Holographic Free Energy and Integrated correlators

Nakwoo Kim
Kyung Hee University

Based on 2411.08615
With Hoseob Shin

Integrated correlators

- By IC, in this talk I mean:
  - the conformally invariant part of, (function of cross ratios only) 4-point correlation functions $\langle \mathcal{O}_{p_1}(x_1)\mathcal{O}_{p_2}(x_2)\mathcal{O}_{p_3}(x_3)\mathcal{O}_{p_4}(x_4)\rangle$
  - of 1/2-BPS operators
  - of *N*=4 SYM
  - integrated over cross ratios $U = \dfrac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2}, V = \dfrac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2}$
  - with appropriate measure (so that the result is finite and identified with free energy of mass deformed SYM)

$\mathcal{N}$=4 SYM

- Interacting QFT with largest amount of supersymmetry

- 1 vector, 4 Weyl spinors, 6 real scalars in adjoint rep of SU(N); R-symmetry SU(4)=SO(6)

- Superconformal (beta function vanishes to all orders)

- Holographic dual of IIB string theory in $AdS_5 \times S^5$:
  $g_{YM}^2 N = (R_5^2/\alpha')^2$ and $g_{YM}^2 = 4\pi g_s$

Why do we integrate?

- 4 point functions contain non-trivial dependence on coordinates and coupling constants g_{YM}, N - even for 1/2-BPS operators whose anom. dim. is protected.

- Of course they can be/have been studied along a number of different directions - SUSY, CFT bootstrap, integrability, localization, holography etc.

- Integrated correlators are related to (mass, coupling) derivatives of mass-deformed SYM free energy.

1/2 BPS operators

- Short multiplets of superconformal algebra.
- Supersymmetry forces their conformal dimensions to be independent of g_{YM}
- Chiral primaries satisfy $\Delta - R = 0$; $\mathcal{O} = \mathrm{Tr}(\phi^{I_1}\phi^{I_2}\ldots\phi^{I_k})S_{I_1 I_2\ldots I_k}$ with totally symmetric and traceless S. Starts at $k=2$, 20 of SO(6); Convenient to introduce "polarization" Y^I satisfying $Y^I Y^I = 0$ and use $\mathcal{O} = \mathrm{Tr}((\phi^I Y^I)^k)$
- Their 2- and 3-point functions are trivial.

Structure of 4pt fn

- 4pt correlators split into the free part and interacting part, and then the interaction part factorizes into R-symmetry part and universal part [Rastelli Zhou 1710.05923; Chester Pufu 2003.08412]

$$\langle S(\vec{x}_1, Y_1)\cdots S(\vec{x}_4, Y_4)\rangle = \frac{1}{\vec{x}_{12}^4 \vec{x}_{34}^4}\vec{\mathcal{S}}\cdot\vec{\mathcal{B}}, \qquad \vec{\mathcal{S}} \equiv \vec{\mathcal{S}}_{\text{free}} + \vec{\mathbf{S}}\mathcal{T},$$

Specifically for $\Delta = 2$ ops

$$\begin{aligned}
\vec{\mathcal{S}}_{\text{free}} &\equiv \begin{pmatrix} 1 & U^2 & \frac{U^2}{V^2} & \frac{1}{c}\frac{U^2}{V} & \frac{1}{c}\frac{U}{V} & \frac{1}{c}U \end{pmatrix}, \\
\vec{\mathbf{S}} &\equiv \begin{pmatrix} V & UV & U & U(U-V-1) & 1-U-V & V(V-U-1) \end{pmatrix}, \\
\mathcal{B} &= \begin{pmatrix} Y_{12}^2 Y_{34}^2 & Y_{13}^2 Y_{24}^2 & Y_{14}^2 Y_{23}^2 & Y_{13}Y_{14}Y_{23}Y_{24} & Y_{12}Y_{14}Y_{23}Y_{34} & Y_{12}Y_{13}Y_{24}Y_{34} \end{pmatrix}.
\end{aligned}$$

Amplitude in double expansion

- In 1/N and $1/(g_{YM}^2 N)$ expansion, restricted as below (using general requirements such as crossing symmetry and analyticity; bootstrap)
- Subleading terms are (via AdS/CFT) related to higher derivative corrections to IIB sugra, and determined from string scattering amplitude.
- Coefficients (*B's*) can be also fixed from integrated correlators.

$$\begin{aligned}\mathcal{T} =& \frac{1}{c}\left[8\mathcal{T}^{\mathrm{SG}} + \lambda^{-\frac{3}{2}} B_0^0 \mathcal{T}^0 + \lambda^{-\frac{5}{2}}\left[B_2^2\mathcal{T}^2 + B_0^2\mathcal{T}^0\right] + \lambda^{-3}\left[B_3^3\mathcal{T}^3 + B_2^3\mathcal{T}^2 + B_0^3\mathcal{T}^0\right] + O(\lambda^{-\frac{7}{2}})\right] \\ &+ \frac{1}{c^2}\left[\lambda^{\frac{1}{2}}\overline{B_0^0}\mathcal{T}^0 + \left[\mathcal{T}^{\mathrm{SG|SG}} + \overline{B}_0^{\mathrm{SG|SG}}\mathcal{T}^0\right] + O(\lambda^{-1})\right] \\ &+ \frac{1}{c^3}\left[\lambda^{\frac{3}{2}}\overline{\overline{B_2^2}}\mathcal{T}^2 + O(\lambda^1)\right] + \frac{1}{c^4}\left[\lambda^3\left[\overline{\overline{\overline{B_3^3}}}\mathcal{T}^3 + \overline{\overline{\overline{B_2^3}}}\mathcal{T}^2 + \overline{\overline{\overline{B_0^3}}}\mathcal{T}^0\right] + O(\lambda^{\frac{5}{2}})\right] + O(c^{-5})\,.\end{aligned}$$

$$
\begin{aligned}
\mathcal{M}(s,t) =& \frac{1}{c}\left[\frac{8}{(s-2)(t-2)(u-2)} + \frac{120\zeta(3)}{\lambda^{\frac{3}{2}}} + \frac{630\zeta(5)}{\lambda^{\frac{5}{2}}}\left[s^2+t^2+u^2-3\right]\right. \\
&\left. + \frac{5040\zeta(3)^2}{\lambda^3}\left[stu - \frac{1}{4}(s^2+t^2+u^2) - 4\right] + O(\lambda^{-3})\right] \\
&+ \frac{1}{c^2}\left[\frac{5\sqrt{\lambda}}{8} + \mathcal{M}^{\mathrm{SG|SG}} + \frac{15}{4} + O(\lambda^{-\frac{3}{2}})\right] \\
&+ \frac{1}{c^3}\left[\frac{7\lambda^{\frac{3}{2}}}{3072}\left[s^2+t^2+u^2-3\right] + O(\lambda)\right] \\
&+ \frac{1}{c^4}\left[\frac{\lambda^3}{221184}\left[stu - \frac{1}{4}(s^2+t^2+u^2) - 4\right] + O(\lambda^{\frac{5}{2}})\right] + O(c^{-5})\,,
\end{aligned}
$$

$c = N^2$

Mass deformation of *N*=4 SYM

- Susy preserving mass deformation of *N*=4 SYM has been also extensively studied.
- *N*=2* : Give mass (m) to hypermultiplet
- *N*=1* : Give mass (m_1, m_2, m_3) to three chiral multiplets
- More realistic cousins of QCD, especially because they may exhibit confinement in IR

4pt with spinors

- 4pt corr are related to 4-th mass derivatives of mass-deformed free energy on 4-sphere

$$S_{\mathcal{N}=4} \to S_{\mathcal{N}=4} + S_m$$

$$\langle P(\vec{x}_1, \bar{X}_1)\bar{P}(\vec{x}_2, X_2)P(\vec{x}_3, \bar{X}_3)\bar{P}(\vec{x}_4, X_4)\rangle = \frac{1}{|\vec{x}_{12}|^6\,|\vec{x}_{34}|^6}\vec{\mathcal{P}}(U,V)\cdot\vec{\mathcal{B}}_P\,,$$

$$S_m = \int \mathrm{d}^4\vec{x}\,\sqrt{g}\sum_{j=1}^{3}\left[m_j(\mathrm{i}\,J_i + K_j) + m_j^2 L_j\right]$$

$$\langle S(\vec{x}_1, Y_1)S(\vec{x}_2, Y_2)\bar{P}(\vec{x}_3, X_3)P(\vec{x}_4, \bar{X}_4)\rangle = \frac{1}{|\vec{x}_{12}|^4\,|\vec{x}_{34}|^6}\vec{\mathcal{R}}(U,V)\cdot\vec{\mathcal{B}}_{SP}\,,$$

$$\left.\frac{\partial^4 F(m)}{\partial m^4}\right|_{m=0} = \int dx_1 dx_2 dx_3 dx_4 (\, C_1\langle SSSS\rangle + C_2\langle SS\bar{P}P\rangle + C_3\langle P\bar{P}P\bar{P}\rangle\,)$$

$$\vec{\mathcal{P}}(U,V) = \vec{\mathcal{P}}_{\text{free}}(U,V) + \vec{\mathbf{P}}(U,V,\partial_U,\partial_V)\mathcal{T}(U,V)\,,$$

$$\vec{\mathcal{R}}(U,V) = \vec{\mathcal{R}}_{\text{free}}(U,V) + \vec{\mathbf{R}}(U,V,\partial_U,\partial_V)\mathcal{T}(U,V)\,,$$

$$\int dU dV\,(\text{measure})\,\mathscr{T}(U,V)$$

Mass terms on 4-sphere

- Mass terms of scalar/spinors of *N*=4 SYM
- L_i do not contribute to 4-pt function
- *N*=2* free energy as sum of integrated 4-pt fn worked out in [Chester Pufu 2003.08412] and was compared to (large N) localization result.

