Quantum Gravity of Open System

A New Genuine Multipartite Entanglement Measure:

from Qubits to Multiboundary Wormholes

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apctp asia pacific center for theoretical physics

The World is a Global Village and the Peoples of the World are One Human Family. May We Strive for Peace and Humanity with the Spirit of Global Cooperation Society.
KYUNG HEE UNIVERSITY

A New Genuine Multipartite Entanglement Measure: from Qubits to Multiboundary Wormholes

Today: 2411.11961

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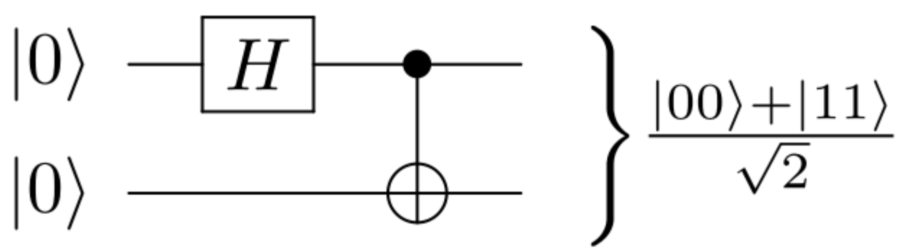
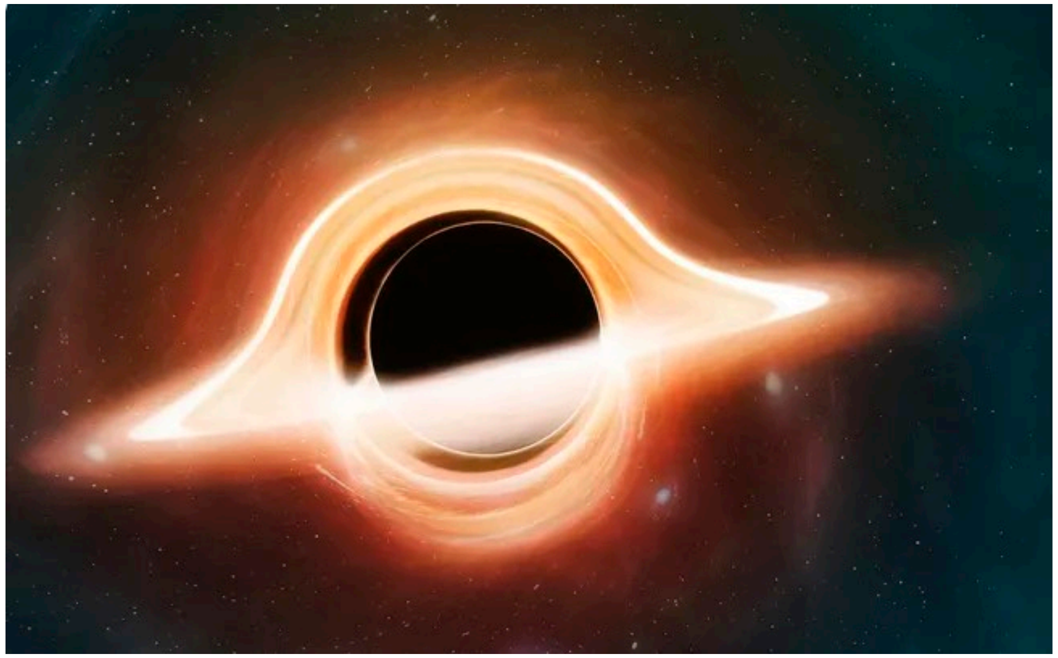
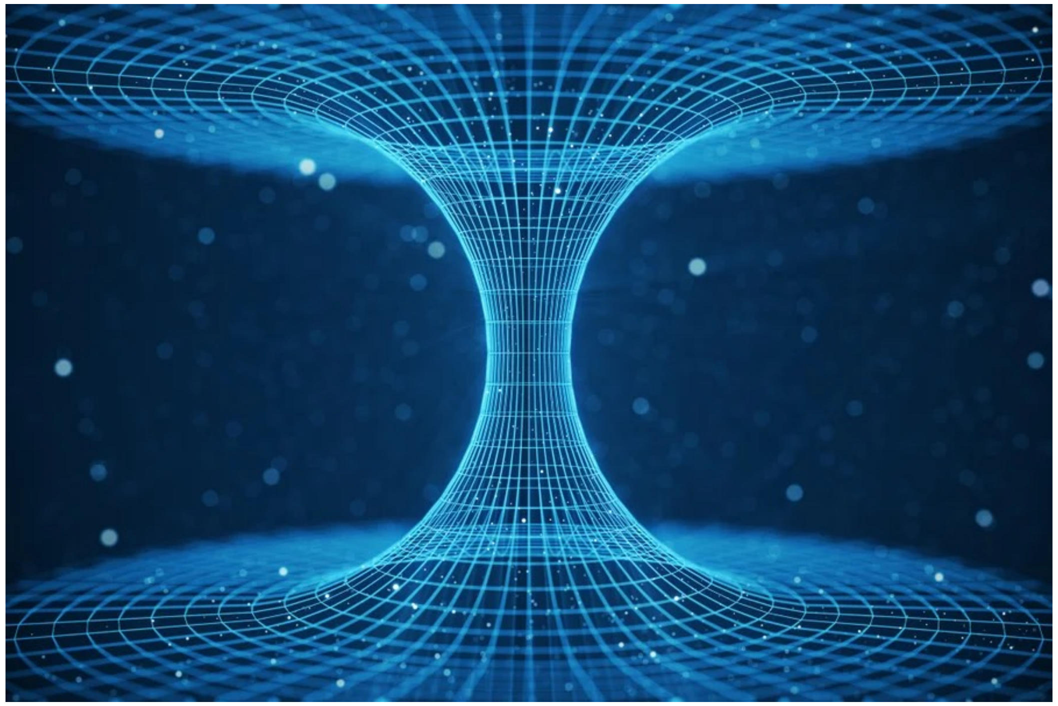


$$|\psi\rangle = \frac{1}{\sqrt{16}}\left(|0000000\rangle + |0001011\rangle + |0011001\rangle + |0110010\rangle + |0110100\rangle\right)$$

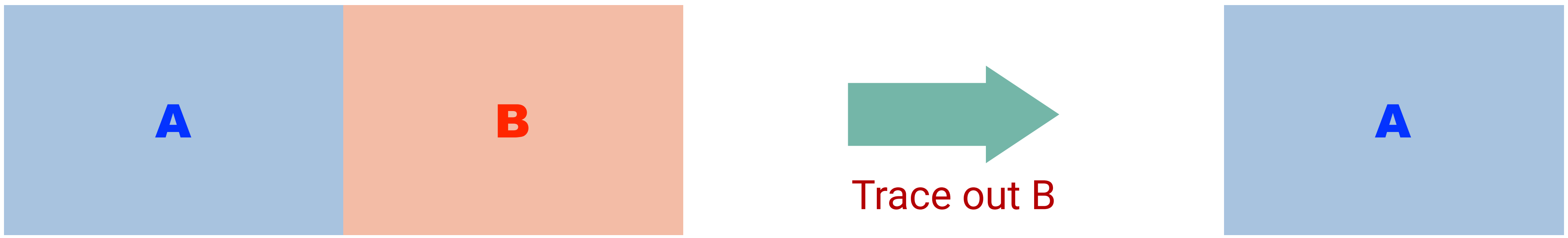
$$+\frac{1}{\sqrt{16}}\left(|1000110\rangle + |1010011\rangle + |1100001\rangle + |1101010\rangle + |1111111\rangle\right)$$

$$-\frac{1}{\sqrt{16}}\left(|0011110\rangle + |0100111\rangle + |0101101\rangle + |1001100\rangle + |1010101\rangle + |1111000\rangle\right)$$





$$|\mathrm{Bell}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$



$$|\text{Bell}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

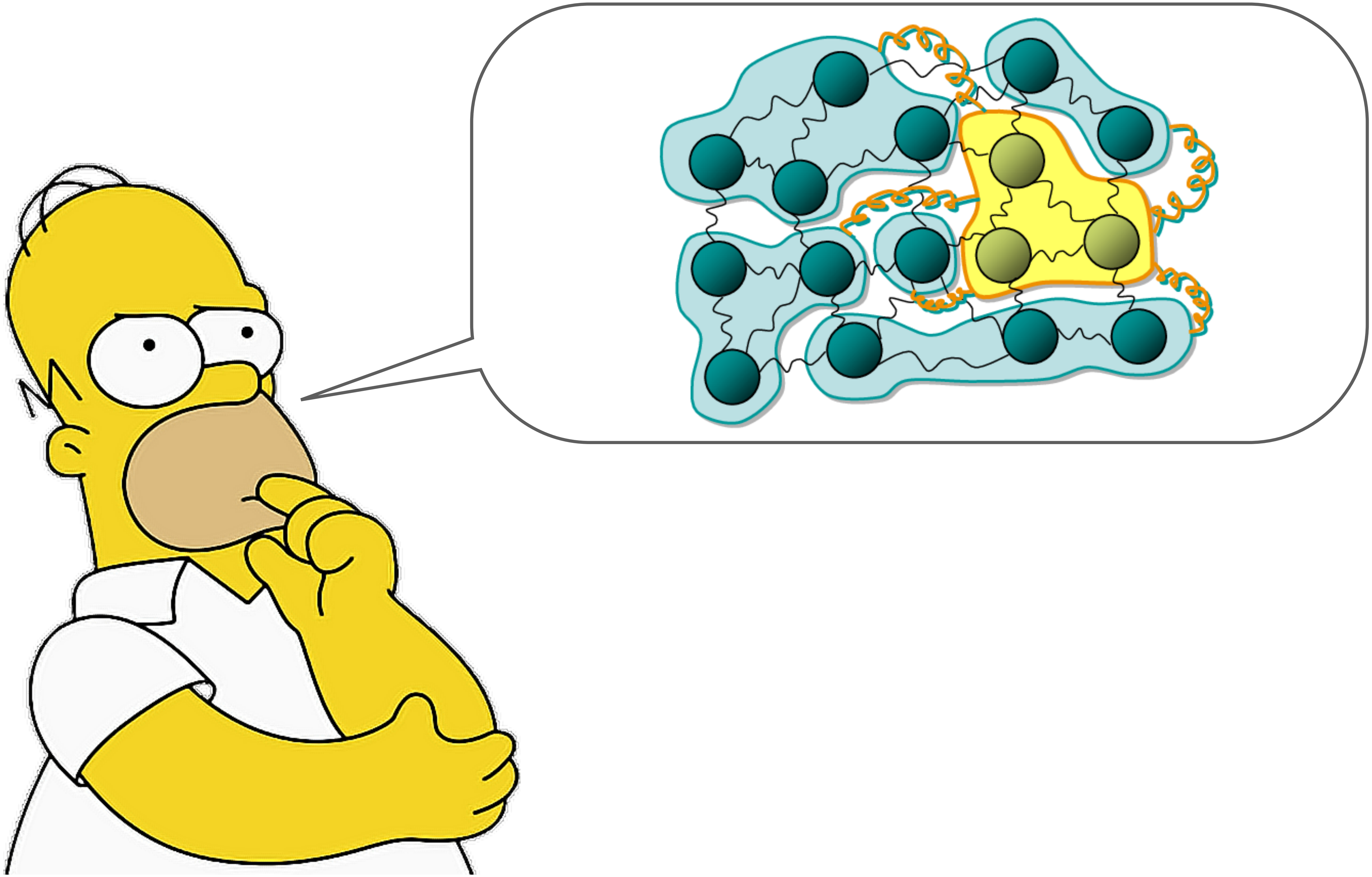
$\rho_A = \mathrm{Tr}_B(\rho)$ **maximally** mixed state

MAXimally mixed state saturates the entanglement entropy.

$$S_A = -\,\mathrm{Tr}(\rho_A \log \rho_A)$$

Good Measure
for bi-partite Entanglement

How about Multi-partite Entanglement?



Which state is more “special”?

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\big(|000\rangle + |111\rangle\big)$$

vs

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}\big(|100\rangle + |010\rangle + |001\rangle\big)$$

or, something else?

We need new MEASURE for Multi-partite Entanglement!!

