{w}^{\mu} u_{\mu} = 0$$

- Projection of the derivative onto the transverse directions:

$$\vec{\nabla} \equiv \vec{w}^{\mu} \nabla_{\mu} \equiv (w_1^{\mu} \nabla_{\mu} \ , \ w_2^{\mu} \nabla_{\mu} \ , \ w_3^{\mu} \nabla_{\mu})$$

Review of Energy Conditions (ECs)

- Most used ECs: null (NEC), weak (WEC), strong (SEC), dominant (DEC).

$$
\begin{aligned}
&\text{(i)} \quad \text{NEC:} \quad T_{\mu\nu}k^\mu k^\nu \geq 0\,,\\
&\text{(ii)} \quad \text{WEC:} \quad T_{\mu\nu}v^\mu v^\nu \geq 0\,,\\
&\text{(iii)} \quad \text{SEC:} \quad \left(T_{\mu\nu} - \tfrac{1}{2}T^\alpha{}_\alpha g_{\mu\nu}\right) v^\mu v^\nu \geq 0\,,\\
&\text{(iv)} \quad \text{DEC:} \quad \text{WEC}\ \&\ F^\mu \equiv -T^{\mu\nu}v_\nu \ \text{causal.}
\end{aligned}
$$

$\forall\ k^\mu,\ v^\mu$

- Chain of implications: $\text{DEC} \Rightarrow \text{WEC} \Rightarrow \text{NEC} \Leftarrow \text{SEC}$

Null Energy Condition

- We can obtain the **general** NEC (without assuming a perfect fluid):

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \left\{ \frac{f_k(g)}{2} \partial_\mu \psi \partial_\nu \psi + g \left[f_v'(g) V(\psi) - \frac{1}{2} f_k'(g) \partial_\alpha \psi \partial^\alpha \psi \right] g_{\mu\nu} \right\}$$

$$\text{NEC:} \quad T_{\mu\nu} k^\mu k^\nu \geq 0 \quad \forall \; k^\mu \; \text{null}$$

$$\Longrightarrow \boxed{f_k \geq 0}$$ (not a ghost field)

- The other ECs $\rightarrow$ **perfect fluid viewpoint + limiting regimes.**

Perfect fluid ECs

- For a perfect fluid, the ECs take the form:

$$
\begin{aligned}
\text{NEC:} \quad & \rho + p \geq 0\,, \\
\text{WEC:} \quad & \rho + p \geq 0\,, \quad \rho \geq 0\,, \\
\text{SEC:} \quad & \rho + p \geq 0\,, \quad \rho + 3p \geq 0\,, \\
\text{DEC:} \quad & \rho \geq |p| \geq 0\,.
\end{aligned}
$$

General ECs (perfect fluid)

- WEC: $$f_k \geq 0 \quad \& \quad \frac{(\partial\psi)^2}{2}\left(f_k - g f_k'\right) + g f_v' V \geq 0$$

- SEC: $$f_k \geq 0 \quad \& \quad \frac{(\partial\psi)^2}{2}\left(f_k + 2g f_k'\right) - 2g f_v' V \geq 0$$

- DEC: $$p \leq 0 \ : \qquad f_v' V - \tfrac{(\partial\psi)^2}{2} f_k' \leq 0 \quad \& \quad f_k \geq 0\,,$$

$$p > 0 \ : \qquad \begin{cases} f_v' V - \dfrac{(\partial\psi)^2}{2} f_k' > 0\,, \\ \dfrac{(\partial\psi)^2}{2}\left(f_k - 2g f_k'\right) + 2g f_v' V \geq 0\,. \end{cases}$$

ECs in potential domination

- EoS $w = -1$ saturates the NEC.
- The other ECs in the potential limit:

$$\text{WEC:} \quad f_v' V \leq 0\,, \quad \text{SEC:} \quad f_v' V \geq 0\,,$$

$$\text{DEC:} \quad \begin{cases} p \leq 0\,: & f_v' V \leq 0\,, \\ p > 0\,: & f_v' V > 0 \quad \& \quad f_v' V \leq 0 \quad \text{(contrad.)} \end{cases}$$

- The kinetic domination limit is more involved → see later in particular cases.