$$J_i \equiv \frac{1}{2}\mathrm{tr}\left(Z_i^2 + \bar{Z}_i^2\right) = N_J\left(S_{IJ}Y_i^I Y_i^J + S_{IJ}Y_i^{*I}Y_i^{*J}\ \right)$$

$$K_i \equiv -\frac{1}{2}tr\left(\chi_i\sigma_2\chi_i + \bar{\chi}_i\sigma_2\bar{\chi}_i\right) = N_K\left(P_{AB}\bar{X}_i^A\bar{X}_i^B + \bar{P}^{AB}X_A^i X_B^i\right)$$

$$L_i \equiv tr(|Z_i|^2)$$

N=1*

- We calculated the IC measure for *N*=1* and compared to dual supergravity result.

$$\mathcal{F}_4 \equiv \left.\frac{\partial^4 F(m,m,0)}{\partial m^4}\right|_{m=0}$$

$\mathcal{N} = 2^*$ integrated correlators

$$\mathcal{F}_{1111} \equiv \left.\frac{\partial^4 F(m_1,m_2,m_3)}{\partial m_1^4}\right|_{m_i=0}$$

$$\mathcal{F}_{1122} \equiv \left.\frac{\partial^4 F(m_1,m_2,m_3)}{\partial m_1^2\,\partial m_2^2}\right|_{m_i=0}$$

$\mathcal{N} = 1^*$ integrated correlators

N=2* Holography

- Motivation: large *N* and large 't Hooft coupling limit of the matrix model from Pestun's localization formula:

$$F_{S^4} = -\frac{N^2}{2}(1+m^2a^2)\log\frac{\lambda(1+m^2a^2)e^{2\gamma+\frac{1}{2}}}{16\pi^2}\,,$$

- Can we reproduce it using holography?: As regularized on-shell action of AdS Einstein gravity plus scalar fields dual to mass terms

Sugra dual of *N*=2*

- Einstein+scalars as consistent truncation of maximal D=5 supergravity, in Euclidean signature

- BPS equations were integrated numerically, and the results are argued to be consistent with localization result. [Bobev, Elvang, Freedman, Pufu 1311.1508]

$$\mathcal{L}_{5\mathrm{D}} = \frac{1}{4\pi G_5}\left[-\frac{R}{4} + \frac{3\partial_\mu\eta\partial^\mu\eta}{\eta^2} + \frac{\partial_\mu z\partial^\mu\tilde{z}}{(1-z\tilde{z})^2} + V\right],$$

$$V \equiv -\frac{1}{L^2}\left(\frac{1}{\eta^4} + 2\eta^2\frac{1+z\tilde{z}}{1-z\tilde{z}} + \frac{\eta^8}{4}\frac{(z-\tilde{z})^2}{(1-z\tilde{z})^2}\right),$$

$$L\frac{\eta'}{\eta} = \frac{\left[1+z^2+\eta^6(z^2-1)\right](LA' \mp e^{-A})}{2(1+z^2)+\eta^6(1-z^2)},$$

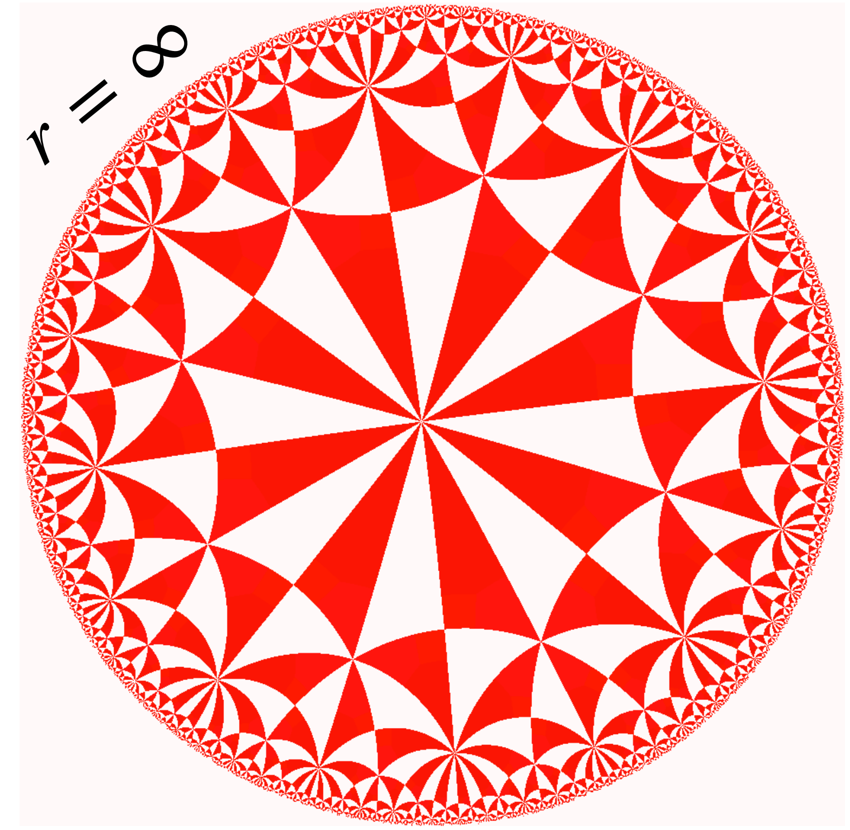
$$L\frac{\eta'}{\eta} = \frac{\left[1+\tilde{z}^2+\eta^6(\tilde{z}^2-1)\right](LA' \pm e^{-A})}{2(1+\tilde{z}^2)+\eta^6(1-\tilde{z}^2)}.$$

$$z' = \frac{3\eta'(z\tilde{z}-1)\left[2(z+\tilde{z})+\eta^6(z-\tilde{z})\right]}{2\eta\left[\eta^6(\tilde{z}^2-1)+\tilde{z}^2+1\right]},$$

$$\tilde{z}' = \frac{3\eta'(z\tilde{z}-1)\left[2(z+\tilde{z})-\eta^6(z-\tilde{z})\right]}{2\eta\left[\eta^6(z^2-1)+z^2+1\right]},$$

$$(\eta')^2 = \frac{\left[\eta^6(z^2-1)+z^2+1\right]\left[\eta^6(\tilde{z}^2-1)+\tilde{z}^2+1\right]}{9L^2\eta^2(z\tilde{z}-1)^2}.$$

$$ds^2 = L^2 e^{2A(r)} ds^2_{S^4} + dr^2\,,$$



$$\begin{aligned} e^{2A} &= \frac{e^{2r}}{4} + \frac{1}{6}(\mu^2 - 3) + \mathcal{O}\big(r^2\, e^{-2r}\big)\ , \\ \eta &= 1 + e^{-2r}\left[\frac{2\mu^2}{3} r + \frac{\mu(\mu+v)}{3}\right] + \mathcal{O}\big(r^2\, e^{-4r}\big)\ , \\ \frac{1}{2}(z + \tilde{z}) &= e^{-2r}\Big[2\mu\, r + v\Big] + \mathcal{O}\big(r^2\, e^{-4r}\big)\ , \\ \frac{1}{2}(z - \tilde{z}) &= \mp\mu\, e^{-r} \mp e^{-3r}\left[\frac{4}{3}\mu\big(\mu^2 - 3\big)\, r + \frac{1}{3}\Big(2v(\mu^2 - 3) + \mu(4\mu^2 - 3)\Big)\right] + \mathcal{O}\big(r^2\, e^{-5r/L}\big)\,. \end{aligned} \tag{4.2}$$

$$v(\mu) = -2\mu - \mu\,\log(1 - \mu^2)\,.$$

Perturbative analysis of Einstein equations

- We proposed a technique to treat scalar fields perturbatively. [NK 1902.00418]

- The BPS eqs are linearized and we just need to solve coupled linear ODEs.

- One can analytically calculate the series expansion form of F(m), from $v(\mu)$.