Measure for Bi-partite Entanglment

$$S_A = -\mathrm{Tr}(\rho_A \log \rho_A)$$

→ **Measure for Multi-partite Entanglment**

Maximally Entangled State

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_n |n\rangle \otimes |n\rangle$$

→ **Maximally Multi-Entangled State**

We propose new MEASURE for

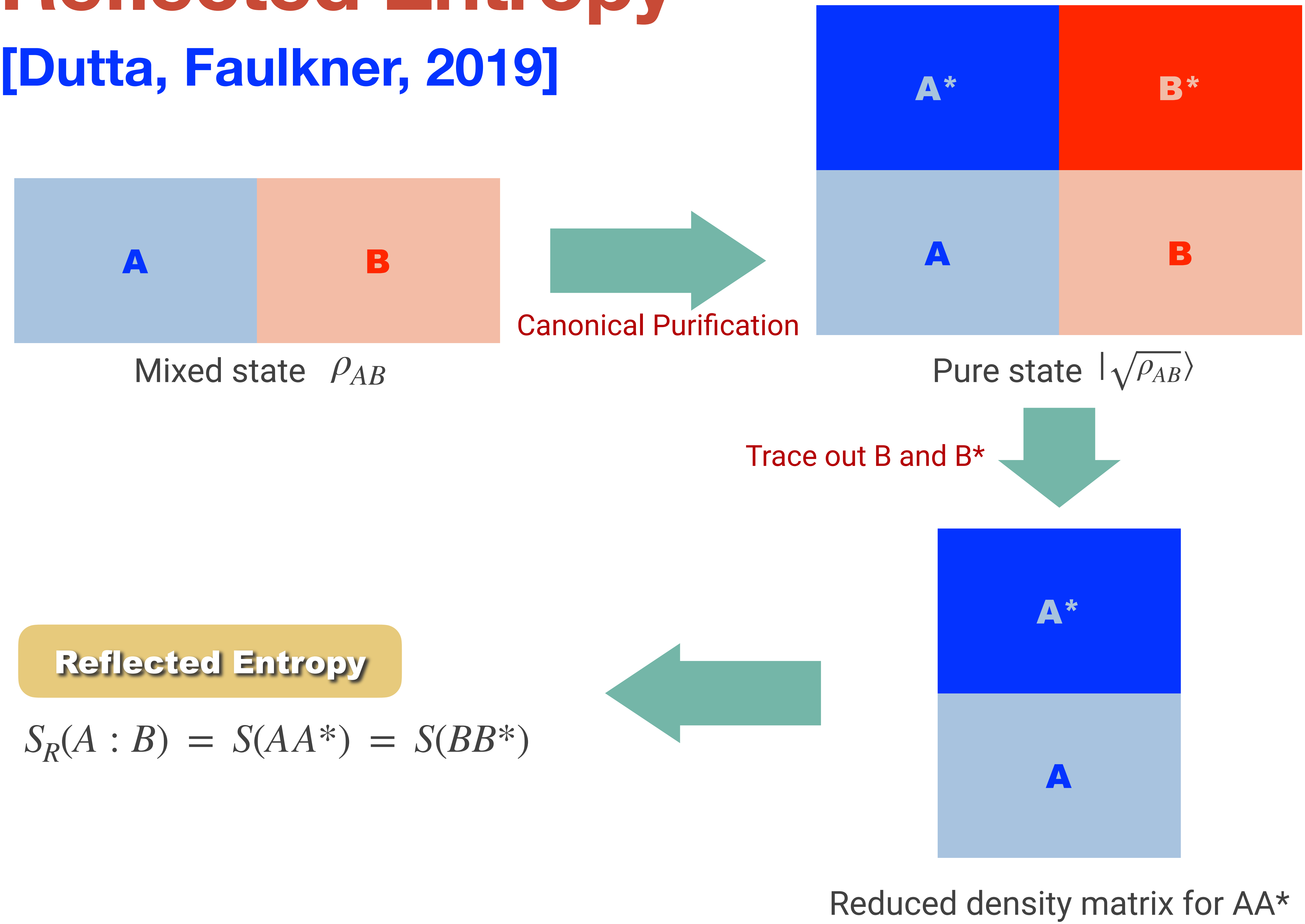
Multi-partite Entanglement.

"L-Entropy" of subsystem A and B

$$\ell_{AB} \equiv 2\min[S(A), S(B)] - S_R(A:B)$$

Reflected Entropy

[Dutta, Faulkner, 2019]



$$S_R(A:B) = S(AA^*) = S(BB^*)$$

Averaged L-Entropy

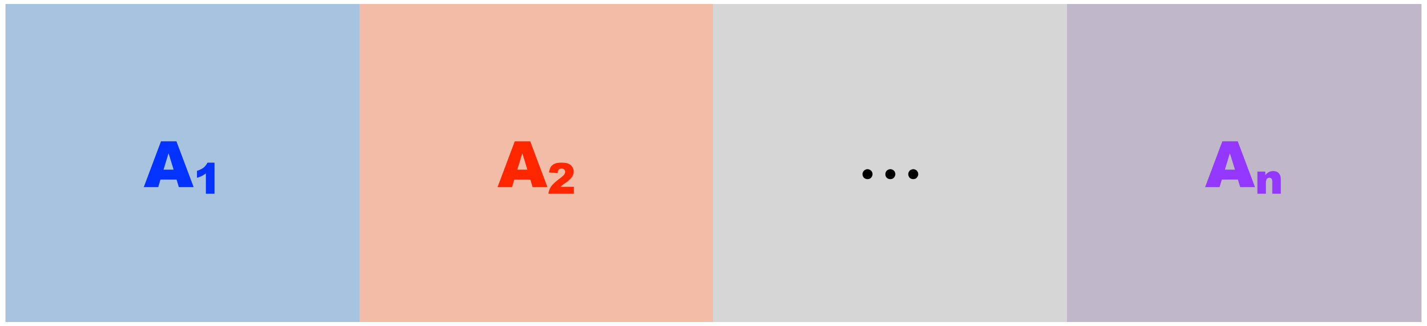
New Measure for Multi-partite Entanglement

✳ The bound for the reflected entropy

$$2\min[S(A), S(B)] \geqq S_R(A:B) \geqq I(A:B)$$

$$\Rightarrow \quad \ell_{AB} \equiv 2\min[S(A), S(B)] - S_R(A:B) \geqq 0$$

n-partite system



Averaged "L-Entropy"

$$\ell_{av} \equiv \prod_{i<j} \left[\ell_{A_iA_j}\right]^{\frac{2}{n(n-1)}}$$

Criteria for Multipartite Entanglement

Genuine Multipartite Entanglement Measure $\mathscr{E}$ (GME)

[Ma, Chen, Chen, Spengler, Gabriel, and Huber, 2011] [Xie, Eberly, 2021]

I. $\mathscr{E} = 0$ for fully-seperable or bi-seperable state

$|000\rangle$ $\quad$ $|\mathrm{Bell}\rangle \otimes |0\rangle$

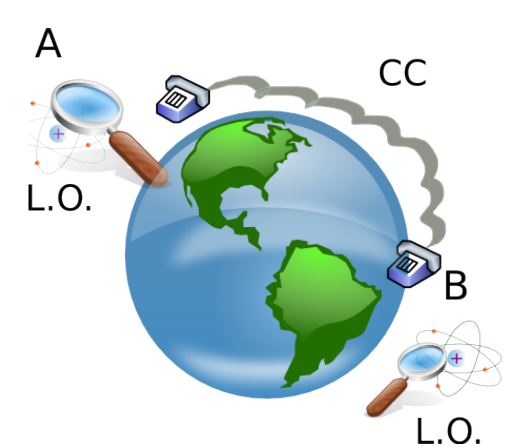
II. $\mathscr{E} > 0$ for non-biseperable state

III. $\mathscr{E}$: invariant under Local Unitary operation.

IV. $\mathscr{E}$: Non-increasing under LOCC [Entanglement Monotone]

Local Operations and Classical Communication

V. $\mathscr{E}(GHZ) > \mathscr{E}(W)$



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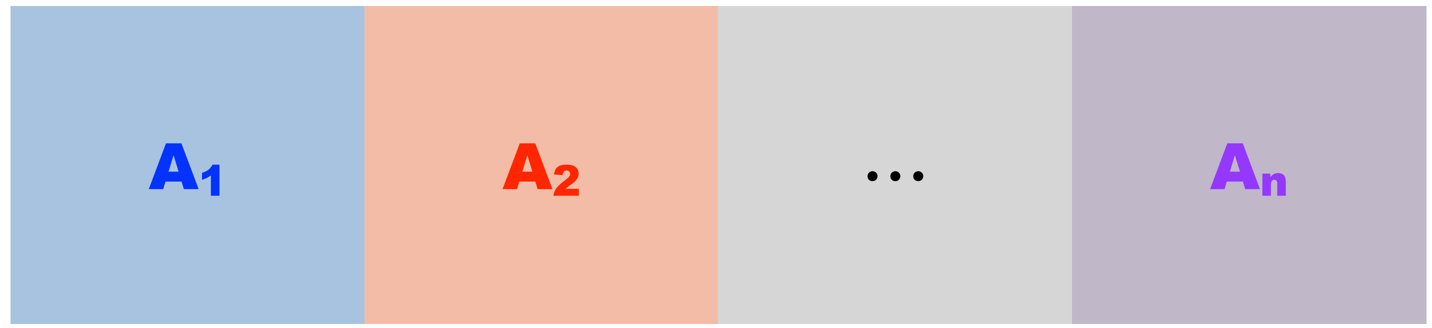
Local Operations and Classical Communication

Our averaged L-entropy satisfies this criteria.

Maximally Multi-entangled State

The Bound of L-entropy

n-partite system



$d_1 = d_2 = \cdots = d_n = d$

✳ In general , the L-entropy is bounded by $2\log[d]$

$$\ell_{av} \leq 2\log[d]$$

✳ Depending on n and d, the bound is not saturated.

For tri-partite system ($n=3$), the averaged L-entropy is bounded by $\log[d]$

which is the averaged L-entropy of (generalized) GHZ state

$$\ell_{av} \leqq \ell_{GHZ} = \log[d] < 2\log[d] \qquad |\psi\rangle_{GGHZ} = \frac{1}{\sqrt{d}}\sum_{j=1}^{d}|j_A j_B j_C\rangle$$

Which states saturate

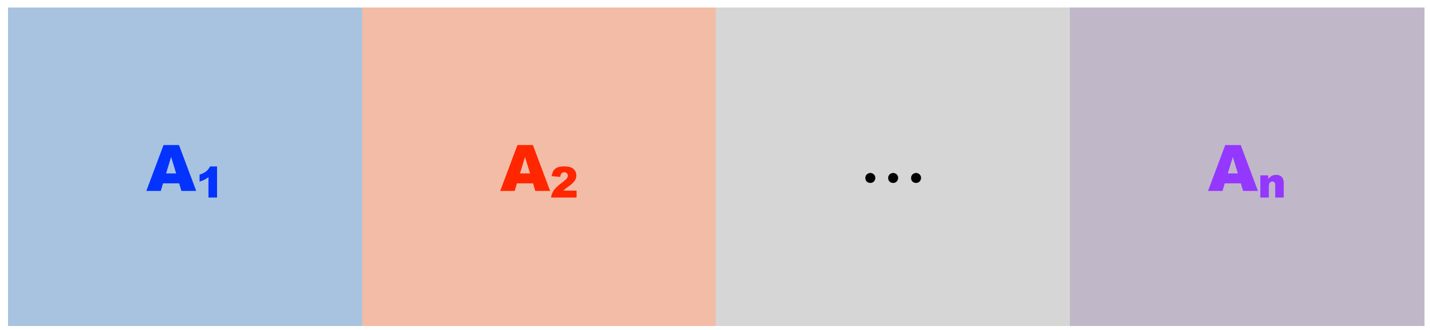
the bound of L-entropy?

$$\ell_{av} \leq 2\log[d]$$

k-Uniform State

Saturates the bound of the L-entropy

n-partite system



✳ ***k*-uniform state**: In n-partite system, the reduced density matrix of any k numbers of subsystems is maximally mixed.

$$\rho_{A_1 A_2 \cdots A_k} = \frac{1}{d^k} \mathbb{I}_{A_1 \cdots A_k} = \frac{1}{d^k} \mathbb{I}_{A_1} \otimes \cdots \otimes \mathbb{I}_{A_k}$$

: **Factorized**

✳ ***k*-uniform state** has maximum L-entropy $2 \log[d]$ $(k \geqq 2)$

$$\ell_{av}(k\text{-uniform}) = 2 \log[d]$$

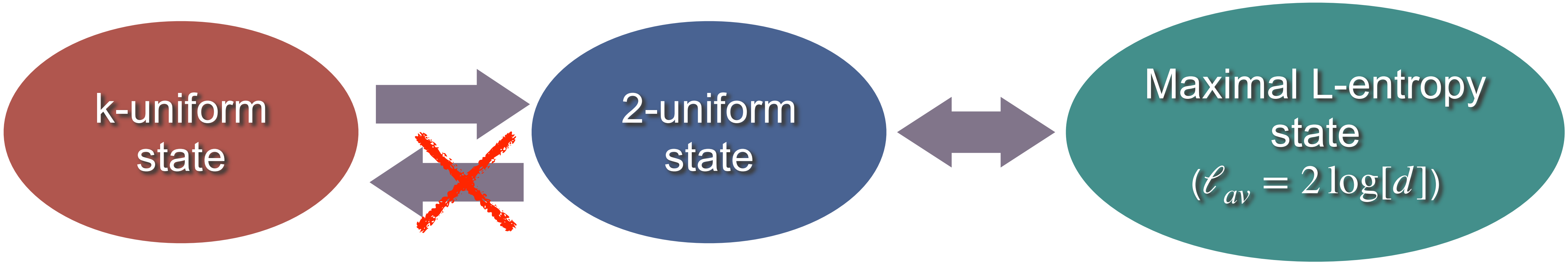
✳ In n-partite system, k-uniform state can exists only if $k \leqq \lfloor \frac{n}{2} \rfloor$ [necessary condition]

Ex) There is no k-uniform state $(k \geqq 2)$ in tri-partite system $(n = 3)$

$$\ell_{av} \leqq \ell_{GHZ} = \log[d] < 2 \log[d]$$

2-Uniform State

L-entropy can capture 2-uniform state



Optimization for 2-uniform State

L-entropy enables the optimization

- L-entropy is a concrete measure for the multi-partite entanglement entropy.
- One can optimize the L-entropy to obtain (approximated) 2-uniform state.