ECs in kinetic domination

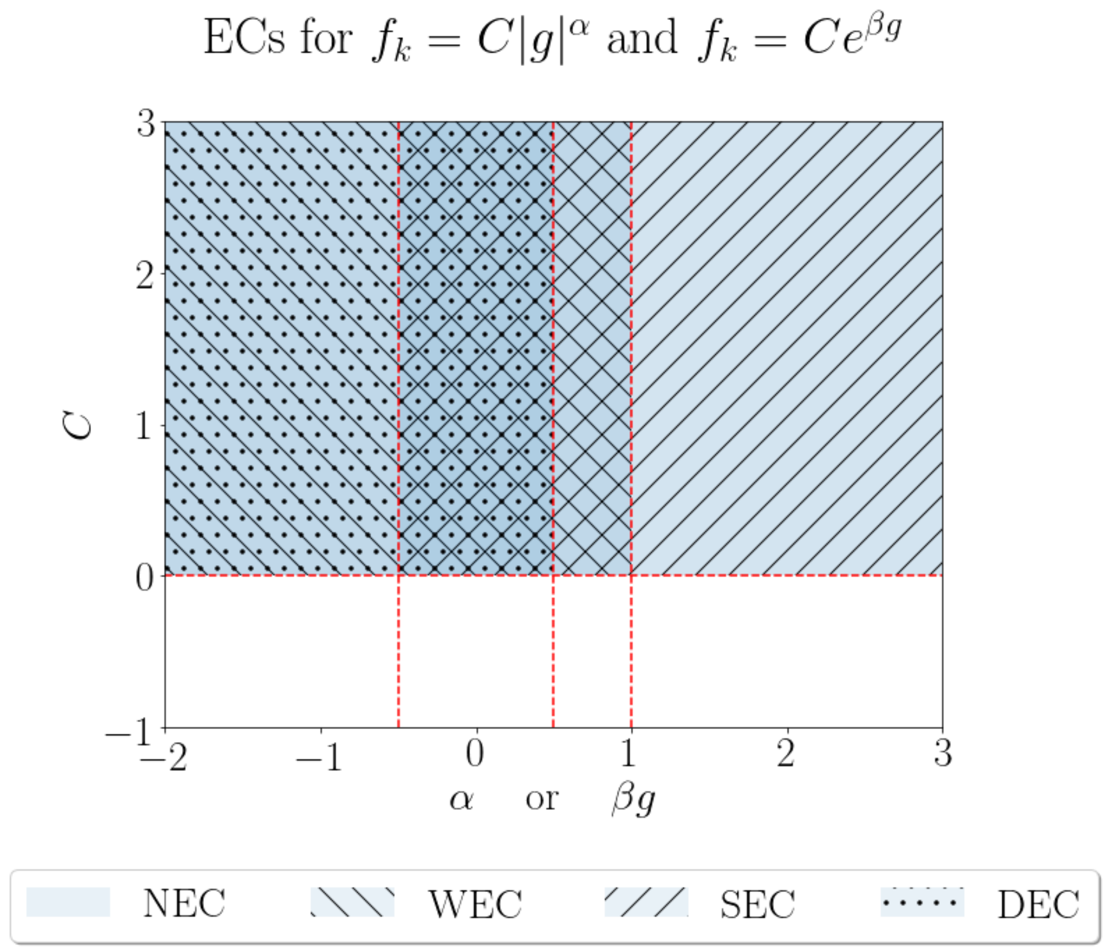
- We already know the NEC: $f_k \geq 0$ (focus on positive in kin. dom.)

- The other ECs:

$$\text{WEC:} \quad f_k \geq 0\,, \quad f_k - g f_k' \geq 0\,,$$

$$\text{SEC:} \quad f_k \geq 0\,, \quad f_k + 2g f_k' \geq 0\,,$$

$$\text{DEC:} \quad \begin{cases} p \leq 0 \;: \quad f_k' \geq 0 \quad \& \quad f_k \geq 0\,, \\ p > 0 \;: \quad f_k' < 0 \quad \& \quad f_k - 2g f_k' \geq 0\,. \end{cases}$$


ECs for $f_k = C|g|^\alpha$ and $f_k = Ce^{\beta g}$
3
2
1
0
−1
C
−2
−1
0
1
2
3
α or βg
NEC
WEC
SEC
DEC