- *N*=1* holographic model (10 scalars) was written down in [Bobev, Elvang, Kol, Olsen, Pufu 1605.00656] and we solved using perturbative method in [NK, SJK 1904.02038]

10-scalar model of *N*=1*

$$\mathcal{L} = -\frac{1}{4}R + 3(\partial\beta_1)^2 + (\partial\beta_2)^2 + \frac{1}{2}\mathcal{K}_{a\bar{b}}\partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} - \mathcal{P}\,,$$

$$\mathcal{K} = -\sum_{a=1}^{4}\log(1 - z^a\bar{z}^a)\,, \quad \mathcal{P} = \frac{1}{8}e^{\mathcal{K}}\left[\frac{1}{6}\partial_{\beta_1}\mathcal{W}\partial_{\beta_1}\overline{\mathcal{W}} + \frac{1}{2}\partial_{\beta_2}\mathcal{W}\partial_{\beta_2}\overline{\mathcal{W}} + \mathcal{K}^{\bar{b}a}\nabla_a\mathcal{W}\nabla_{\bar{b}}\overline{\mathcal{W}} - \frac{8}{3}\mathcal{W}\overline{\mathcal{W}}\right],$$

$$\begin{aligned}\mathcal{W} \equiv\ & \frac{1}{L}e^{2\beta_1+2\beta_2}\left(1 + z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4 + z_1z_2z_3z_4\right)\\ &+ \frac{1}{L}e^{2\beta_1-2\beta_2}\left(1 - z_1z_2 + z_1z_3 - z_1z_4 - z_2z_3 + z_2z_4 - z_3z_4 + z_1z_2z_3z_4\right)\\ &+ \frac{1}{L}e^{-4\beta_1}\left(1 + z_1z_2 - z_1z_3 - z_1z_4 - z_2z_3 - z_2z_4 + z_3z_4 + z_1z_2z_3z_4\right)\end{aligned}$$

Holographic *N*=1*

- Numerical results in Bobev et al. is now partly determined exactly.
- Consistency with *N*=2 Localization result. ($A + 2B = -1/8$)

$$\left.\frac{F}{N^2}\right|_{\mathcal{O}(m^4)} = \mathcal{A}(m_1^4 + m_2^4 + m_3^4) + \mathcal{B}(m_1^2 + m_2^2 + m_3^2)^2.$$

$$\mathcal{A} = +\frac{1}{40} - \frac{2\pi^4}{525} \approx -0.346082$$

$$\mathcal{B} = -\frac{3}{40} + \frac{\pi^4}{525} \approx +0.110541$$

- What we have done in 2411.08615:
  - Worked out the expression for IC relevant to A, B in previous page (i.e. measure for $\mathcal{N}$=1* IC)
  - Proposed a method to evaluate the integration for IC: Even for $\mathcal{N}$=2*, the integral had been only numerically checked to be consistent with localization result.
  - We have successfully reproduced the $\mathcal{N}$=1* holographic result: $B = -\frac{3}{40} + \frac{\pi^4}{525}$

Setup

Cross ratios

$$U \equiv \frac{\vec{x}_{12}^2 \vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{24}^2} \qquad V \equiv \frac{\vec{x}_{14}^2 \vec{x}_{23}^2}{\vec{x}_{13}^2 \vec{x}_{24}^2}$$

$\langle SSSS \rangle$

$$\vec{\mathbf{S}} \equiv \begin{pmatrix} V & UV & U & U(U-V-1) & 1-U-V & V(V-U-1) \end{pmatrix}$$

$\langle SS\bar{P}P \rangle$

$$\begin{aligned}
\mathbf{R}_1(U,V,\partial_U,\partial_V) = \frac{1}{8}\big[&2U(U-V-3)\partial_U V + UV(2-U+2V)\partial_V^2 V \\
&- U^2(U-2-2V)\partial_U^2 V - (4V^2-4+U[1+U-5V])\partial_V V \\
&-U(U-2-2V)(U+V-1)\partial_V\partial_U V + 8V\big]\,, \\
\mathbf{R}_2(U,V,\partial_U,\partial_V) = \frac{1}{4}\big[&(4V+2UV-2-2V^2)\partial_V V + UV(V-1)\partial_V^2 V \\
&+U(1+U-V)\partial_U V + U(V-1)(U+V-1)\partial_V\partial_U V \\
&+U^2(V-1)\partial_U^2 V\big]\,, \\
\mathbf{R}_3(U,V,\partial_U,\partial_V) = \frac{1}{8}\big[&U(1+U-V)\partial_V V + U^2 V\partial_V^2 V \\
&+U^2(U+V-1)\partial_V\partial_U V + U^3\partial_U^2 V\big]\,.
\end{aligned}$$

$\langle P\bar{P}P\bar{P} \rangle$

$$\begin{aligned}
\mathbf{P}_1 = \frac{1}{4}\big[&U^4V^2\partial_U^2\partial_V^2 + 2U^4V^2\partial_U^3\partial_V + U^4V^2\partial_U^4 + U^4\partial_U^2 + 3U^4V\partial_U^2\partial_V + 3U^4V\partial_U^3 \\
&+ 2U^3V^3\partial_U\partial_V^3 + 4U^3V^3\partial_U^2\partial_V^2 + 2U^3V^3\partial_U^3\partial_V + 12U^3V^2\partial_U\partial_V^2 + 15U^3V^2\partial_U^2\partial_V - 2U^3V^2\partial_U^2\partial_V^2 \\
&+ 3U^3V^2\partial_U^3 - 2U^3V^2\partial_U^3\partial_V + 2U^3\partial_U + 14U^3V\partial_U\partial_V + 7U^3V\partial_U^2 - 2U^3\partial_U^2 - 6U^3V\partial_U^2\partial_V \\
&- 3U^3V\partial_U^3 + U^2V^4\partial_V^4 + 2U^2V^4\partial_U\partial_V^3 + U^2V^4\partial_U^2\partial_V^2 + 9U^2V^3\partial_V^3 + 9U^2V^3\partial_U\partial_V^2 \\
&- 2U^2V^3\partial_U\partial_V^3 - 2U^2V^3\partial_U^2\partial_V^2 + 19U^2V^2\partial_V^2 + 2U^2V^2\partial_U\partial_V - 9U^2V^2\partial_U\partial_V^2 - 2U^2V^2\partial_U^2 \\
&- 3U^2V^2\partial_U^2\partial_V + U^2V^2\partial_U^2\partial_V^2 + 4V\left(2U^2 - 4UV + 3U + 2V^2 - 5V + 3\right)\partial_V - 4U^2V\partial_U \\
&+ U^2\partial_U - 5U^2V\partial_U\partial_V + 3U^2V\partial_U^2 + U^2\partial_U^2 + 3U^2V\partial_U^2\partial_V - 3UV^4\partial_V^3 - 3UV^4\partial_U\partial_V^2 \\
&+ 4V^4\partial_V^2 - 17UV^3\partial_V^2 + 3UV^3\partial_V^3 - 4UV^3\partial_U\partial_V + 6UV^3\partial_U\partial_V^2 - 8V^3\partial_V^2 + 15UV^2\partial_V^2 \\
&+ 2UV^2\partial_U + 13UV^2\partial_U\partial_V - 3UV^2\partial_U\partial_V^2 + 4V^2\partial_V^2 - 3UV\partial_U - 3U\partial_U - 9UV\partial_U\partial_V + 4\big]\,,
\end{aligned}$$

Setup

[Chester, Pufu, 2003.08412]
$\mathcal{N}=2$ preserving mass deformation

- We consider the $\mathcal{N}=1$ preserving mass deformation on S^4.
- Giving three masses to three chiral multiplets $(Z_i\,,\chi_i)$, $\mathrm{i}=1,2,3$

$$S_{\mathcal{N}=4} \to S_{\mathcal{N}=4} + S_m$$

$$S_m = \int \mathrm{d}^4\vec{x}\,\sqrt{g}\sum_{j=1}^{3}\left[m_j\left(\mathrm{i}\,J_i + K_j\right) + m_j^2 L_j\right]$$

- We handle the operators J_i and K_i using polarization vectors.

$$J_i \equiv \frac{1}{2}\mathrm{tr}\left(Z_i^2 + \bar{Z}_i^2\right) = N_J\left(S_{IJ}Y_i^I Y_i^J + S_{IJ}Y_i^{*I}Y_i^{*J}\right)$$

$$K_i \equiv -\frac{1}{2}\mathrm{tr}(\chi_i\sigma_2\chi_i + \bar{\chi}_i\sigma_2\bar{\chi}_i) = N_K\left(P_{AB}\bar{X}_i^A\bar{X}_i^B + \bar{P}^{AB}X_A^i X_B^i\right)$$

$$L_i \equiv tr(|Z_i|^2)$$

Does not appear in the 4-pt function

$$Y_1 = \frac{1}{\sqrt{2}}(1,0,0,\mathrm{i},0,0) \qquad X_1 = \bar{X}_1 = (1,0,0,0)$$

$$Y_2 = \frac{1}{\sqrt{2}}(0,1,0,0,\mathrm{i},0) \qquad X_2 = \bar{X}_2 = (0,1,0,0)$$

$$Y_3 = \frac{1}{\sqrt{2}}(0,0,1,0,0,\mathrm{i}) \qquad X_3 = \bar{X}_3 = (0,0,1,0)$$

Setup

- We calculate **three integrated correlators** obtained by taking **four mass derivative** of the mass deformed S^4 free energy.

$$\mathcal{F}_4 \equiv \frac{\partial^4 F(m,m,0)}{\partial m^4}\bigg|_{m=0}$$

$\mathcal{N} = 2^*$ integrated correlators

[Chester, Pufu, 2003.08412]
$\mathcal{N} = 2$ **preserving mass deformation**

$$\mathcal{F}_{1111} \equiv \frac{\partial^4 F(m_1,m_2,m_3)}{\partial m_1^4}\bigg|_{m_i=0}$$

$$\mathcal{F}_{1122} \equiv \frac{\partial^4 F(m_1,m_2,m_3)}{\partial m_1^2\,\partial m_2^2}\bigg|_{m_i=0}$$

$\mathcal{N} = 1^*$ integrated correlators

- We expect a good match with the four-mass derivative of the holographic $\mathcal{N} = 1^*$ free energy. [N. Kim, S. Kim, 1904.02038]

$$F/N^2 = A_1(\mu_1^4+\mu_2^4+\mu_3^4) + A_2(\mu_1^2+\mu_2^2+\mu_3^2)^2 + \mathcal{O}(\mu^4)$$

Integrated correlator

- Integrated correlators can be split into two parts.

$$\mathcal{F}_4 = (\mathcal{F}_4)_{free} + (\mathcal{F}_4)_T\,, \qquad \mathcal{F}_{1111} = (\mathcal{F}_{1111})_{free} + (\mathcal{F}_{1111})_T\,, \qquad \mathcal{F}_{1122} = (\mathcal{F}_{1122})_{free} + (\mathcal{F}_{1122})_T$$

- Free parts are calculated using a different method.