Algorithm

Choose a random state $|\psi_0\rangle$

↓

Compare L-entropy of the following states

$$|\psi_0\rangle - |\epsilon\rangle \qquad |\psi_0\rangle \qquad |\psi_0\rangle + |\epsilon\rangle$$

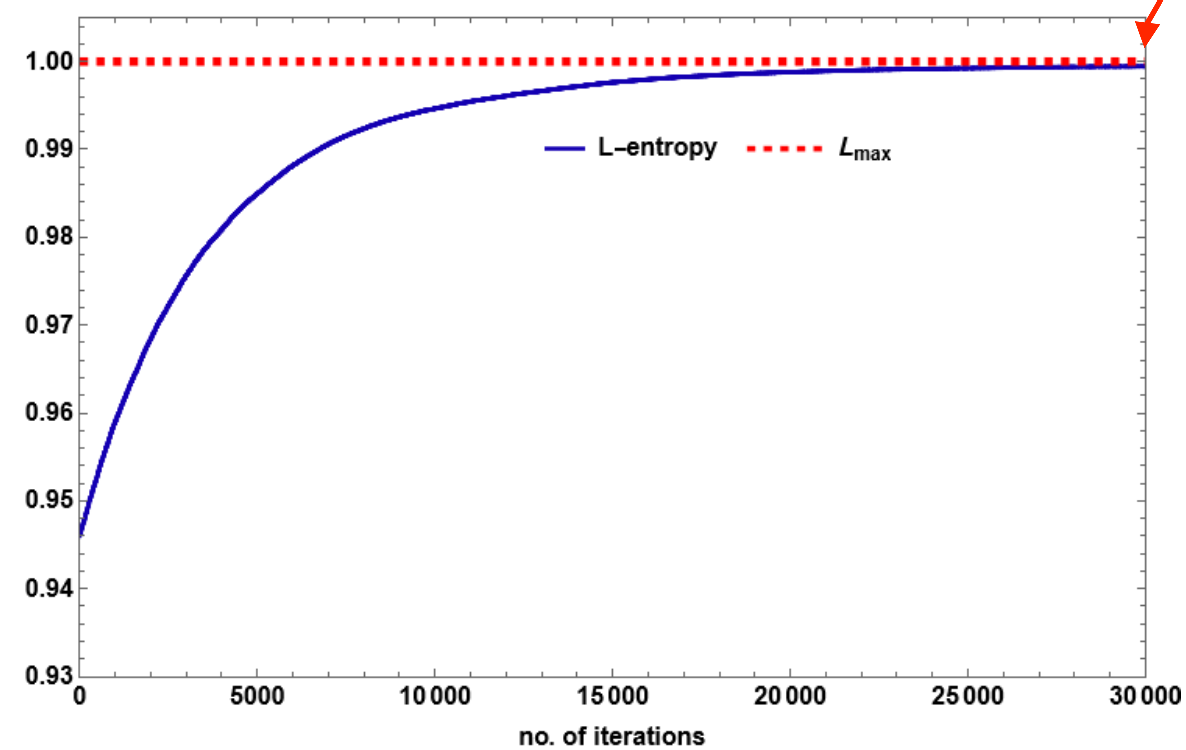
random state (perturbation)

↓

Choose the largest L-entropy state to define $|\psi_1\rangle$

L-entropy: 0.999447

→ approximated 2-uniform state



8 partie d=2

Optimization for 2-uniform State

L-entropy enables the optimization

- Sometimes, we can get the exact 2-uniform states from the optimization.

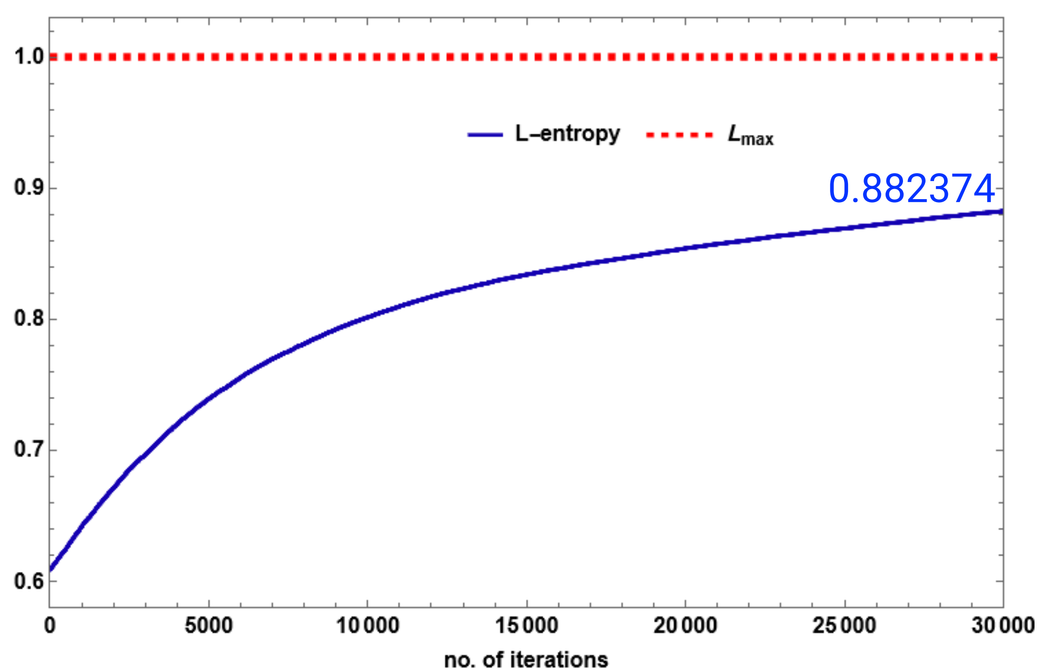
$$|\psi\rangle = \frac{1}{\sqrt{8}}(|00000\rangle + |01100\rangle + |10001\rangle + |11101\rangle - |00111\rangle - |01011\rangle - |11010\rangle - |11101\rangle)$$

$$|\psi\rangle = \frac{1}{\sqrt{8}}(|000100\rangle + |011000\rangle + |011111\rangle + |101110\rangle + |110010\rangle - |000011\rangle - |101001\rangle - |110101\rangle)$$

$$|\psi\rangle = \frac{1}{\sqrt{8}}(|0000011\rangle + |0010100\rangle + |0101110\rangle + |0111001\rangle + |1001101\rangle + |1011010\rangle + |1100000\rangle + |1110111\rangle)$$

$$|\psi\rangle = \frac{1}{\sqrt{9}}(|0121\rangle + |0202\rangle + |1022\rangle + |1100\rangle + |2001\rangle + |2112\rangle - |0010\rangle - |1211\rangle - |2220\rangle)$$

- The optimization does not always work.



4 parties d=6

$$k \leq \lfloor \frac{n}{2} \rfloor$$ [necessary condition for k-uniform state]

The case of 4-partite 2-uniform state ($k = 2$, $n = 4$) is unclear.

- ✓ $d = 2$: does not exist
- ✓ $d = 3,4,5$: exist
- ✓ $d = 6$: open question

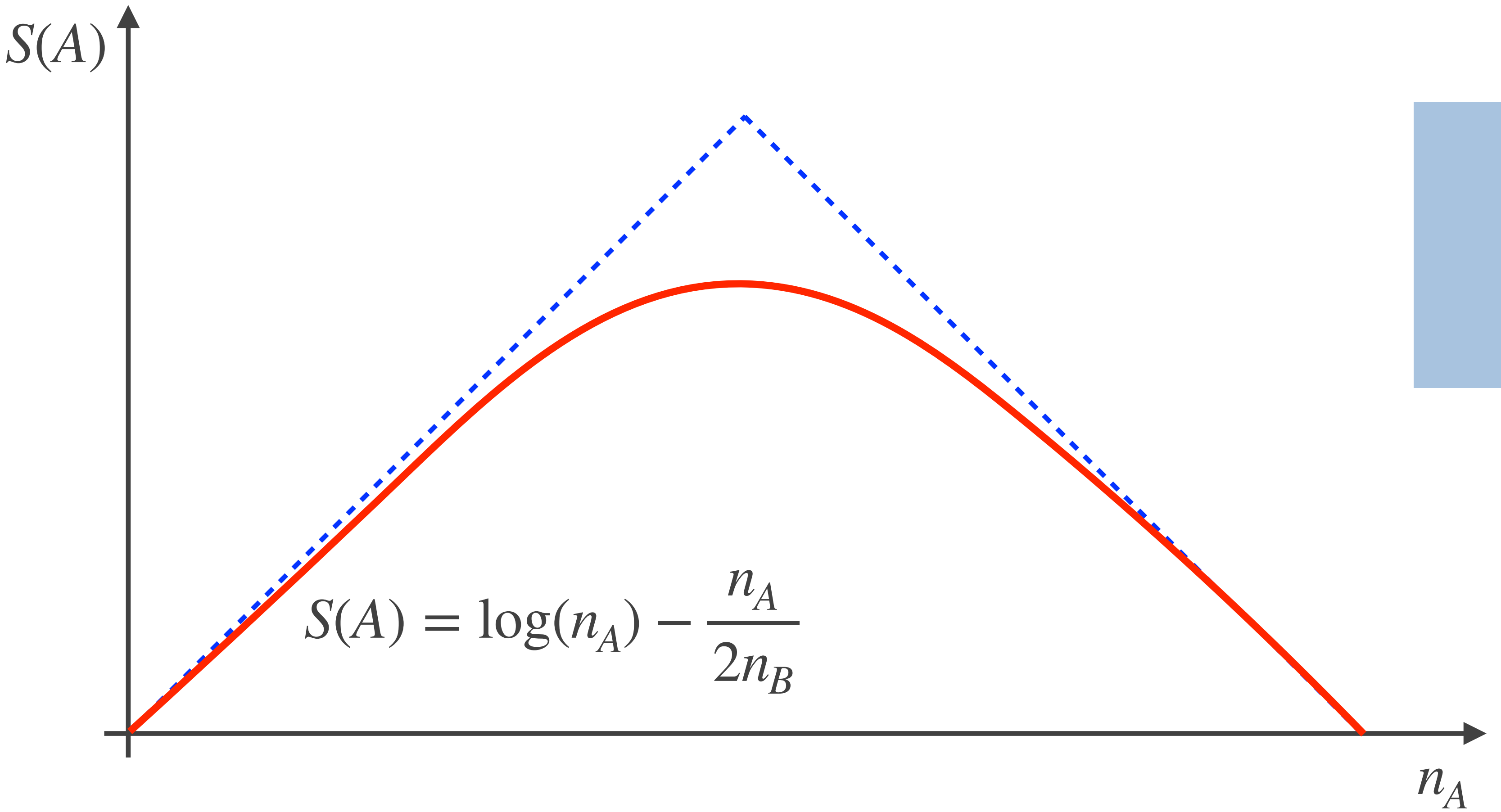
The 2-uniform state is

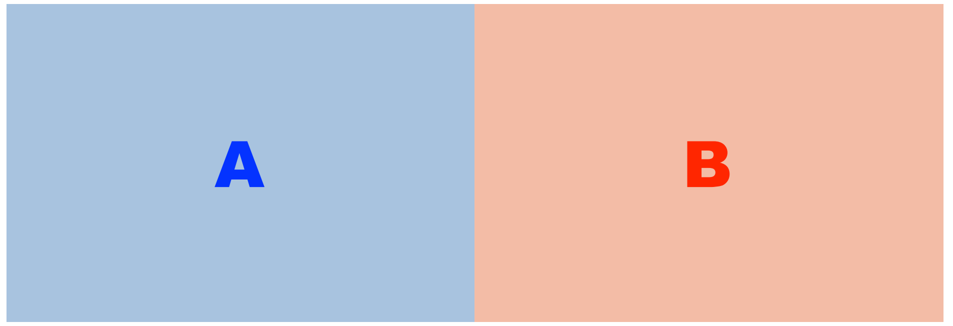
maximally multi-entangled

with respect to averaged L-entropy.

Page Curve

A typical state (random state) is maximally (bi-partite) entangled





Is a typical state

maximally multi-entangled?

Mostly, Yes.