- For the $\mathcal{N} = 2^*$ free part $(\mathcal{F}_4)_{free}$, it is computed through the 1-loop determinant of the partition function $Z_{S^4,free}$ of a hypermultiplet of mass m. [Chester, Pufu, 2003.08412]

$$(\mathcal{F}_4)_{free} = -\frac{\partial^4}{\partial m^4}\log Z_{S^4,free}\Big|_{m=0} = 48\,\zeta(3)\,c \qquad , \qquad c = \frac{N^2-1}{4}$$

- The hypermultiplet consists of two chiral multiplets with the same mass.

$$(\mathcal{F}_4)_{free} = -48\,\zeta(3)\,c\,, \qquad (\mathcal{F}_{1111})_{free} = -24\,\zeta(3)\,c\,, \qquad (\mathcal{F}_{1122})_{free} = 0$$

Integrated correlator

- Schematic form of the interaction part of the integrated correlators.

$$\left.\begin{array}{l} -(\mathcal{F}_4)_\mathcal{T} \\ -(\mathcal{F}_{1111})_\mathcal{T} \\ -(\mathcal{F}_{1122})_\mathcal{T} \end{array}\right\} = \left\langle \left(\int d^4\vec{x}\,\sqrt{g}\,(iJ+K) \right)^4 \right\rangle \Bigg|_\mathcal{T}$$

$$= \int \left(\prod_{i=1}^{4} d^4\vec{x}_i\, \Omega^4(\vec{x}_i) \right) \frac{[\Omega(\vec{x}_1)\Omega(\vec{x}_2)]^{-\Delta_A}\,[\Omega(\vec{x}_3)\Omega(\vec{x}_4)]^{-\Delta_B}}{|\vec{x}_{12}|^{2\Delta_A}\,|\vec{x}_{34}|^{2\Delta_B}} \underbrace{\mathcal{D}(U,V,\partial_U,\partial_V)}[\mathcal{T}(U,V)]$$

specific linear combination of $\vec{\mathbf{S}}$, $\vec{\mathbf{P}}$, $\vec{\mathbf{R}}$.

Cross ratios

$$U \equiv \frac{\vec{x}_{12}^2\vec{x}_{34}^2}{\vec{x}_{13}^2\vec{x}_{24}^2} \quad V \equiv \frac{\vec{x}_{14}^2\vec{x}_{23}^2}{\vec{x}_{13}^2\vec{x}_{24}^2}$$

S^4 metric

$$g_{\mu\nu} = \Omega^2\,\delta_{\mu\nu}$$

$$\Omega(\vec{x}) = \frac{1}{1+\frac{\vec{x}^2}{4}}$$

Scaling dimensions

$$\begin{cases} \langle SSSS\rangle \to (\Delta_A,\Delta_B) = (2,2) \\ \langle SS\bar{P}P\rangle \to (\Delta_A,\Delta_B) = (2,3) \\ \langle P\bar{P}P\bar{P}\rangle \to (\Delta_A,\Delta_B) = (3,3) \end{cases}$$

Integrated correlator

- Using the symmetry of the integral

$$\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Longrightarrow$$

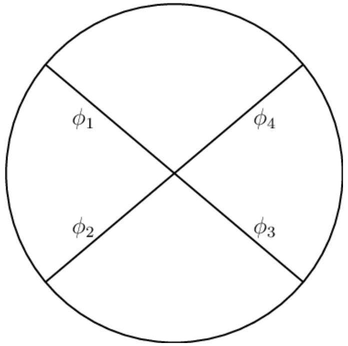
16-fold integration

4-fold $\Longrightarrow$ $\mathrm{Vol}(S^4)$

3-fold $\Longrightarrow$ $\mathrm{Vol}(S^3)$

2-fold $\Longrightarrow$ $\mathrm{Vol}(S^2)$

5-fold $\Longrightarrow$ $D_{r_1r_2r_3r_4}(x_1,x_2,x_3,x_4) = \int \frac{dz_0\, d^4\vec{z}}{z_0^5} \prod_{i=1}^{4} K_{r_i}(z,\vec{x}_i)$



(Contact Witten diagram in AdS_5 of four scalar)

$$\left.\begin{array}{l} -(\mathcal{F}_4)_{\mathcal{T}} \\ -(\mathcal{F}_{1111})_{\mathcal{T}} \\ -(\mathcal{F}_{1122})_{\mathcal{T}} \end{array}\right\} \Rightarrow$$

$$\frac{\mathrm{Vol}(S^4)\mathrm{Vol}(S^3)\mathrm{Vol}(S^2)\,\pi^2\,\Gamma(6-\Delta_A-\Delta_B)}{2^{2\Delta_A+2\Delta_B-11}\,\Gamma(4-\Delta_A)^2\,\Gamma(4-\Delta_B)^2}$$

(conformal invariant part of $D_{r_1r_2r_3r_4}(x_1,x_2,x_3,x_4)$)

$$\times \int dr\, d\theta\, r^3\,(\sin\theta)^2\, \frac{\mathcal{D}(U,V,\partial_U,\partial_V)[\mathcal{T}(U,V)]}{U^{\Delta_A}} \overbrace{\overline{D}_{4-\Delta_A,4-\Delta_A,4-\Delta_B,4-\Delta_B}(U,V)}$$

2-fold integration

$$U = 1 + r^2 - 2r\cos\theta$$
$$V = r^2$$

Integrated correlator

- Integration by parts

$$U = 1 + r^2 - 2r\cos\theta$$
$$V = r^2$$

$$\left.\begin{array}{l} -(\mathcal{F}_4)_{\mathcal{T}} \\ -(\mathcal{F}_{1111})_{\mathcal{T}} \\ -(\mathcal{F}_{1122})_{\mathcal{T}} \end{array}\right\} \Rightarrow \frac{\mathrm{Vol}(S^4)\mathrm{Vol}(S^3)\mathrm{Vol}(S^2)\,\pi^2\,\Gamma(6-\Delta_A-\Delta_B)}{2^{2\Delta_A+2\Delta_B-11}\,\Gamma(4-\Delta_A)^2\,\Gamma(4-\Delta_B)^2}$$
$$\times \int dr\,d\theta\,r^3\,(\sin\theta)^2\,\frac{\boxed{\mathcal{D}(U,V,\partial_U,\partial_V)}[\mathcal{T}(U,V)]}{U^{\Delta_A}}\,\overline{D}_{4-\Delta_A,4-\Delta_A,4-\Delta_B,4-\Delta_B}(U,V)$$

More schematically

$$\Rightarrow \boxed{\int dr\,d\theta\,r^3\sin^2\theta\,\frac{\mathcal{T}(U,V)}{U^2}\mathcal{D}'(U,V,\partial_U,\partial_V)\left[\overline{D}_{r_1r_1r_2r_2}(U,V)\right]}$$

Integrated correlator

- Mellin transformation

$$f(U,V) = \int \frac{ds\,dt}{(2\pi i)^2} U^s\, V^t \Gamma(-s)^2\Gamma(-t)^2\Gamma(-u)^2\mathcal{M}(s,t,u)\Big|_{u=-s-t}$$

$$U \equiv \frac{\vec{x}_{12}^2\vec{x}_{34}^2}{\vec{x}_{13}^2\vec{x}_{24}^2} \qquad V \equiv \frac{\vec{x}_{14}^2\vec{x}_{23}^2}{\vec{x}_{13}^2\vec{x}_{24}^2}$$

$\boxed{\text{Permutations of } (x_1, x_2, x_3, x_4)}$ $\Rightarrow$ $\boxed{\text{Nontrivial changes of } (U,V)}$

$$(U,V) \rightarrow \begin{cases} (U,V) \\ (U/V\,,1/V) \\ (V,U) \\ (V/U\,,1/U) \\ (1/V\,,U/V) \\ (1/U\,,V/U) \end{cases}$$

$\Rightarrow$ $\boxed{\text{Exchanges of } (s,t,u)}$

$$\mathcal{M}(s,t,u) \rightarrow \begin{cases} \mathcal{M}\,(s,t,u) \\ \mathcal{M}\,(s,u,t) \\ \mathcal{M}\,(t,s,u) \\ \mathcal{M}(t,u,s) \\ \mathcal{M}(u,s,t) \\ \mathcal{M}(u,t,s) \end{cases}$$

- To facilitate the application of **crossing symmetry**, we use the **Mellin transform** of the $\bar{D}$-function.