But not always.

n-partite Random State ($n \geqq 5$)

Estimate the reflected entropy by resolvent technique

[Akers, Faulkner, Lin and Rath, 2021]

n-partite system

✴ Reflected Entropy of random state

$$S_R(A:B) = \frac{d^2 + 4d^2\log(d) - 2d^2\log\left(\frac{d^2}{4d_{\overline{AB}}}\right)}{8d_{\overline{AB}}} + O\left(\frac{1}{d_{\overline{AB}}^2}\right)$$

cf. Entanglement entropy of random state [Page, 1993]

$$S_A \approx \log[d_A] - \frac{d_A}{2d_{\overline{A}}}$$

n-partite Random State ($n \geqq 5$)

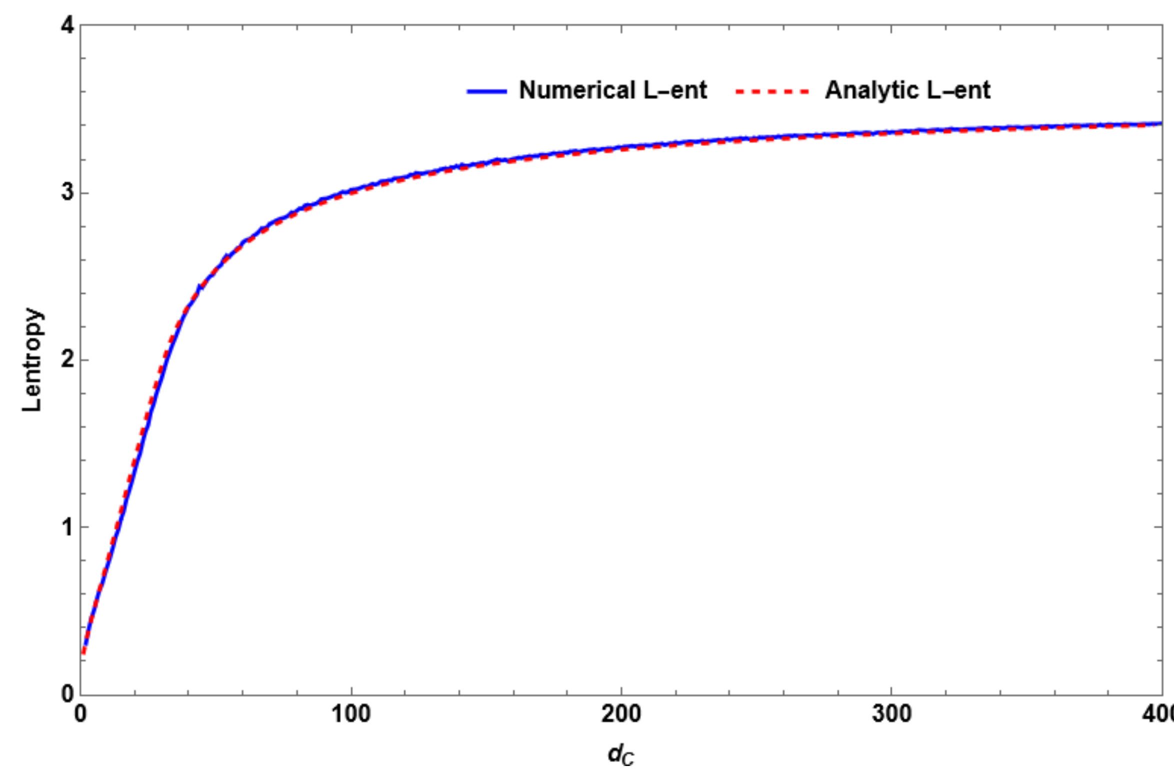
Estimate the reflected entropy by resolvent technique

n-partite system

$$\ell_{AB} \equiv 2\min[S(A), S(B)] - S_R(A:B)$$

$$= 2\log[d] - \frac{8 + d^2 + 4d^2\log(d) - 2d^2\log(\frac{d^2}{4d_{\overline{AB}}})}{8d_{\overline{AB}}}$$

$$+ O\left(\frac{1}{d_{\overline{AB}}^2}\right)$$

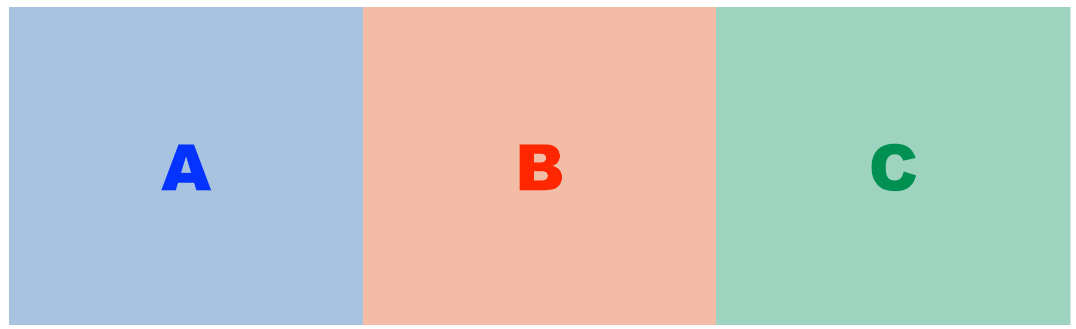


$d_A = d_B = 6\ ,\ \ d_{\overline{AB}} = d_C$

3-partite Random State

It is NOT 2-uniform state

3-partite system



$d_A = d_B = d_C = d$

✳ Mutual Information

$= S(C)$ ∵ Total system ABC is pure

$$I(A:B) = S(A) + S(B) - S(AB) \approx \log[d] \neq 0$$

➡ The subsystem A and B cannot be factorized.

✳ L-entorpy is smaller than the L-entropy of GHZ state

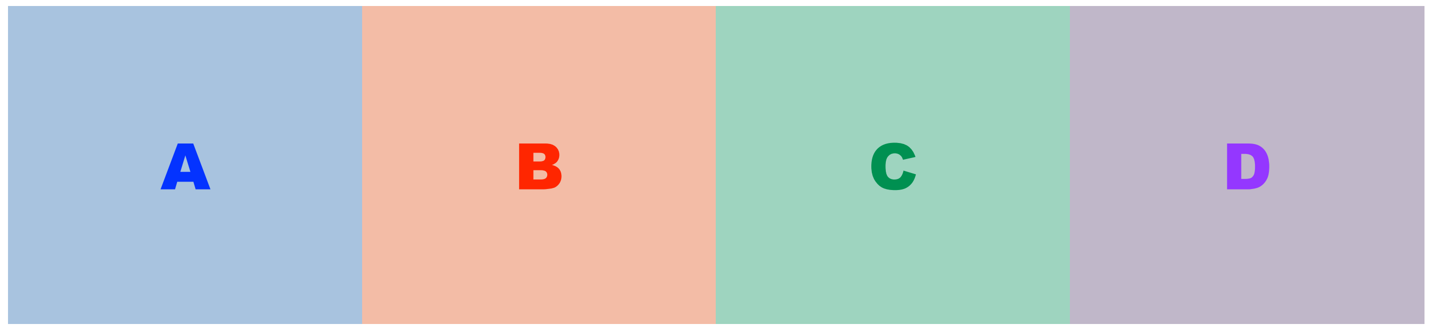
$$\ell_{AB} = \frac{1}{2} + \frac{2\log[d] - 5}{2d} + O(\frac{1}{d^2})$$: The leading contribution is independent of d

$$\ll \log[d]$$: L-entropy of GHZ state

4-partite Random State

NOT 2-uniform state

4-partite system



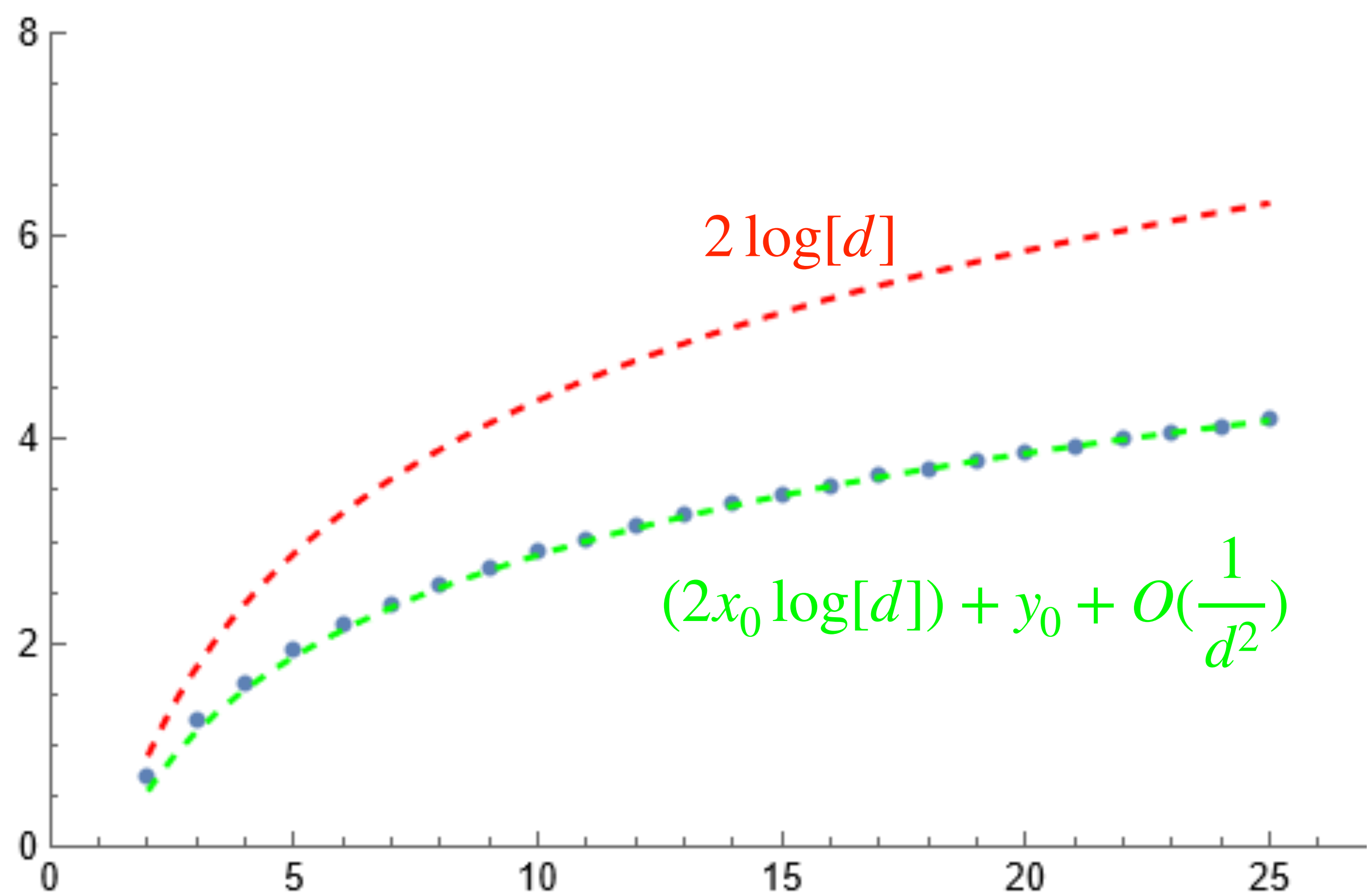
✳ L-entropy of 4-partite random state by resolvent technique

$$\ell_{AB} = (2x_0 \log[d]) + y_0 + O(\frac{1}{d^2})$$

$$x_0 \approx 0.720$$

$$y_0 \approx -0.453$$

: smaller than Maximum value



Maximally Mixed State

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_n |E_n\rangle \otimes |E_n\rangle$$

Introduce **Temperature**

Thermofield Double(TFD) State

(Canonical Purification of Thermal State)

$$|TFD(\beta)\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\frac{\beta}{2}E_n} |E_n\rangle \otimes |E_n\rangle$$

Black Hole in Gravity

What is Finite Temperature version of Multi-entangled State?

Thermal Pure Quantum (TPQ) State

Pure state reproducing Thermal Expectation Value

[Sugiura and Shimizu, 2013]

- ✳ In a given Hilbert space $\mathscr{H}$, we choose a random state $|\psi\rangle$.

  Then, we define the TPQ state $|\Psi_\beta\rangle$ by

$$|\Psi\rangle \equiv e^{-\frac{\beta}{2}H}|\psi\rangle$$

- ✳ The random average of the expectation value with respect to the TPQ state yields the thermal expectation value.

$$\frac{\overline{\langle\Psi_\beta|\mathcal{O}|\Psi_\beta\rangle}}{\overline{\langle\Psi_\beta|\Psi_\beta\rangle}} = \frac{1}{Z(\beta)}\mathrm{Tr}\left(\mathcal{O}\,e^{-\beta H}\right)$$

$$\frac{\overline{\langle \Psi_\beta | \mathcal{O} | \Psi_\beta \rangle}}{\overline{\langle \Psi_\beta | \Psi_\beta \rangle}} = \frac{1}{Z(\beta)} \mathrm{Tr}\left(\mathcal{O}\, e^{-\beta H}\right)$$

This result looks nice.

One might think it is similar to TFD state.

$$\langle TFD(\beta) | \mathcal{O} | TFD(\beta) \rangle = \frac{1}{Z(\beta)} \mathrm{Tr}\left(\mathcal{O}\, e^{-\beta H}\right)$$

But, it is different.

TPQ state $|\Psi_\beta\rangle \in \mathcal{H}$ *VS* $|TFD(\beta)\rangle \in \mathcal{H} \otimes \mathcal{H}$

: Not purification of thermal state

: purification of thermal state

Then, why not consider

the random state

in enlarged Hilbert space?

TPQ-like State in Enlarged Hilbert Space

- ✳ For n-partite system, consider a random state in the n copy of Hilbert space:

$$|\psi\rangle \in \underbrace{\mathscr{H} \otimes \cdots \otimes \mathscr{H}}_{n}$$

➡ Define TPQ-like state:

$$|\Psi_\alpha\rangle \equiv \prod_{i=1}^{n} e^{-\frac{\beta}{2}H^{(k)}} |\psi\rangle \in \underbrace{\mathscr{H} \otimes \cdots \otimes \mathscr{H}}_{n}$$

- ✳ Then, the random average of the expectation value still reproduce the thermal one!

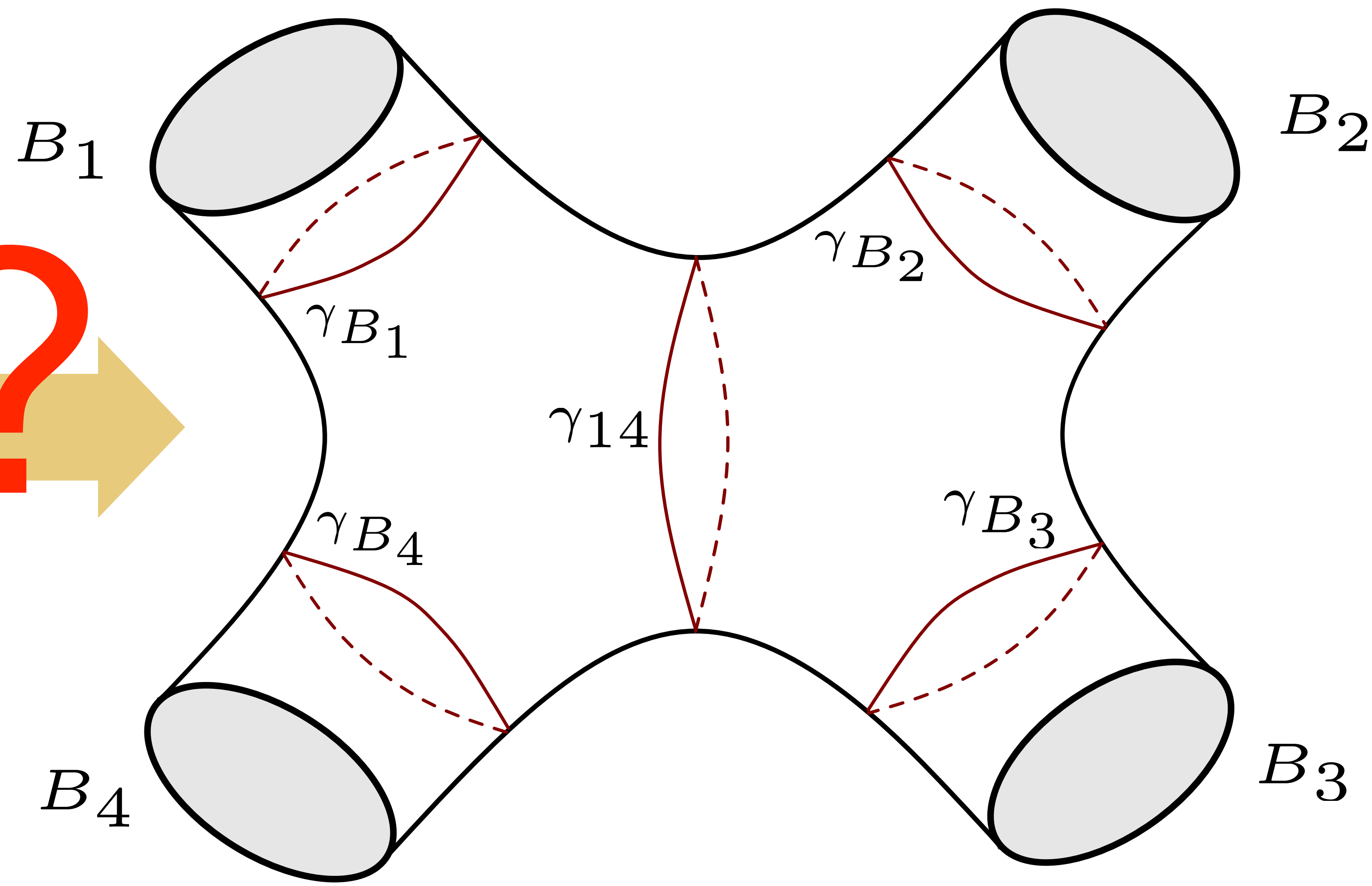
$$\frac{\overline{\langle \Psi_\beta | \mathcal{O}_j | \Psi_\beta \rangle}}{\overline{\langle \Psi_\beta | \Psi_\beta \rangle}} = \frac{1}{Z(\beta)} \mathrm{Tr}\left(\mathcal{O}_j \, e^{-\beta H}\right)$$

when $\mathcal{O}_j$ acts only on j_{th} Hilbert space.

Holographic Dual of TPQ-like State

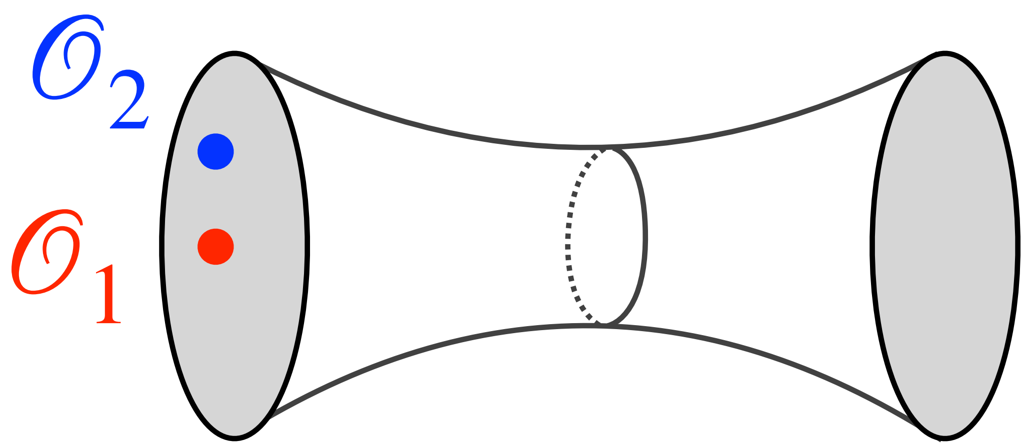
Microstate of Black Hole or Multi-boundary Wormhole?

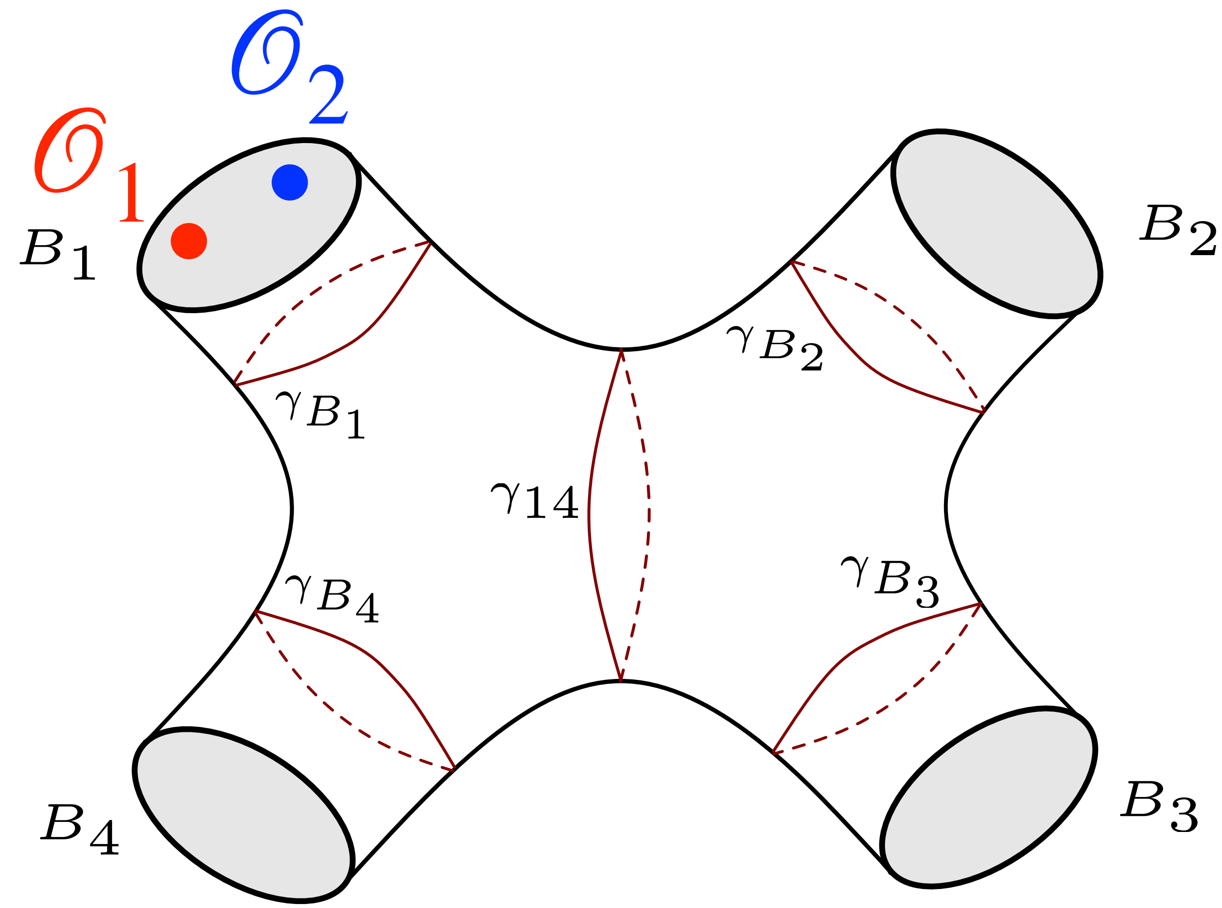
$$|\Psi_\alpha\rangle \equiv \prod_{i=1}^{n} e^{-\frac{\alpha}{2}H^{(k)}}|\psi\rangle$$



Factorization Problem

TPQ-like State still looks problematic

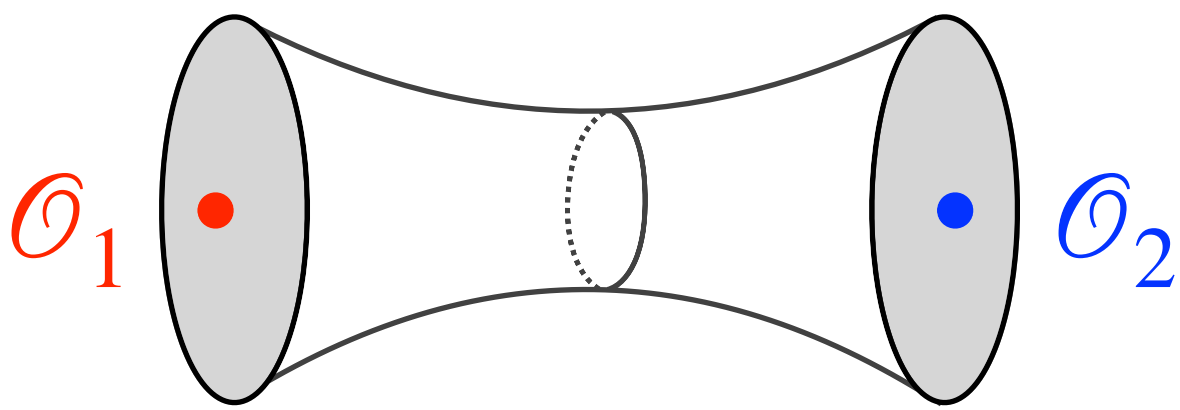


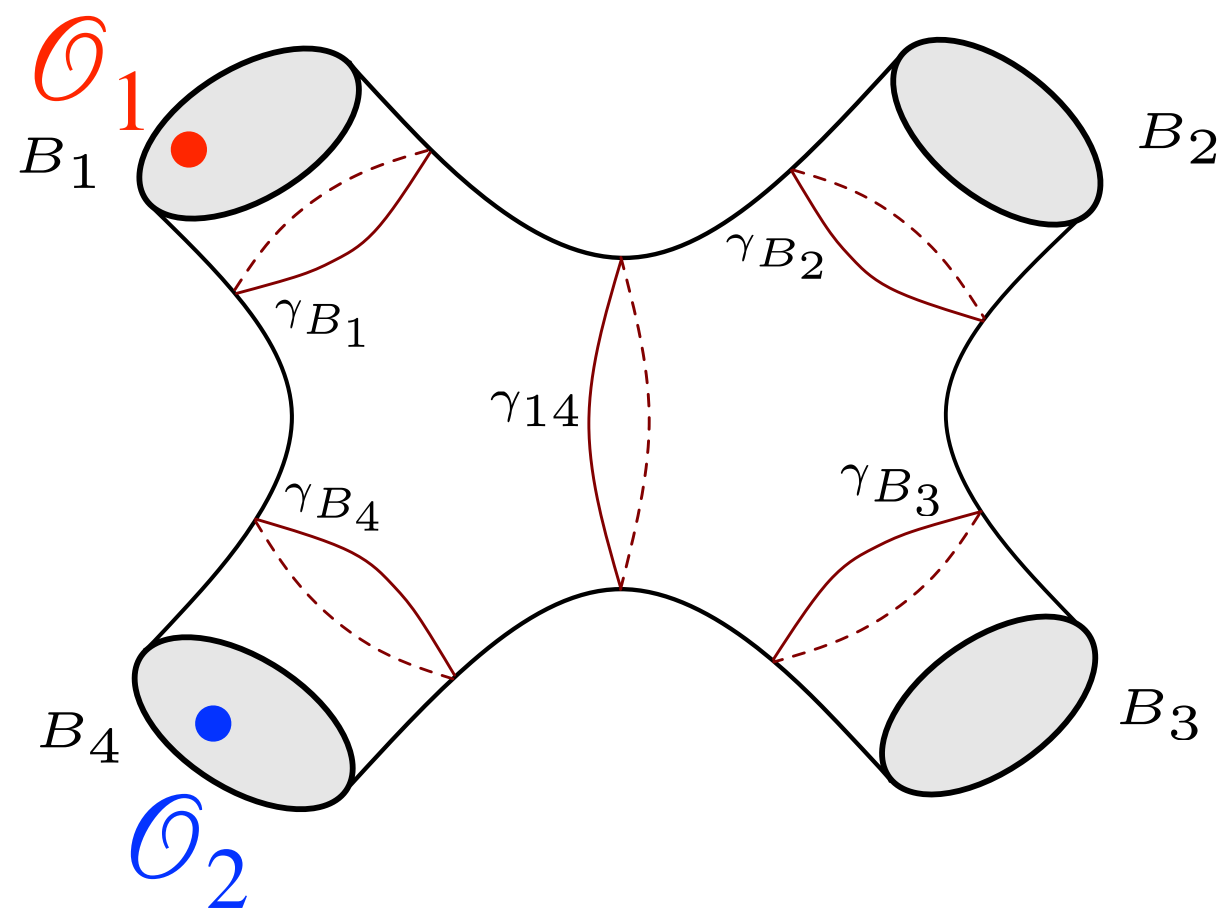


$$\frac{\overline{\langle \Psi_\beta | \mathcal{O}_1 \mathcal{O}_2 | \Psi_\beta \rangle}}{\overline{\langle \Psi_\beta | \Psi_\beta \rangle}} = \frac{1}{Z(\beta)} \mathrm{Tr}\left(\mathcal{O}_1 \mathcal{O}_2 \, e^{-\beta H} \right)$$