$$\bar{D}_{r_1,r_1,r_2,r_2}(U,V) \equiv \int \frac{ds\,dt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{t}{2}} \Gamma\left(-\frac{s+2(r_1-r_2)}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma\left(-\frac{t}{2}\right)^2\Gamma\left(\frac{s+t+2r_1}{2}\right)^2$$

Integrated correlator

Mellin transform of the $\overline{D}$-function

$$\bar{D}_{r_1,r_1,r_2,r_2}(U,V) \equiv \int \frac{ds\,dt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{t}{2}} \Gamma\left(-\frac{s+2(r_1-r_2)}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma\left(-\frac{t}{2}\right)^2 \Gamma\left(\frac{s+t+2r_1}{2}\right)^2$$

- Using **crossing symmetry**

$$\left.\begin{array}{l} -(\mathcal{F}_4)_{\mathcal{T}} \\ -(\mathcal{F}_{1111})_{\mathcal{T}} \\ -(\mathcal{F}_{1122})_{\mathcal{T}} \end{array}\right\} \Rightarrow \int dr\, d\theta\, r^3 \sin^2\theta \frac{\mathcal{T}(U,V)}{U^2} \underbrace{\mathcal{D}'(U,V,\partial_U,\partial_V)\left[\overline{D}_{r_1 r_1 r_2 r_2}(U,V)\right]}$$

Mellin transformation

$$\Rightarrow \int dr\, d\theta\, r^3 \sin^2\theta \frac{\mathcal{T}(U,V)}{U^2} \underbrace{\int \frac{ds\,dt}{(2\pi i)^2} U^s V^t \Gamma(-s)^2 \Gamma(-t)^2 \Gamma(-u)^2 Q(s,t,u)}$$

Integrated correlator

- Using **crossing symmetry**

Change $(U,V) \to \left\{(U,V), \left(\frac{U}{V}, \frac{1}{V}\right), (V,U), \left(\frac{V}{U}, \frac{1}{U}\right), \left(\frac{1}{V}, \frac{U}{V}\right), \left(\frac{1}{U}, \frac{V}{U}\right)\right\}$

$$\Rightarrow \int dr\, d\theta\, r^3 \sin^2\theta \frac{\mathcal{T}(U,V)}{U^2} \int \frac{ds\, dt}{(2\pi i)^2} U^s V^t \Gamma(-s)^2\Gamma(-t)^2\Gamma(-u)^2 Q(s,t,u)$$

$$\mathcal{T}\left(\frac{U}{V}, \frac{1}{V}\right) = \mathcal{T}\left(\frac{V}{U}, \frac{1}{U}\right) = V^2 \mathcal{T}(U,V)$$

$$\mathcal{T}\left(\frac{1}{U}, \frac{V}{U}\right) = \mathcal{T}(U,V)$$

$$\mathcal{T}\left(\frac{1}{V}, \frac{U}{V}\right) = \mathcal{T}(V,U) = \frac{V^2}{U^2}\mathcal{T}(U,V)$$

$$\Rightarrow \underbrace{\int dr\, d\theta\, r^3 \sin^2\theta \frac{\mathcal{T}(U,V)}{U^2}}_{\text{Invariant}} \int \frac{ds\, dt}{(2\pi i)^2} U^s V^t \Gamma(-s)^2\Gamma(-t)^2\Gamma(-u)^2$$

$$\underbrace{\times \frac{1}{6}\{Q(s,t) + Q(s,u) + Q(t,s) + Q(t,u) + Q(u,s) + Q(u,t)\}}_{\text{symmetrize}}$$

Integrated correlator

Mellin transform of the $\overline{D}$-function

$$\bar{D}_{r_1,r_1,r_2,r_2}(U,V) \equiv \int \frac{ds\,dt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{t}{2}} \Gamma\left(-\frac{s+2(r_1-r_2)}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma\left(-\frac{t}{2}\right)^2 \Gamma\left(\frac{s+t+2r_1}{2}\right)^2$$

- Returning to position space (U,V), we finally obtain

$$\left.\begin{array}{l} -(\mathcal{F}_4)_{\mathcal{T}} \\ -(\mathcal{F}_{1111})_{\mathcal{T}} \\ -(\mathcal{F}_{1122})_{\mathcal{T}} \end{array}\right\} \Rightarrow \int dr\, d\theta\, r^3 \sin^2\theta \frac{\mathcal{T}(U,V)}{U^2} \int \frac{ds\,dt}{(2\pi i)^2} U^s V^t \Gamma(-s)^2\Gamma(-t)^2\Gamma(-u)^2 \times \frac{1}{6}\{Q(s,t)+Q(s,u)+Q(t,s)+Q(t,u)+Q(u,s)+Q(u,t)\}$$

$$-(\mathcal{F}_4)_{\mathcal{T}} = \frac{32c^2}{\pi}\int dr\, d\theta\, r^3 \sin^2\theta \frac{\mathcal{T}(U,V)}{U^2}(1+U+V)\overline{D}_{1111}$$

$$-(\mathcal{F}_{1111})_{\mathcal{T}} = \frac{48c^2}{\pi}\int dr\, d\theta\, r^3 \sin^2\theta \frac{\mathcal{T}(U,V)}{U^2}\frac{1}{6}\left[(2+2U+2V)\overline{D}_{1111} - 2\left(U\overline{D}_{1212}+V\overline{D}_{2121}+U\overline{D}_{2112}+V\overline{D}_{1221}+U\overline{D}_{2211}+V\overline{D}_{1122}\right)\right]$$

$$-(\mathcal{F}_{1122})_{\mathcal{T}} = \frac{32c^2}{\pi}\int dr\, d\theta\, r^3 \sin^2\theta \frac{\mathcal{T}(U,V)}{U^2}\frac{1}{6}\left(U\overline{D}_{1212}+V\overline{D}_{2121}+U\overline{D}_{2112}+V\overline{D}_{1221}+U\overline{D}_{2211}+V\overline{D}_{1122}\right)$$

- We find that $\boxed{-(\mathcal{F}_{1111})_{\mathcal{T}} = 3(\mathcal{F}_{1122})_{\mathcal{T}} - \frac{1}{2}(\mathcal{F}_4)_{\mathcal{T}}}$

Integrated correlator

- We perform the evaluation using a symmetry-broken expression of $-(\mathcal{F}_{1122})_{\mathcal{T}}$.

$$\boxed{\begin{aligned} -(\mathcal{F}_4)_{\mathcal{T}} &= \frac{32c^2}{\pi}\int dr\, d\theta\, r^3 \sin^2\theta \frac{\mathcal{T}(U,V)}{U^2}(1+U+V)\overline{D}_{1111} \\ -(\mathcal{F}_{1122})_{\mathcal{T}} &= \frac{32c^2}{\pi}\int dr\, d\theta\, r^3 \sin^2\theta \frac{\mathcal{T}(U,V)}{U^2}\overline{D}_{2112} \end{aligned}}$$

$$\boxed{-(\mathcal{F}_{1111})_{\mathcal{T}} = 3(\mathcal{F}_{1122})_{\mathcal{T}} - \frac{1}{2}(\mathcal{F}_4)_{\mathcal{T}}}$$

Integrated correlator

- We perform the evaluation using a symmetry-broken expression of $-(\mathcal{F}_{1122})_{\mathcal{T}}$.

$$-(\mathcal{F}_4)_{\mathcal{T}} = \frac{32c^2}{\pi}\int dr\, d\theta\, r^3 \sin^2\theta \frac{\mathcal{T}(U,V)}{U^2}(1+U+V)\overline{D}_{1111}$$
$$-(\mathcal{F}_{1122})_{\mathcal{T}} = \frac{32c^2}{\pi}\int dr\, d\theta\, r^3 \sin^2\theta \frac{\mathcal{T}(U,V)}{U^2}\overline{D}_{2112}$$

$$-(\mathcal{F}_{1111})_{\mathcal{T}} = 3(\mathcal{F}_{1122})_{\mathcal{T}} - \frac{1}{2}(\mathcal{F}_4)_{\mathcal{T}}$$

- In the **supergravity limit**, $\mathcal{T}(U,V) \to \frac{8}{c}\mathcal{T}^{SG}(U,V) = -\frac{U^2}{c}\overline{D}_{2422}(U,V)$

$$-(\mathcal{F}_4)_{\mathcal{T}}\Big|_{\mathcal{T}\to\frac{8}{c}\mathcal{T}^{SG}} = -\frac{32c}{\pi}\int dr\, d\theta\, r^3 \sin^2\theta\, \overline{D}_{2422}(1+U+V)\overline{D}_{1111}$$
$$-(\mathcal{F}_{1122})_{\mathcal{T}}\Big|_{\mathcal{T}\to\frac{8}{c}\mathcal{T}^{SG}} = -\frac{32c}{\pi}\int dr\, d\theta\, r^3 \sin^2\theta\, \overline{D}_{2422}\overline{D}_{2112}$$

- $\overline{D}_{abcd}$ is expressed as a derivative of $\overline{D}_{1111}$.

$$\overline{D}_{2422} = \partial_U\partial_V\,(1+U\partial_U+V\partial_V)\,\overline{D}_{1111}\ , \qquad \overline{D}_{2112} = -V\partial_V\overline{D}_{1111}$$

Integral Evaluation

$U = r^2 = z\bar{z}$

$V = 1 + r^2 - 2r\cos\theta = (1-z)(1-\bar{z})$

- $\overline{D}_{1111}(U,V)$ can be expressed in various forms.

$$\boxed{\overline{D}_{1111}(U,V)}$$

One-loop box Feynman diagram

$$\frac{1}{\pi^2} x_{13}^2 x_{24}^2 \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

Bloch-Wigner dilogarithm expression

$$\frac{1}{z-\bar{z}} \left(\log(z\bar{z}) \log(\frac{1-z}{1-\bar{z}}) + 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) \right)$$

We use this !