Factorization Problem

TPQ-like State still looks problematic

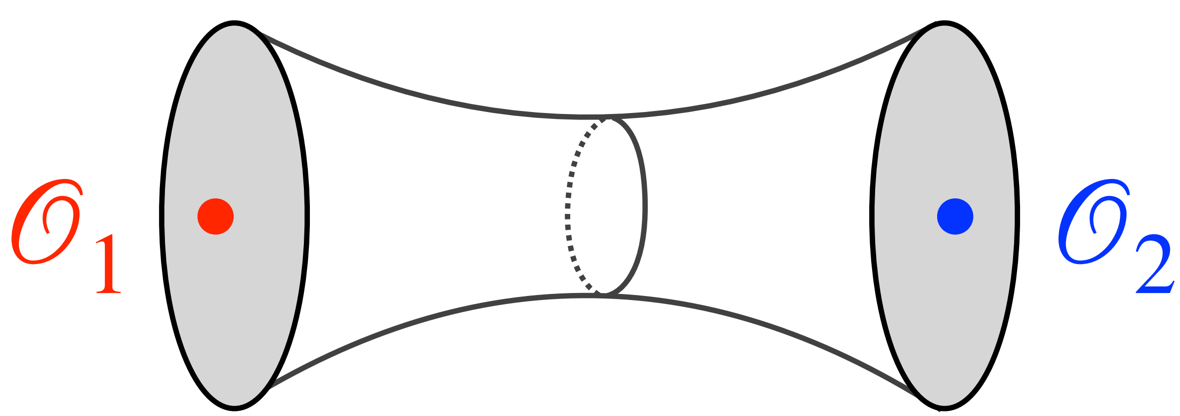




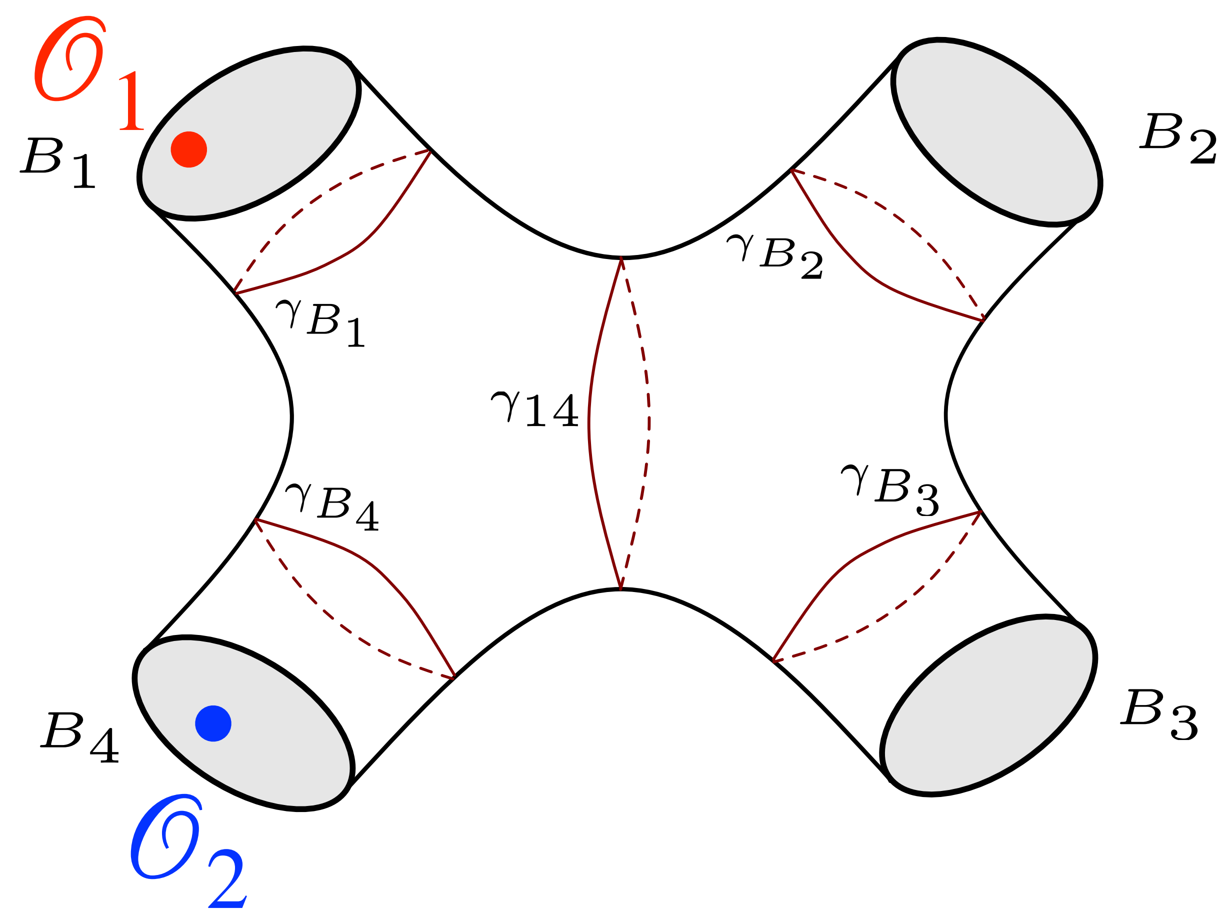
$$\frac{\overline{\langle \Psi_\beta | \mathcal{O}_1 \mathcal{O}_2 | \Psi_\beta \rangle}}{\overline{\langle \Psi_\beta | \Psi_\beta \rangle}} = \frac{1}{[Z(\beta)]^2} \mathrm{Tr}\left(\mathcal{O}_1\, e^{-\beta H}\right) \mathrm{Tr}\left(\mathcal{O}_2\, e^{-\beta H}\right)$$

Factorization Problem

TPQ-like State still looks problematic



This should not be factorized.

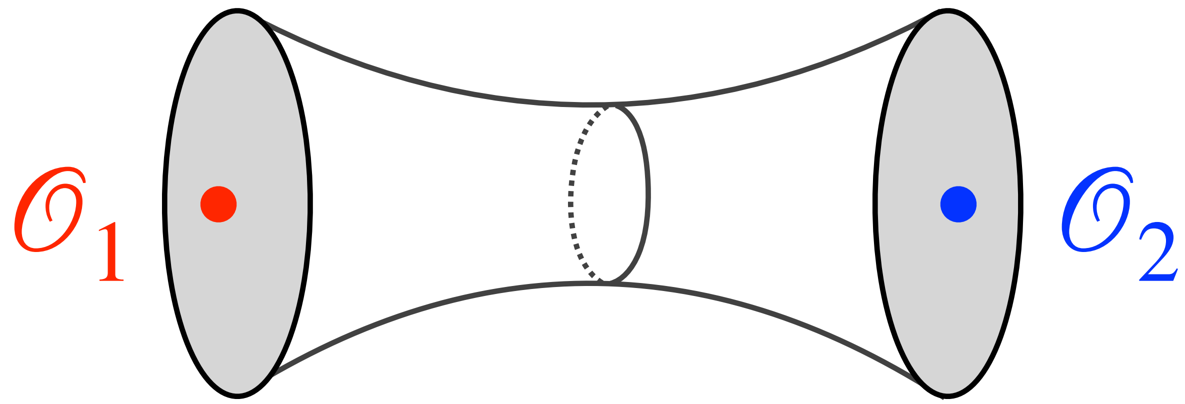


$$\frac{\overline{\langle \Psi_\beta | \mathcal{O}_1 \mathcal{O}_2 | \Psi_\beta \rangle}}{\overline{\langle \Psi_\beta | \Psi_\beta \rangle}} = \frac{1}{[Z(\beta)]^2} \mathrm{Tr}\left(\mathcal{O}_1\, e^{-\beta H}\right) \mathrm{Tr}\left(\mathcal{O}_2\, e^{-\beta H}\right)$$

: Factorized!!

Is it Contradiction?

Or, is TPQ-like state not holographic dual to black hole?

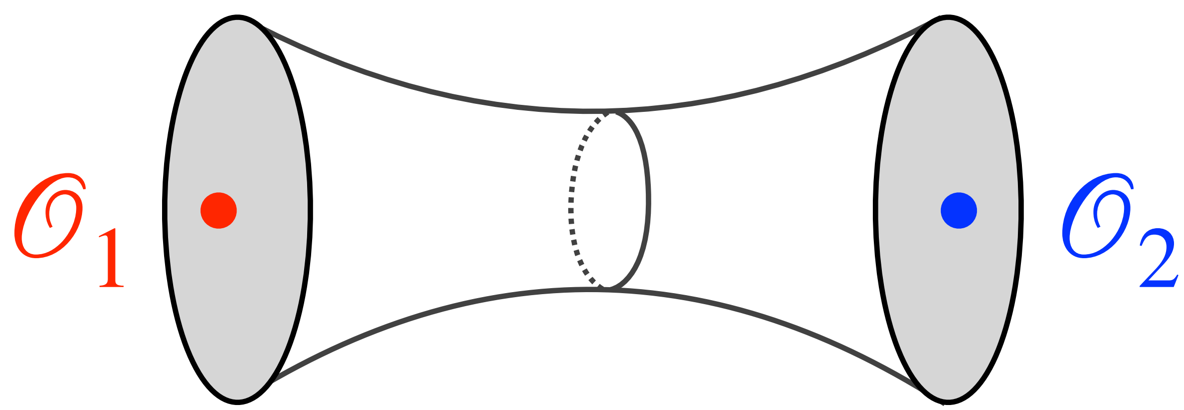


vs

$$\frac{\overline{\langle \Psi_\beta | \mathcal{O}_1 \mathcal{O}_2 | \Psi_\beta \rangle}}{\overline{\langle \Psi_\beta | \Psi_\beta \rangle}} = \frac{\mathrm{Tr}\left(\mathcal{O}_1 \, e^{-\beta H}\right) \mathrm{Tr}\left(\mathcal{O}_2 \, e^{-\beta H}\right)}{[Z(\beta)]^2}$$

This should not be factorized.

Factorized!!



vs

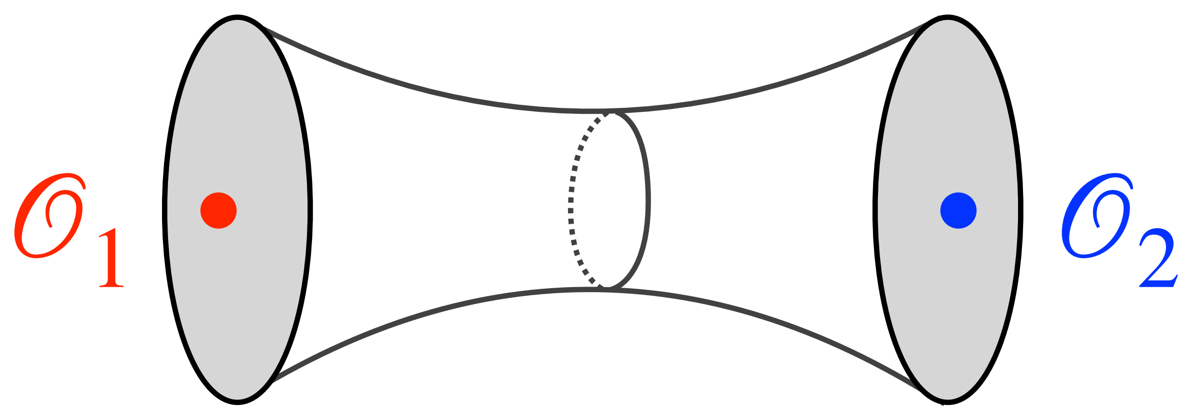
Assumption:
Operator is state-independent

$$\frac{\overline{\langle \Psi_\beta | \mathcal{O}_1 \mathcal{O}_2 | \Psi_\beta \rangle}}{\overline{\langle \Psi_\beta | \Psi_\beta \rangle}} = \frac{\mathrm{Tr}\left(\mathcal{O}_1\, e^{-\beta H}\right) \mathrm{Tr}\left(\mathcal{O}_2\, e^{-\beta H}\right)}{[Z(\beta)]^2}$$

This should not be factorized.

Factorized!!

Operator is state-dependent
in the black hole



This should not be factorized.