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} (2 - n\log(z\bar{z})) \frac{z^n - \bar{z}^n}{z - \bar{z}}}$$

$$\int \frac{ds\, dt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{t}{2}} \Gamma\left(-\frac{s}{2}\right)^2 \Gamma\left(-\frac{t}{2}\right)^2 \Gamma\left(\frac{s+t+2}{2}\right)^2$$

Mellin-Barnes representation

$$\frac{1}{z-\bar{z}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{n}{(\nu^2 + \frac{n^2}{4})^2} (z\bar{z})^{i\nu} \left(\frac{z}{\bar{z}}\right)^{n/2}$$

Fourier-Mellin form

Analytical evaluation

Integrated correlators

$$-(\mathcal{F}_4)_{\mathcal{T}}\Big|_{\mathcal{T}\to\frac{8}{c}\mathcal{T}^{SG}} = -\frac{32c}{\pi}\int dr\, d\theta\, r^3 \sin^2\theta \overline{D}_{2422}(1+U+V)\overline{D}_{1111}$$

$$-(\mathcal{F}_{1122})_{\mathcal{T}}\Big|_{\mathcal{T}\to\frac{8}{c}\mathcal{T}^{SG}} = -\frac{32c}{\pi}\int dr\, d\theta\, r^3 \sin^2\theta \overline{D}_{2422}\overline{D}_{2112}$$

- Analytic approach to the integral

$$\frac{1}{2\pi}\int_0^1 dr \int_0^{2\pi} d\theta\, r^3 \sin^2\theta\, f(U,V)\,\overline{D}_{ABCD}(U,V)\,\overline{D}_{abcd}(U,V)$$

We use this

$$\overline{D}_{1111} = \sum_{n=1}^{\infty} \frac{2r^{n-1}}{n^2}(1-n\log r)\frac{\sin n\theta}{\sin\theta}$$

$$U = r^2$$

$$V = 1 + r^2 - 2r\cos\theta$$

1. *Set the integration range* $[\, r : 0\sim 1\, ,\ \ \theta : 0\sim 2\pi\,]$

2. *Derive* $A_{n_1\, n_2}$
   a. θ*-integration*
   b. r*-integration*

Integrated correlator $\displaystyle = \sum_{n_1,n_2=1}^{\infty} A_{n_1\, n_2}$

3. *Regularization*

Analytical evaluation($\mathcal{N}$=2*)

$$I_4[\mathcal{T}^{SG}] = -\frac{1}{8c}(\mathcal{F}_4)_{\mathcal{T}}\Big|_{\mathcal{T}\to\frac{8}{c}\mathcal{T}^{SG}}$$

1. *Set the integration range* [r : 0~1 , θ : 0~2π]

$$I_4[\mathcal{T}^{SG}] = -\frac{4}{\pi}\int_0^1 dr \int_0^{2\pi} d\theta\, r^3 \sin^2\theta (1+U+V)\overline{D}_{1111}(U,V)\overline{D}_{2422}(U,V)$$

We use this

$$\boxed{\overline{D}_{1111} = \sum_{n=1}^{\infty} \frac{2r^{n-1}}{n^2}(1-n\log r)\frac{\sin n\theta}{\sin\theta}}$$

Analytical evaluation($\mathcal{N}$=2*)

$$I_4[\mathcal{T}^{SG}] = -\frac{1}{8c}(\mathcal{F}_4)_{\mathcal{T}}\Big|_{\mathcal{T}\to\frac{8}{c}\mathcal{T}^{SG}}$$

1. *Set the integration range* [r : 0~1 , θ : 0~2π]

$$I_4[\mathcal{T}^{SG}] = -\frac{4}{\pi}\int_0^1 dr \int_0^{2\pi} d\theta\, r^3 \sin^2\theta (1+U+V)\overline{D}_{1111}(U,V)\overline{D}_{2422}(U,V)$$

We use this

$$\boxed{\overline{D}_{1111} = \sum_{n=1}^{\infty} \frac{2r^{n-1}}{n^2}(1-n\log r)\frac{\sin n\theta}{\sin\theta}}$$

$$\overline{D}_{2422} = \partial_U \partial_V \left(1 + U\partial_U + V\partial_V\right)\overline{D}_{1111}$$

$$\partial_U = \frac{z-1}{z-\bar{z}}\frac{\partial}{\partial z} - \frac{\bar{z}-1}{z-\bar{z}}\frac{\partial}{\partial \bar{z}}$$

$$\partial_V = \frac{-z}{z-\bar{z}}\frac{\partial}{\partial z} + \frac{\bar{z}}{z-\bar{z}}\frac{\partial}{\partial \bar{z}}$$

2. *Derive* $A_{n_1 n_2}$

$$I_4[\mathcal{T}^{SG}] = \sum_{n_1,n_2=1}^{\infty}\int_0^1 dr \int_0^{2\pi}\frac{d\theta}{2\pi}\left[\frac{\cos(n_1 \pm n_2 + \mathbb{Z})}{\sin^6\theta} + \dots\right]$$

a. θ-*integration*

We use $\displaystyle \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos(2k\theta)}{\sin^6\theta}d\theta = -\frac{4}{15}\sqrt{k^2}(k^2-1)(k^2-4)$

b. r-*integration*

Analytical evaluation($\mathcal{N}$=2*)

- We obtain

$$I_4[\mathcal{T}^{SG}] = \sum_{n_1=n_2} d_{n_1,n_2} + \sum_{\substack{n_1+n_2=\text{even} \\ n_1>n_2}} A_{n_1,n_2} + \sum_{\substack{n_1+n_2=\text{odd} \\ n_1>n_2}} B_{n_1,n_2}$$

$$\begin{aligned}
d_{n_1,n_2} =& \frac{6(n-1)}{n^3}\,, \\
A_{n_1,n_2} =& \big(12n_1^9 + 72n_2n_1^8 + 132n_2^2n_1^7 - 80n_1^7 + 12n_2^3n_1^6 - 288n_2n_1^6 - 228n_2^4n_1^5 - 96n_2^2n_1^5 \\
& + 128n_1^5 - 228n_2^5n_1^4 + 640n_2^3n_1^4 + 256n_2n_1^4 + 12n_2^6n_1^3 + 752n_2^4n_1^3 - 640n_2^2n_1^3 \\
& + 132n_2^7n_1^2 + 96n_2^5n_1^2 - 1024n_2^3n_1^2 + 72n_2^8n_1 - 192n_2^6n_1 - 256n_2^4n_1 + 512n_2^2n_1 \\
& + 12n_2^9 - 64n_2^7 + 256n_2^3\big)\big/\big(n_1^2n_2\,(n_1+n_2-2)^2\,(n_1+n_2)^2\,(n_1+n_2+2)^2\,\big)\,, \\
B_{n_1,n_2} =& \big(-12n_1^7 - 48n_2n_1^6 - 24n_2^2n_1^5 + 8n_1^5 + 84n_2^3n_1^4 - 16n_2n_1^4 + 84n_2^4n_1^3 \\
& - 40n_2^2n_1^3 + 4n_1^3 - 24n_2^5n_1^2 - 88n_2^3n_1^2 - 48n_2^6n_1 - 80n_2^4n_1 \\
& - 12n_2^7 - 8n_2^5 + 20n_2^3\big)\big/\big(n_1^2n_2\,(n_1+n_2-1)^2\,(n_1+n_2+1)^2\,\big)\,,
\end{aligned}$$

- For several small values of n_1 and n_2

$n_2=8$	0	0	0	0	0	0	0	$\frac{21}{256}$
$n_2=7$	0	0	0	0	0	0	$\frac{36}{343}$	$-\frac{4513}{6272}$
$n_2=6$	0	0	0	0	0	$\frac{5}{36}$	$-\frac{1951}{2058}$	$\frac{14853}{6272}$
$n_2=5$	0	0	0	0	$\frac{24}{125}$	$-\frac{98}{75}$	$\frac{245177}{77175}$	$-\frac{100181}{17640}$
$n_2=4$	0	0	0	$\frac{9}{32}$	$-\frac{1919}{1000}$	$\frac{2699}{600}$	$-\frac{86258}{11025}$	$\frac{497573}{44100}$
$n_2=3$	0	0	$\frac{4}{9}$	$-\frac{149}{48}$	$\frac{10361}{1500}$	$-\frac{1157}{100}$	$\frac{179371}{11025}$	$-\frac{4738}{225}$
$n_2=2$	0	$\frac{3}{4}$	$-\frac{107}{18}$	$\frac{1757}{144}$	$-\frac{34753}{1800}$	$\frac{70957}{2700}$	$-\frac{325249}{9800}$	$\frac{1997}{50}$
$n_2=1$	0	-17	$\frac{271}{9}$	$-\frac{1529}{36}$	$\frac{12479}{225}$	$-\frac{3665}{54}$	$\frac{394087}{4900}$	$-\frac{2318}{25}$
	$n_1=1$	$n_1=2$	$n_1=3$	$n_1=4$	$n_1=5$	$n_1=6$	$n_1=7$	$n_1=8$

Diverge !

Analytical evaluation($\mathcal{N}$=2*)

3. *Regularization*

- We use Abel's summation for regularization.

$$\sum_{k=1}^{\infty} a_k = \lim_{t\to 1} \sum_{k=1}^{\infty} a_k t^k = \lim_{\theta \to 0} \sum_{k=1}^{\infty} a_k e^{ik\theta}$$

Oscillating factor $t = e^{i\theta}$

- We perform regularization twice. **The first sum** in the double sum proceeds as *Figure2.*

$$I_4[\mathcal{T}^{SG}] = \sum_{n_1=n_2} d_{n_1,n_2} + \sum_{\substack{n_1+n_2=\text{even} \\ n_1>n_2}} A_{n_1,n_2} + \sum_{\substack{n_1+n_2=\text{odd} \\ n_1>n_2}} B_{n_1,n_2}$$

$$= \pi^2 - 6\zeta(3) + \sum_{\substack{n_1+n_2=\text{even} \\ n_1>n_2}} A_{n_1,n_2} + \sum_{\substack{n_1+n_2=\text{odd} \\ n_1>n_2}} B_{n_1,n_2}$$

$n_2=8$	0	0	0	0	0	0	0	$\frac{21}{256}$
$n_2=7$	0	0	0	0	0	0	$\frac{36}{343}$	$-\frac{4513}{6272}$
$n_2=6$	0	0	0	0	0	$\frac{5}{36}$	$-\frac{1951}{2058}$	$\frac{14853}{6272}$
$n_2=5$	0	0	0	0	$\frac{24}{125}$	$-\frac{98}{75}$	$\frac{245177}{77175}$	$-\frac{100181}{17640}$
$n_2=4$	0	0	0	$\frac{9}{32}$	$-\frac{1919}{1000}$	$\frac{2699}{600}$	$-\frac{86258}{11025}$	$\frac{497573}{44100}$
$n_2=3$	0	0	$\frac{4}{9}$	$-\frac{149}{48}$	$\frac{10361}{1500}$	$-\frac{1157}{100}$	$\frac{179371}{11025}$	$-\frac{4738}{225}$
$n_2=2$	0	$\frac{3}{4}$	$-\frac{107}{18}$	$\frac{1757}{144}$	$-\frac{34753}{1800}$	$\frac{70957}{2700}$	$-\frac{325249}{9800}$	$\frac{1997}{50}$
$n_2=1$	0	-17	$\frac{271}{9}$	$-\frac{1529}{36}$	$\frac{12479}{225}$	$-\frac{3665}{54}$	$\frac{394087}{4900}$	$-\frac{2318}{25}$
	$n_1=1$	$n_1=2$	$n_1=3$	$n_1=4$	$n_1=5$	$n_1=6$	$n_1=7$	$n_1=8$

Diverge

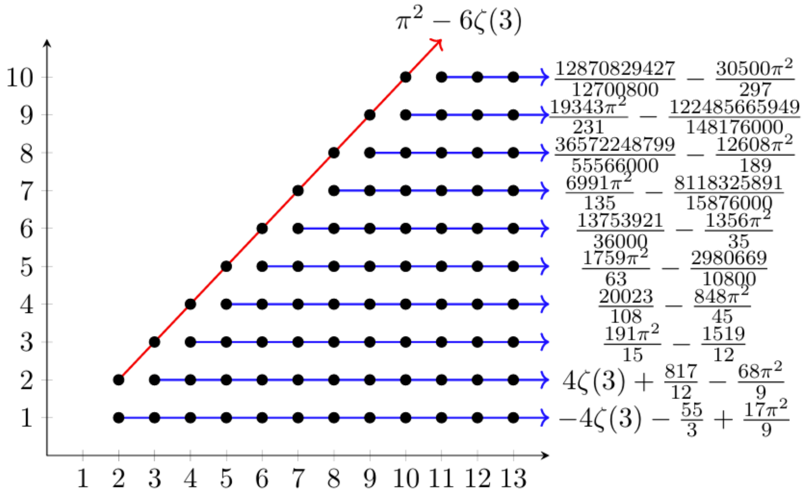


Figure 2: The sum of the integrated correlator $I_4[\mathcal{T}^{SG}]$ is performed first horizontally and then vertically. The outermost diagonal line is handled separately.

Analytical evaluation($\mathcal{N}$=2*)

- We obtain

$$I_4[\mathcal{T}^{SG}] = \frac{199}{4} - \frac{14\pi^2}{3} - 6\zeta(3) + \sum_{n=3,5,\dots}^{\infty} C_n + \sum_{n=4,6,\dots}^{\infty} C_n$$

$$\begin{aligned}
\sum_{n=3,5,\dots}^{\infty} C_n = \sum_{k=1,2,\dots}^{\infty} \Bigg[& -\frac{1}{2k-1} - \frac{6}{2k+1} - \frac{4}{2k+3} + \frac{2}{k} + 6 \\
& + \left(4k^2 + 4k + \frac{4}{3(2k-1)} - \frac{4}{3(2k+3)} + \frac{11}{3}\right)\pi^2 \\
& + \left(24k^2 + 24k - \frac{2}{k+1} + \frac{4}{2k-1} - \frac{4}{2k+3} + \frac{2}{k} + 22\right) H_k^{(2)} \\
& + \left(-48k^2 - 48k + \frac{2}{k+1} - \frac{12}{2k-1} + \frac{12}{2k+3} - \frac{2}{k} - 44\right) H_{2k+1}^{(2)} \Bigg], \\
\sum_{n=4,6,\dots}^{\infty} C_n = \sum_{k=1,2,\dots}^{\infty} \Bigg[& -\frac{3}{k+1} - \frac{1}{2(k+2)} + \frac{4}{2k+3} - \frac{3}{(k+1)^2} - \frac{2}{k} - 6 \\
& + \left(-4k^2 - 8k - \frac{4}{3(2k+1)} + \frac{4}{3(2k+3)} - \frac{20}{3}\right)\pi^2 \\
& + \left(-24k^2 - 48k + \frac{2}{k+2} - \frac{4}{2k+1} + \frac{4}{2k+3} - \frac{2}{k} - 40\right) H_k^{(2)} \\
& + \left(48k^2 + 96k - \frac{2}{k+2} + \frac{12}{2k+1} - \frac{12}{2k+3} + \frac{2}{k} + 80\right) H_{2k+1}^{(2)} \Bigg],
\end{aligned}$$

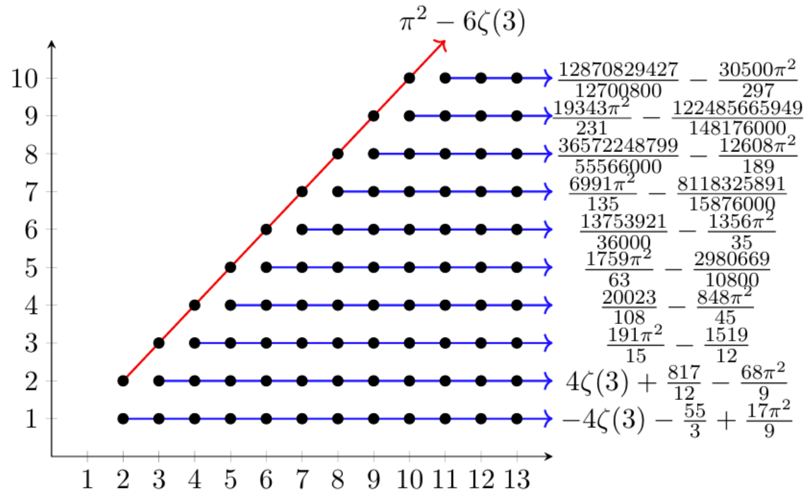


Figure 2: The sum of the integrated correlator $I_4[\mathcal{T}^{SG}]$ is performed first horizontally and then vertically. The outermost diagonal line is handled separately.