Operator is state-dependent
in the black hole

VS

~~**Assumption**:~~
~~Operator is state-independent~~

$$\frac{\overline{\langle\Psi_\beta|\mathcal{O}_1\mathcal{O}_2|\Psi_\beta\rangle}}{\overline{\langle\Psi_\beta|\Psi_\beta\rangle}} = \frac{\mathrm{Tr}\left(\mathcal{O}_1\, e^{-\beta H}\right)\mathrm{Tr}\left(\mathcal{O}_2\, e^{-\beta H}\right)}{[Z(\beta)]^2}$$

~~**Factorized!!**~~

Multi-partite Thermal Pure Quantum State

Definition

* Reference Hamiltonian H and its eigenstates $\{|E_j\rangle\}$

$$H|E_j\rangle = E_j|E_j\rangle$$

Multi-partite Thermal Pure Quantum State

Definition

- ✳ Reference Hamiltonian H and its eigenstates $\{|E_j\rangle\}$

$$H|E_j\rangle = E_j|E_j\rangle$$

- ✳ Schmidt decomposition of n-partite random state $|\psi\rangle$ for each k-th party.

$$|\psi\rangle \approx \frac{1}{\sqrt{d}} \sum_j |\sigma_j^{(k)}\rangle_{\{k\}} \otimes |\omega_j^{(k)}\rangle_{\{1,2,\cdots,n\}/\{k\}}$$

random state is 1-uniform state

k-th party

rest of them

Multi-partite Thermal Pure Quantum State

Definition

- Reference Hamiltonian H and its eigenstates $\{|E_j\rangle\}$

$$H|E_j\rangle = E_j|E_j\rangle$$

- Schmidt decomposition of n-partite random state $|\psi\rangle$ for each k-th party.

$$|\psi\rangle \approx \frac{1}{\sqrt{d}} \sum_j |\sigma_j^{(k)}\rangle_{\{k\}} \otimes |\omega_j^{(k)}\rangle_{\{1,2,\cdots,n\}/\{k\}}$$

random state is 1-uniform state

k-th party

rest of them

- Define a unitary matrix $U^{(k)}$ mapping $|\sigma_j^{(k)}\rangle$ to $|E_j\rangle$

$$U^{(k)} \equiv \sum_j |E_j\rangle\langle\sigma_j^{(k)}|$$

$$|\sigma_j^{(k)}\rangle \longrightarrow |E_j\rangle$$

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- ✳ Define the Hamiltonian $H^{(k)}$ of the k-th party

$$H^{(k)} \equiv U^{(k)\dagger} H U^{(k)}$$

$|\sigma_j^{(k)}\rangle$ is the eigenstate of $H^{(k)}$

$$H^{(k)}|\sigma_j^{(k)}\rangle = E_j|\sigma_j^{(k)}\rangle$$

Multi-partite Thermal Pure Quantum State

Effective Temperature

- Define multi-partite Thermal Pure Quantum (MTPQ) state:

$$|\Psi_\alpha\rangle \equiv \prod_{i=1}^{n} e^{-\frac{\alpha}{2}H^{(k)}} |\psi\rangle$$

: the parameter $\alpha = \alpha(\beta)$ is related to the inverse temperature β.

Multi-partite Thermal Pure Quantum State

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- ✳ We compare the reduced density matrix with the thermal density matrix.

$$\rho = |\Psi_\alpha\rangle\langle\Psi_\alpha|$$

$$\rho^{(k)} \equiv \mathrm{tr}_{\{1,2,\cdots,n\}/\{k\}}\rho = \sum_j |\zeta_j^{(k)}\rangle\lambda_j^{(k)}\langle\zeta_j^{(k)}|$$

(diagonalization of) reduced density matrix

same state

$$\rho^{(k)}_{\mathrm{thermal}}(\beta) \equiv |\zeta_j^{(k)}\rangle\frac{e^{-\beta E_j}}{Z(\beta)}\langle\zeta_j^{(k)}|$$

energy eigenvalue of reference Hamiltonian

thermal density matrix

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(diagonalization of) reduced density matrix

$$\rho^{(k)}_{\mathrm{thermal}}(\beta) \equiv |\zeta_j^{(k)}\rangle\frac{e^{-\beta E_j}}{Z(\beta)}\langle\zeta_j^{(k)}|$$

thermal density matrix

✳ For the given α, we determine the effective inverse temperature β by minimizing the relative entropy between the reduced density matrix and the thermal density matrix.

$$S(\rho^{(k)}||\rho^{(k)}_{\mathrm{thermal}}(\beta)) = \mathrm{tr}\left(\rho^{(k)}\log\rho^{(k)}\right) - \mathrm{tr}\left(\rho^{(k)}\log\rho^{(k)}_{\mathrm{thermal}}(\beta)\right)$$

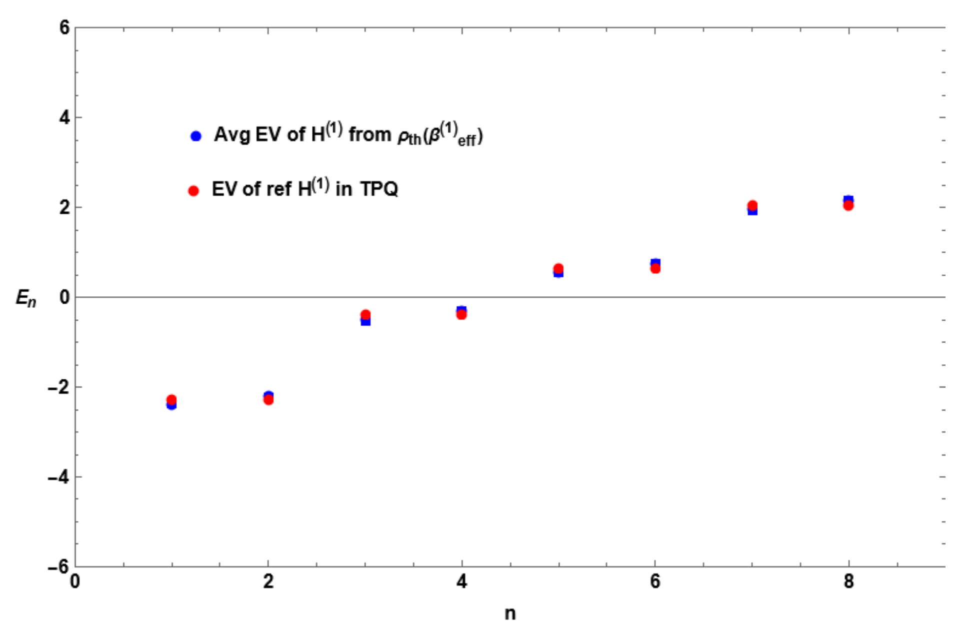
Multi-partite Thermal Pure Quantum State

Spectrum from Reduced Density Matrix

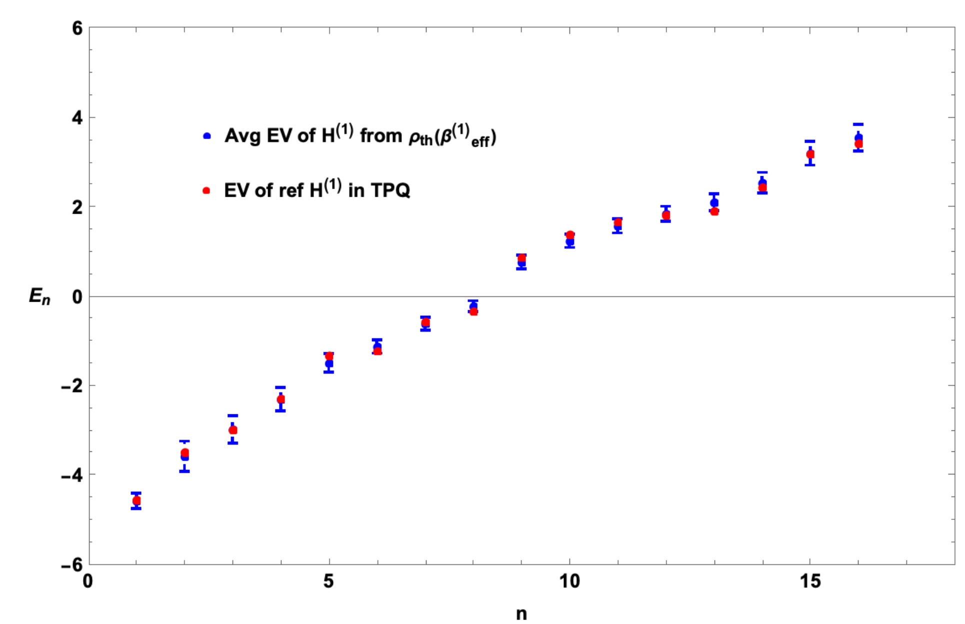
- After determining the effective (inverse) temperature, one can extract the spectrum $\{\widetilde{E}_j\}$ from the reduced density matrix

$$\rho^{(k)} = \sum_j |\zeta_j^{(k)}\rangle \frac{e^{-\beta_k \widetilde{E}_j}}{\widetilde{Z}(\beta_k)} \langle \zeta_j^{(k)}|$$

- Comparison the spectrum $\{\widetilde{E}_j\}$ from $\rho^{(k)}$ and the spectrum $\{E_j\}$ of the reference Hamiltonian



5-party of N=6 SYK model



4-party of N=8 SYK model

Multi-partite Thermal Pure Quantum State

State-dependence

reduced density matrix

$$\rho^{(k)} = \sum_j |\zeta_j^{(k)}\rangle \frac{e^{-\beta_k \widetilde{E}_j}}{\widetilde{Z}(\beta_k)} \langle \zeta_j^{(k)}|$$

- ✳ Define a unitary matrix $W^{(k)}$ mapping $|\zeta_j^{(k)}\rangle$ to $|E_j\rangle$.

$$W^{(k)} \equiv \sum_j |E_j\rangle\langle\zeta_j^{(k)}|$$

$$|\zeta_j^{(k)}\rangle \longrightarrow |E_j\rangle$$

Multi-partite Thermal Pure Quantum State

State-dependence

reduced density matrix

$$\rho^{(k)} = \sum_j |\zeta_j^{(k)}\rangle \frac{e^{-\beta_k \widetilde{E}_j}}{\widetilde{Z}(\beta_k)} \langle \zeta_j^{(k)}|$$

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$$|\zeta_j^{(k)}\rangle \longrightarrow |E_j\rangle$$

* For an operator $\mathcal{O}$ in the reference system, we define a state-dependent operator $\mathcal{O}^{(k)}$ in the k-th party.

$$\mathcal{O}^{(k)} \equiv W^{(k)\dagger}\mathcal{O}W^{(k)}$$

Multi-partite Thermal Pure Quantum State

State-dependence

reduced density matrix

$$\rho^{(k)} = \sum_j |\zeta_j^{(k)}\rangle \frac{e^{-\beta_k \widetilde{E}_j}}{\widetilde{Z}(\beta_k)} \langle \zeta_j^{(k)}|$$

- Define a unitary matrix $W^{(k)}$ mapping $|\zeta_j^{(k)}\rangle$ to $|E_j\rangle$.

$$W^{(k)} \equiv \sum_j |E_j\rangle\langle\zeta_j^{(k)}|$$

$$|\zeta_j^{(k)}\rangle \longrightarrow |E_j\rangle$$

- For an operator $\mathcal{O}$ in the reference system, we define a state-dependent operator $\mathcal{O}^{(k)}$ in the k-th party.

$$\mathcal{O}^{(k)} \equiv W^{(k)\dagger} \mathcal{O} W^{(k)}$$

- Then, one can show that the expectation value of $\mathcal{O}^{(k)}$ with respect to the MTPQ state becomes the thermal expectation value of the corresponding operator $\mathcal{O}$ in the reference system.

$$\langle \Psi_\alpha | \mathcal{O}^{(k)} | \Psi_\alpha \rangle = \frac{1}{\widetilde{Z}(\beta_k)} \sum_j \langle E_j | \mathcal{O} | E_j \rangle$$

: (at least) “thermal 1-uniform state”

Example of MTPQ State

Bi-partite Random State

- The Schmidt decomposition of bi-partite random state

$$|\psi\rangle \approx \frac{1}{\sqrt{d}} \sum_j |\sigma_j^{(1)}\rangle |\sigma_j^{(2)}\rangle$$

- Define (state-dependent) unitary operator and the Hamiltonian for each party.

$$U^{(k)} \equiv \sum_j |E_j\rangle\langle\sigma_j^{(k)}| \qquad H^{(k)} = U^{(k)\dagger} H U^{(k)} \qquad (k = 1,2)$$

- Construction of MTPQ state:

$$|\Psi_\alpha\rangle = e^{-\frac{\alpha}{2}(H^{(1)}+H^{(2)})}|\psi\rangle \approx \frac{1}{\sqrt{d}} \sum_j e^{-\alpha E_j} |\sigma_j^{(1)}\rangle |\sigma_j^{(2)}\rangle$$

- The effective (inverse) temperature is easy to determined by $\beta = 2\alpha$

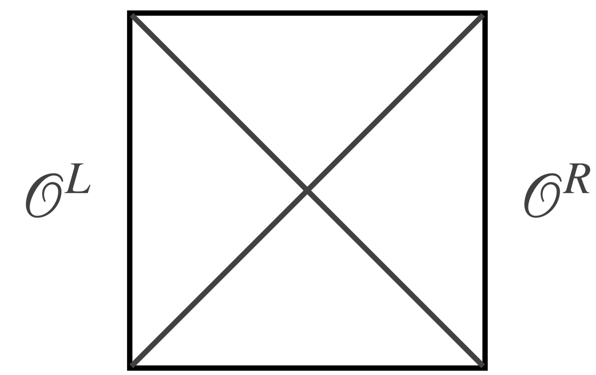
$$|\Psi_{\frac{\beta}{2}}\rangle \approx \frac{\sqrt{Z(\beta)}}{\sqrt{d}} |TFD(\beta)\rangle$$