Analytical evaluation($\mathcal{N}$=2*)

- In **the second regularization**, we encounter

$$\sum_{k=1}^{\infty} \frac{1}{(a\,k+b)^c}\, t^{2k+d}\, H^{(r)}_{p\,k+q}$$

- To carry out this summation, We need a closed form of

$$\mathcal{H}^r_{p\,q}(t) \equiv \sum_{k=1}^{\infty} t^{2k}\, H^{(r)}_{p\,k+q}$$

$$\boxed{c<0} \qquad \sum_{k=1}^{\infty} k^{-c} t^{2k+d} H^{(r)}_{p\,k+q} = t^d \left(\frac{t}{2}\partial_t\right)^{-c} \mathcal{H}^r_{p\,q}(t)$$

$$\boxed{c>0} \qquad \sum_{k=1}^{\infty} \frac{1}{(a\,k+b)^c}\, t^{2k+d}\, H^{(r)}_{p\,k+q} = \left(\frac{2}{a}\right)^c t^{d-\frac{2b}{a}} \int_0^t \frac{dt_c}{t_c} \cdots \int_0^{t_2} \frac{dt_1}{t_1} t_1^{\frac{2b}{a}}\, \mathcal{H}^r_{p\,q}(t_1)$$

$$\begin{aligned}
\sum_{n=3,5,\ldots}^{\infty} C_n = \sum_{k=1,2,\ldots}^{\infty} \Bigg[& -\frac{1}{2k-1} - \frac{6}{2k+1} - \frac{4}{2k+3} + \frac{2}{k} + 6 \\
& + \left(4k^2 + 4k + \frac{4}{3(2k-1)} - \frac{4}{3(2k+3)} + \frac{11}{3}\right)\pi^2 \\
& + \left(24k^2 + 24k - \frac{2}{k+1} + \frac{4}{2k-1} - \frac{4}{2k+3} + \frac{2}{k} + 22\right) H^{(2)}_k \\
& + \left(-48k^2 - 48k + \frac{2}{k+1} - \frac{12}{2k-1} + \frac{12}{2k+3} - \frac{2}{k} - 44\right) H^{(2)}_{2k+1} \Bigg],
\end{aligned}$$

$$\begin{aligned}
\sum_{n=4,6,\ldots}^{\infty} C_n = \sum_{k=1,2,\ldots}^{\infty} \Bigg[& -\frac{3}{k+1} - \frac{1}{2(k+2)} + \frac{4}{2k+3} - \frac{3}{(k+1)^2} - \frac{2}{k} - 6 \\
& + \left(-4k^2 - 8k - \frac{4}{3(2k+1)} + \frac{4}{3(2k+3)} - \frac{20}{3}\right)\pi^2 \\
& + \left(-24k^2 - 48k + \frac{2}{k+2} - \frac{4}{2k+1} + \frac{4}{2k+3} - \frac{2}{k} - 40\right) H^{(2)}_k \\
& + \left(48k^2 + 96k - \frac{2}{k+2} + \frac{12}{2k+1} - \frac{12}{2k+3} + \frac{2}{k} + 80\right) H^{(2)}_{2k+1} \Bigg],
\end{aligned}$$

Analytical evaluation($\mathcal{N}$=2*)

- We obtain

$$I_4[\mathcal{T}^{SG}] = \frac{199}{4} - \frac{14\pi^2}{3} - 6\zeta(3) + \sum_{n=3,5,\ldots}^{\infty} C_n + \sum_{n=4,6,\ldots}^{\infty} C_n$$

$$\sum_{k=1,2,\ldots}^{\infty} C_{2k+1}\, t^{2k+1}\Big|_{t\to e^{i\theta}} = \frac{\log(\theta)}{2} - 9\zeta(3) - \frac{95\pi^2}{36} - \frac{i\pi}{4} + \frac{64}{3} + \frac{43\log(2)}{2},$$

$$\sum_{k=1,2,\ldots}^{\infty} C_{2k+2}\, t^{2k+2}\Big|_{t\to e^{i\theta}} = \frac{\log(\theta)}{2} + 9\zeta(3) + \frac{263\pi^2}{36} - \frac{i\pi}{4} - \frac{817}{12} - \frac{43\log(2)}{2}$$

Oscillating factor $t = e^{i\theta}$

$$\boxed{I_4[\mathcal{T}^{SG}] = 3 - 6\zeta(3) + \log(\theta) - \frac{i\pi}{2}}$$

$=$

up to the **log divergence** and the imaginary part

[Chester, Pufu, 2003.08412]

$$I_4[\mathcal{T}^{SG}] = 3 - 6\zeta(3)$$

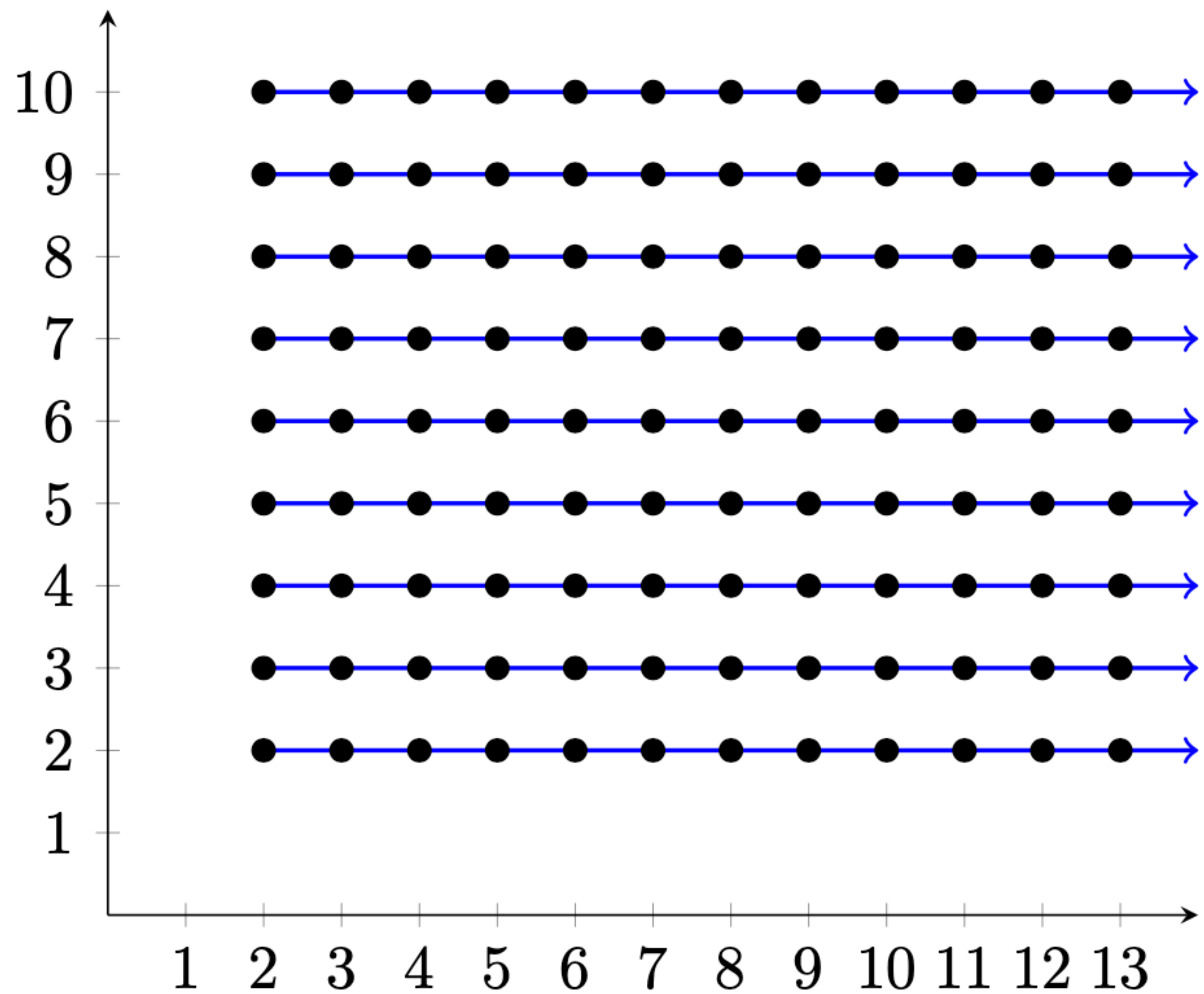
numerically high precision

- For $\mathcal{N}$=1* integrated correlators, we obtain

$$I_{1111}[\mathcal{T}^{\rm SG}] = \frac{3}{5} + \frac{4}{175}\pi^4 - 3\zeta(3) - i\frac{24}{5}\pi\zeta(3)$$

$$I_{1122}[\mathcal{T}^{\rm SG}] = \frac{3}{10} - \frac{4}{525}\pi^4 + i\frac{8}{5}\pi\zeta(3)$$

good match..



$$\frac{625765701782861}{16669800000} - \frac{2423520\pi^2}{637} + \frac{1584\zeta(3)}{175}$$

$$-\frac{6162599354819}{250047000} + \frac{326938240\pi^2}{130977} + \frac{512\zeta(3)}{63}$$

$$\frac{813465748553}{52920000} - \frac{428544\pi^2}{275} + \frac{36\zeta(3)}{5}$$

$$-\frac{104506346759}{11576250} + \frac{23510912\pi^2}{25725} + \frac{1536\zeta(3)}{245}$$

$$\frac{2192167}{450} - \frac{4448\pi^2}{9} + \frac{16\zeta(3)}{3}$$

$$-\frac{111923909}{47250} + \frac{1466816\pi^2}{6125} + \frac{768\zeta(3)}{175}$$

$$\frac{1514621}{1512} - \frac{4992\pi^2}{49} + \frac{24\zeta(3)}{7}$$

$$-\frac{136328}{405} + \frac{23744\pi^2}{675} - \frac{128\zeta(3)}{15}$$

$$\frac{144}{5} - \frac{96\pi^2}{25} + \frac{48\zeta(3)}{5}$$

Discussion

- Our method for evaluation of IC involving $\bar{D}_{r_1,r_2,r_3,r_4}(U,V)$ is admittedly rather inefficient. A Mellin representation technique is desirable.

- Log divergence in general, as we re-sum the series.

- Correlators with heavy or non-local operators?

- Higher order results in holography and integrated 6- ,8-point functions?