Example of MTPQ State

Bi-partite Random State

- ✳ For a given operator $\mathcal{O}$ in the reference system, we define state-dependent operator

$$\mathcal{O}^L = U^{(1)\dagger}\mathcal{O}U^{(1)}$$
$$\mathcal{O}^R = U^{(2)\dagger}\mathcal{O}U^{(2)}$$



- ✳ The expectation value of $\mathcal{O}^R$ and $\mathcal{O}^L$ with respect to MTPQ state (with proper normalization) gives the thermal expectation value of them.

$$\langle \Psi_{\frac{\beta}{2}} | \mathcal{O}^R | \Psi_{\frac{\beta}{2}} \rangle = \frac{1}{Z}\sum_j e^{-\beta E_j}\langle E_j | \mathcal{O} | E_j \rangle$$

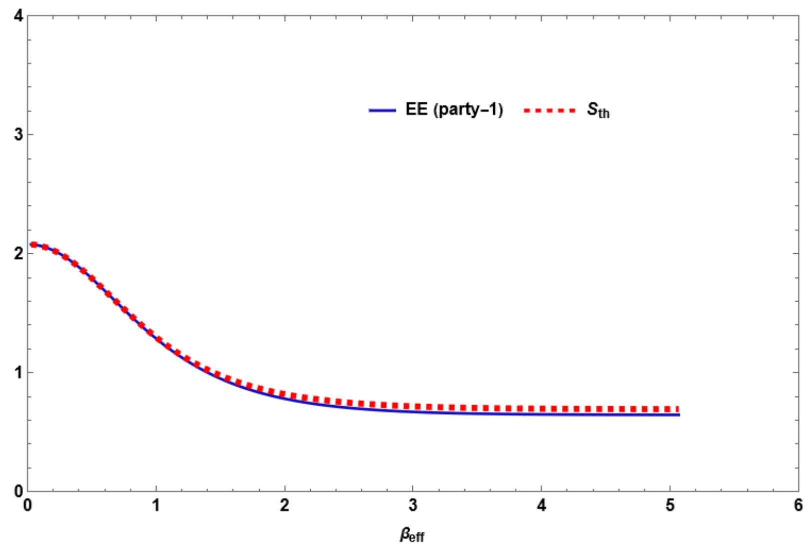
: thermal 1-uniform state

- ✳ The expectation value of $\mathcal{O}^R\mathcal{O}^L$ with respect to MTPQ state also reproduces the expected thermal expectation value.

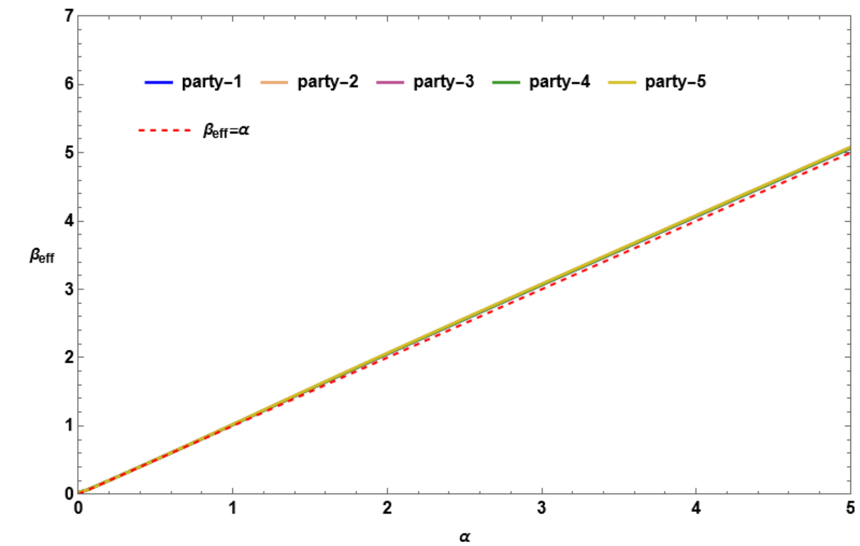
$$\langle \Psi_{\frac{\beta}{2}} | \mathcal{O}_1^L\mathcal{O}_2^R | \Psi_{\frac{\beta}{2}} \rangle = \frac{1}{Z}\sum_{j,k} e^{-\frac{\beta}{2}(E_j+E_k)}\langle E_j | \mathcal{O}_1 | E_k \rangle\langle E_j | \mathcal{O}_2 | E_k \rangle$$

Example of MTPQ State

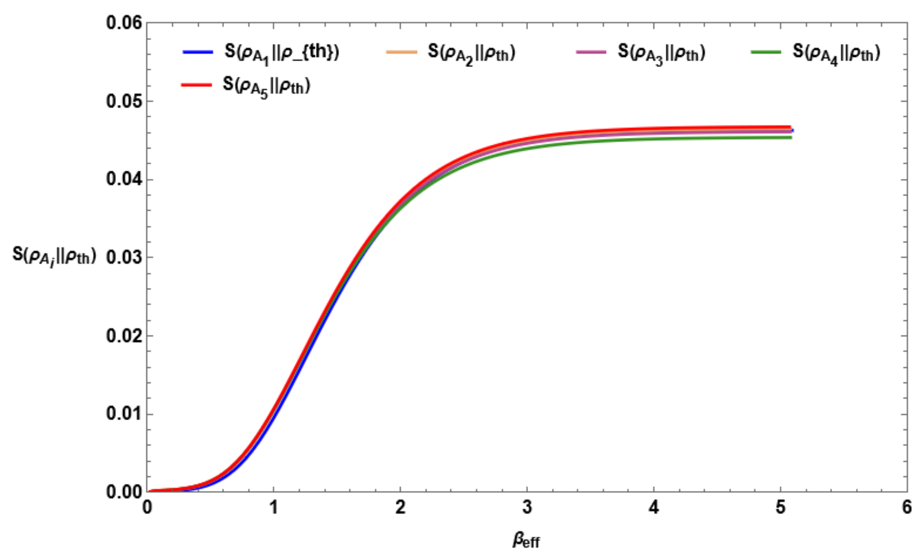
5-party 3 qubit SYK Model (N=6)



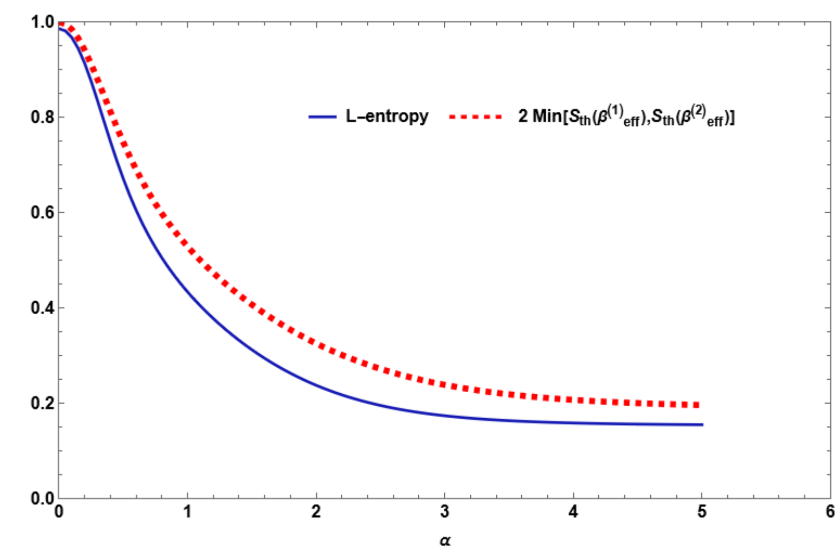
EE of one party vs thermal entropy



parameter α vs effective (inverse) temperature



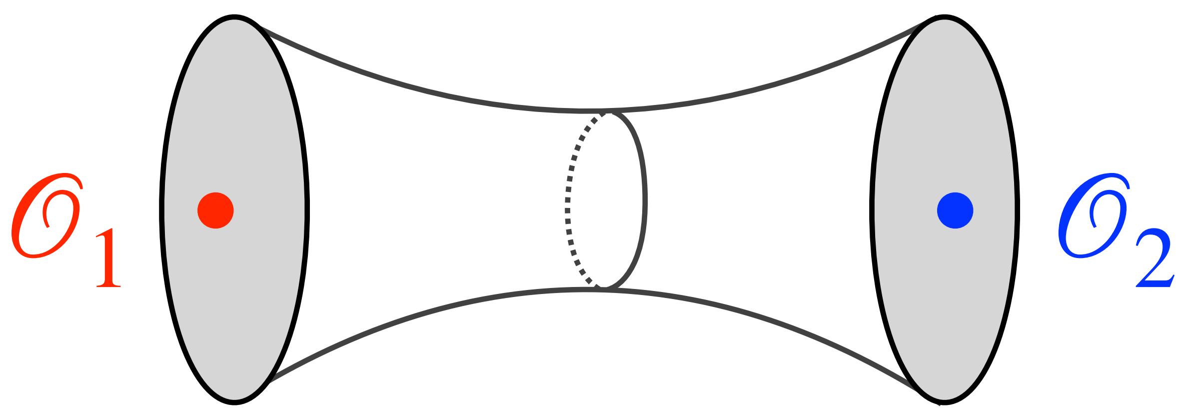
relative entropy between reduced density matrix
and thermal density matrix

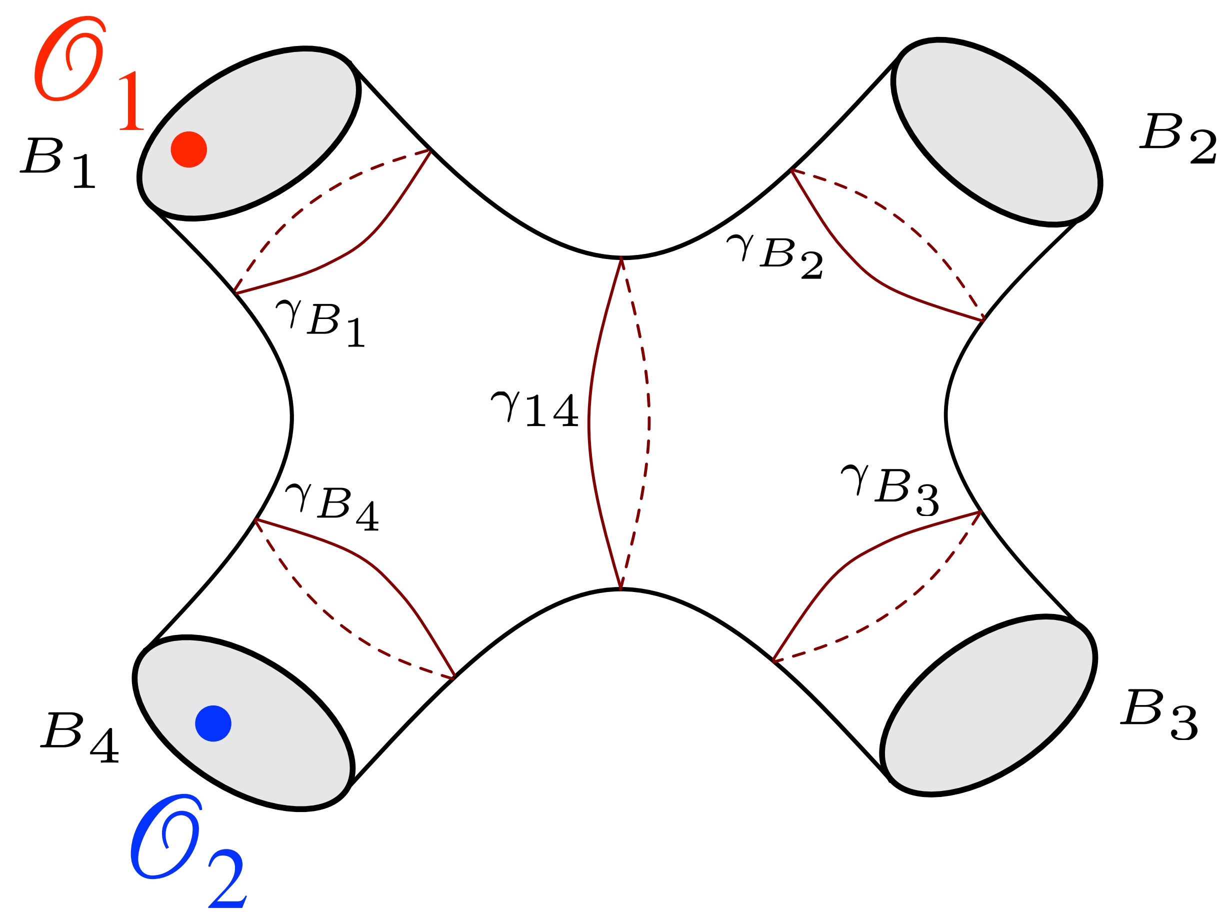


L-entropy vs $2\min(S_{th}^{(1)}, S_{th}^{(2)})$

Future Works

Further Confirmations of Holographic Dual wormhole of MTPQ State





Thank You